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Homotopy relations for digital images

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Abstract. We introduce three generalizations of homotopy equivalence in digital images, to allow us to express whether a bounded and an unbounded digital image with standard adjacencies are similar with respect to homotopy.

We show that these three generalizations are not equivalent to ordinary homotopy equivalence, and give several examples. We show that, like homotopy equivalence, our three generalizations imply isomorphism of fundamental groups, and are preserved under wedges and Cartesian products.

Keywords: digital topology, digital image, homotopy, fundamental group

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1 Introduction

In *digital topology*, we study geometric and topological properties of digital images via tools adapted from geometric and algebraic topology. Prominent among these tools are digital versions of continuous functions and homotopy. Digital homotopy can be thought of as the topology of animated digital images.

In Euclidean topology, finite and infinite spaces can have the same homotopy type. E.g., the Euclidean line $\mathbf{R}$ and a point have the same homotopy type. The analogous statement is not true in digital topology; e.g., the digital line $\mathbf{Z}$ with the c_1 adjacency and a single point do not have the same digital homotopy type, despite sharing homotopy properties such as having trivial fundamental groups. We introduce in this paper the notions of digital homotopy similarity, same long homotopy type, and same real homotopy type, all of which are more general than digital homotopy equivalence and whose pointed versions are less general than having isomorphic fundamental groups. These notions allow the possibility of considering a bounded and an unbounded digital image as similar with respect to homotopy.

2 Preliminaries

We say a connected digital image X has *bounded diameter* if there is a positive integer n such that if x and y are members of X, then there is a path in X from x to y of length at most n.

Much of the material in this section is quoted or paraphrased from other papers in digital topology, such as [4].

2.1 General Properties

Let $\mathbf{N}$ be the set of natural numbers, $\mathbf{N}^* = \{0\} \cup \mathbf{N}$, and let $\mathbf{Z}$ denote the set of integers. Then $\mathbf{Z}^n$ is the set of lattice points in Euclidean $n-$dimensional space.

Adjacency relations commonly used for digital images include the following [10]. Two points p and q in $\mathbf{Z}^2$ are $8-$ *adjacent* if they are distinct and differ by at most 1 in each coordinate; p and q in $\mathbf{Z}^2$ are $4-$ *adjacent* if they are 8-adjacent and differ in exactly one coordinate. Two points p and q in $\mathbf{Z}^3$ are $26-$*adjacent* if they are distinct and differ by at most 1 in each coordinate; they are $18-$ *adjacent* if they are 26-adjacent and differ in at most two coordinates; they are $6-$*adjacent* if they are 18-adjacent and differ in exactly one coordinate. For $k \in \{4, 8, 6, 18, 26\}$, a $k-$ *neighbor* of a lattice point p is a point that is $k-$adjacent to p.

The adjacencies discussed above are generalized as follows. Let u, n be positive integers, $1 \leq u \leq n$. Distinct points $p, q \in \mathbf{Z}^n$ are called c_u-*adjacent*, or c_u-*neighbors*, if there are at most u distinct coordinates j for which $|p_j - q_j| = 1$, and for all other coordinates j, $p_j = q_j$. The notation c_u represents the number of points $q \in \mathbf{Z}^n$ that are adjacent to a given point $p \in \mathbf{Z}^n$ in this sense. Thus the values mentioned above: if $n = 1$ we have $c_1 = 2$; if $n = 2$ we have $c_1 = 4$ and $c_2 = 8$; if $n = 3$ we have $c_1 = 6$, $c_2 = 18$, and $c_3 = 26$. Yet more general adjacency relations are discussed in [8].

Let κ be an adjacency relation defined on $\mathbf{Z}^n$. A digital image $X \subset \mathbf{Z}^n$ is $\kappa-$ *connected* [8] if and only if for every pair of points $\{x, y\} \subset X$, $x \neq y$, there exists a set $\{x_0, x_1, \ldots, x_c\} \subset X$ such that $x = x_0$, $x_c = y$, and x_j and x_{j+1} are $\kappa-$neighbors, $i \in \{0, 1, \ldots, c-1\}$. A κ-*component* of X is a maximal κ-connected subset of X.

Often, we must assume some adjacency relation for the *white pixels* in $\mathbf{Z}^n$, *i.e.*, the pixels of $\mathbf{Z}^n \setminus X$ (the pixels that belong to X are sometimes referred to as *black pixels*). In this paper, we are not concerned with adjacencies between white pixels.

Definition 1. [1] Let $a, b \in \mathbf{Z}$, $a < b$. A digital interval is a set of the form

$$[a, b]_{\mathbf{Z}} = \{z \in \mathbf{Z} \mid a \leq z \leq b\}$$

in which $2-$adjacency is assumed. $\square$

Definition 2. ([2]; see also [11]) Let $X \subset \mathbf{Z}^{n_0}$, $Y \subset \mathbf{Z}^{n_1}$. Let $f : X \to Y$ be a function. Let κ_j be an adjacency relation defined on $\mathbf{Z}^{n_j}$, $i \in \{0, 1\}$. We say f is $(\kappa_0, \kappa_1)-$continuous if for every κ_0-connected subset A of X, $f(A)$ is a κ_1-connected subset of Y. $\square$

See also [5, 6], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

We say a function satisfying Definition 2 is *digitally continuous*. This definition implies the following.

Proposition 1. ([2]; see also [11]) *Let X and Y be digital images. Then the function $f : X \to Y$ is (κ_0, κ_1)-continuous if and only if for every $\{x_0, x_1\} \subset X$ such that x_0 and x_1 are κ_0-adjacent, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent.* $\square$

For example, if κ is an adjacency relation on a digital image Y, then $f : [a, b]_{\mathbf{Z}} \to Y$ is $(2, \kappa)-$continuous if and only if for every $\{c, c + 1\} \subset [a, b]_{\mathbf{Z}}$, either $f(c) = f(c + 1)$ or $f(c)$ and $f(c + 1)$ are $\kappa-$adjacent. If some function $f : [0, k]_{\mathbf{Z}} \to Y$ is $(2, \kappa)-$continuous, we say f is a *κ-path* from $f(0)$ to $f(k)$ of length k.

We have the following.

Proposition 2. [2] *Composition preserves digital continuity,* i.e.*, if $f : X \to Y$ and $g : Y \to Z$ are, respectively, $(\kappa_0, \kappa_1)-$continuous and $(\kappa_1, \kappa_2)-$continuous functions, then the composite function $g \circ f : X \to Z$ is $(\kappa_0, \kappa_2)-$continuous.* $\square$

Digital images (X, κ) and (Y, λ) are $(\kappa, \lambda) -$ *isomorphic* (called $(\kappa, \lambda) -$ *homeomorphic* in [1, 3]) if there is a bijection $h : X \to Y$ that is (κ, λ)-continuous, such that the function $h^{-1} : Y \to X$ is (λ, κ)-continuous.

2.2 Digital homotopy

A homotopy between continuous functions may be thought of as a continuous deformation of one of the functions into the other over a finite time period.

Definition 3. ([2]; see also [9]) Let X and Y be digital images. Let $f, g : X \to Y$ be (κ, κ')-continuous functions. Suppose there is a positive integer m and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;

- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by

$$F_x(t) \;=\; F(x, t) \text{ for all } t \in [0, m]_{\mathbf{Z}}$$

is $(2, \kappa')-$continuous. That is, $F_x(t)$ is a path in Y.

- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by

$$F_t(x) \;=\; F(x, t) \text{ for all } x \in X$$

is $(\kappa, \kappa')-$continuous.

Then F is a digital $(\kappa, \kappa')-$homotopy between f and g, and f and g are digitally $(\kappa, \kappa')-$homotopic in Y. If for some $x \in X$ we have $F(x, t) = F(x, 0)$ for all $t \in [0, m]_{\mathbf{Z}}$, we say F holds x fixed, and F is a pointed homotopy. $\square$

We indicate a pair of homotopic functions as described above by $f \simeq_{\kappa,\kappa'} g$. When the adjacency relations κ and κ' are understood in context, we say f and g are *digitally homotopic* to abbreviate "digitally $(\kappa, \kappa')-$homotopic in Y," and write $f \simeq g$.

Proposition 3. [9, 2] *Digital homotopy is an equivalence relation among digitally continuous functions $f : X \to Y$.* $\square$

Definition 4. [3] Let $f : X \to Y$ be a (κ, κ')-continuous function and let $g : Y \to X$ be a (κ', κ)-continuous function such that

$$f \circ g \simeq_{\kappa',\kappa'} 1_X \text{ and } g \circ f \simeq_{\kappa,\kappa} 1_Y.$$

Then we say X and Y have the same (κ, κ')-homotopy type and that X and Y are (κ, κ')-homotopy equivalent, denoted $X \simeq_{\kappa,\kappa'} Y$ or as $X \simeq Y$ when κ and κ' are understood. If for some $x_0 \in X$ and $y_0 \in Y$ we have $f(x_0) = y_0$, $g(y_0) = x_0$, and there exists a homotopy between $f \circ g$ and 1_X that holds x_0 fixed, and a homotopy between $g \circ f$ and 1_Y that holds y_0 fixed, we say (X, x_0, κ) and (Y, y_0, κ') are pointed homotopy equivalent and that (X, x_0) and (Y, y_0) have the same pointed homotopy type, denoted $(X, x_0) \simeq_{\kappa,\kappa'} (Y, y_0)$ or as $(X, x_0) \simeq (Y, y_0)$ when κ and κ' are understood. $\square$

It is easily seen, from Proposition 3, that having the same homotopy type (respectively, the same pointed homotopy type) is an equivalence relation among digital images (respectively, among pointed digital images).

For $p \in Y$, we denote by $\overline{p}$ the constant function $\overline{p} : X \to Y$ defined by $\overline{p}(x) = p$ for all $x \in X$.

Definition 5. A digital image (X, κ) is κ-contractible [9, 1] if its identity map is (κ, κ)-homotopic to a constant function $\overline{p}$ for some $p \in X$. If the homotopy of the contraction holds p fixed, we say (X, p, κ) is pointed κ-contractible. $\square$

When κ is understood, we speak of *contractibility* for short. It is easily seen that X is contractible if and only if X has the homotopy type of a one-point digital image.

2.3 Fundamental group

Inspired by the fundamental group of a topological space, several researchers [12, 10, 2, 4] have developed versions of a fundamental group for digital images. These are not all equivalent; however, it is shown in [4] that the version of the fundamental group developed in that paper is equivalent to the version in [2].

In the following, we present the version of the digital fundamental group developed in [4].

Given a digital image X, a continuous function $f : \mathbf{N}^* \to X$ is an *eventually constant path* or *EC path* if there is some point $c \in X$ and some $N \geq 0$ such that $f(x) = c$ whenever $x \geq N$. We abbreviate the latter by $f(\infty) = c$. The endpoints of an EC path f are the two points $f(0)$ and $f(\infty)$. If f is an EC path and $f(0) = f(\infty)$, we say f is an *EC loop*, and $f(0)$ is called the *basepoint* of this loop.

We say that a homotopy $H : [0, k]_{\mathbf{Z}} \times \mathbf{N}^* \to X$ between EC paths is an *EC homotopy* when the function $H_s : \mathbf{N}^* \to X$ defined by $H_s(t) = H(s, t)$ is an EC path for all $s \in [0, k]_{\mathbf{Z}}$. To indicate an EC homotopy, we write $f \simeq^{EC} g$, or $f \simeq_\kappa^{EC} g$ if it is desirable to state the adjacency κ of X. We say an EC homotopy H *holds the endpoints fixed* when $H_t(0) = f(0) = g(0)$ and there is a $c \in \mathbf{N}$ such that $n \geq c$ implies $H_t(n) = f(n) = g(n)$ for all t.

Given an EC loop $f : \mathbf{N}^* \to X$, we let

$$N_f = \min\{m \in \mathbf{N}^* \mid n \geq m \text{ implies } f(n) = f(m)\}.$$

For $x_0 \in X$, suppose $f_0, f_1 : \mathbf{N}^* \to X$ are x_0-based EC loops. Define an x_0-based EC loop $f_0 * f_1 : \mathbf{N}^* \to X$ via

$$f_0 * f_1(n) = \begin{cases} f_0(n) & \text{if } 0 \leq n \leq N_{f_0}; \\ f_1(n - N_{f_0}) & \text{if } N_{f_0} \leq n. \end{cases}$$

Given an x_0-based EC loop $f : \mathbf{N}^* \to X$, we denote by $[f]_X$, or $[f]$ when X is understood, the equivalence class of EC loops that are homotopic to f in X holding the endpoints fixed. We let $\Pi_1^\kappa(X, x_0)$ be the set of all such sets $[f]$. The $*$ operation enables us to define an operation on $\Pi_1^\kappa(X, x_0)$ via

$$[f] \cdot [g] = [f * g].$$

This operation is well defined, and makes $\Pi_1^\kappa(X, x_0)$ into a group in which the identity element is the class $[\overline{x_0}]$ of the constant loop $\overline{x_0}$ and in which inverse elements are given by $[f]^{-1} = [f^{-1}]$, where $f^{-1} : \mathbf{N}^* \to X$ is the EC loop defined by

$$f^{-1}(n) = \begin{cases} f(N_f - n) & \text{if } 0 \le n \le N_f; \\ x_0 & \text{if } n \ge N_f. \end{cases}$$

3 Homotopically similar images

In Euclidean topology, it is often possible to say that a bounded space X and an unbounded space Y have the same homotopy type. For example, a single point and n-dimensional Euclidean space $\mathbf{R}^n$ have the same homotopy type, roughly since the points of $\mathbf{R}^n$ can be moved continuously within $\mathbf{R}^n$ over a finite time interval to a single point. However, Definition 3 does not permit a digital image with unbounded diameter to have the homotopy type of an image with bounded diameter, since the second factor of the domain of a homotopy is a finite interval $[0, m]_{\mathbf{Z}}$. In this paper, we seek to circumvent this limitation. One of the ways we do so depends on the following.

Definition 6. Let X and Y be digital images. We say (X, κ) and (Y, λ) are homotopically similar, denoted $X \simeq^s_{\kappa,\lambda} Y$, if there exist subsets $\{X_j\}_{j=1}^{\infty}$ of X and $\{Y_j\}_{j=1}^{\infty}$ of Y such that:

- $X = \bigcup_{j=1}^{\infty} X_j$, $Y = \bigcup_{j=1}^{\infty} Y_j$, and, for all j, $X_j \subset X_{j+1}$, $Y_j \subset Y_{j+1}$.
- There are continuous functions $f_j : X_j \to Y_j$, $g_j : Y_j \to X_j$ such that $g_j \circ f_j \simeq_{\kappa,\kappa} 1_{X_j}$ and $f_j \circ g_j \simeq_{\lambda,\lambda} 1_{Y_j}$.
- For $v \le w$, $f_w|X_v \simeq_{\kappa,\lambda} f_v$ in Y_v and $g_w|Y_v \simeq_{\lambda,\kappa} g_v$ in X_v.

If all of these homotopies are pointed with respect to some $x_1 \in X_1$ and $y_1 \in Y_1$, we say (X, x_1) and (Y, y_1) are pointed homotopically similar, denoted $(X, x_1) \simeq^s_{\kappa,\lambda} (Y, y_1)$ or $(X, x_1) \simeq^s (Y, y_1)$ when κ and λ are understood. $\square$

Proposition 4. *If $X \simeq_{\kappa,\lambda} Y$, then $X \simeq^s_{\kappa,\lambda} Y$. If $(X, x_1) \simeq_{\kappa,\lambda} (Y, y_1)$, then $(X, x_1) \simeq^s_{\kappa,\lambda} (Y, y_1)$.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ realize a homotopy equivalence between (X, κ) and (Y, λ), or a pointed homotopy equivalence between (X, κ, x_1) and (Y, λ, y_1). Then corresponding to Definition 6, we can take, for all j, $X_j = X$, $Y_j = Y$, $f_j = f$, $g_j = g$. $\boxed{QED}$

Although Definition 6 does not require it, we often choose the X_j and Y_j to be finite sets. Example 1 has an image with bounded diameter and an image with unbounded diameter that are not homotopy equivalent but are pointed homotopically similar.

Theorem 1. *Let X and Y be finite digital images. Then $X \simeq_{\kappa,\lambda} Y$ if and only if $X \simeq^s_{\kappa,\lambda} Y$, and $(X, x_0) \simeq_{\kappa,\lambda} (Y, y_0)$ if and only if $(X, x_0) \simeq^s_{\kappa,\lambda} (Y, y_0)$.*

Proof. That $X \simeq_{\kappa,\lambda} Y$ implies $X \simeq^s_{\kappa,\lambda} Y$ is shown in Proposition 4. To show the converse: if $X \simeq^s_{\kappa,\lambda} Y$, let $\{X_j, Y_j, f_j, g_j\}_{j=1}^{\infty}$ be as in Definition 6. Since X and Y are finite, there exists a positive integer m such that $i \geq m$ implies $X = X_i$ and $Y = Y_i$. Then $f_m : X = X_m \to Y_m = Y$ and $g_m : Y = Y_m \to X_m = X$ satisfy

$$g_m \circ f_m \simeq_{\kappa,\kappa} 1_X, \;\; f_m \circ g_m \simeq_{\lambda,\lambda} 1_Y.$$

Thus, $X \simeq_{\kappa,\lambda} Y$.

A similar argument yields the pointed assertion. $\boxed{QED}$

In Example 3, we show that two digital images with unbounded diameters can be homotopically similar but not homotopy equivalent.

Theorem 2. *Homotopic similarity and pointed homotopic similarity are reflexive and symmetric relations among digital images.*

Proof. The assertion follows easily from Definition 6. $\boxed{QED}$

Remark 1. At the current writing, we do not have an answer for the following: Is the homotopy similarity (unpointed or pointed) of digital images a transitive relation?

This appears to be a difficult problem. We need a positive resolution to this question if we are to conclude that homotopic similarity is an equivalence relation. Notice that if $A \simeq^s B$ via subsets $\{A_i\}_{i=1}^{\infty}$ of A and $\{B_i\}_{i=1}^{\infty}$ of B, and $B \simeq^s C$ via subsets $\{B'_i\}_{i=1}^{\infty}$ of B and $\{C_i\}_{i=1}^{\infty}$ of C, we would have $A \simeq^s C$ if $B_i = B'_i$ for infinitely many i, but one can easily construct examples for which the latter is not satisfied.

We show transitivity for the following special case.

Theorem 3. *Let B be finite. Let $A \simeq^s B \simeq^s C$. Then $A \simeq^s C$. If $(A, a_0) \simeq^s (B, b_0) \simeq^s (C, c_0)$, then $(A, a_0) \simeq^s (C, c_0)$.*

Proof. We sketch a proof for the unpointed assertion. A similar argument yields the pointed assertion.

Let $A \simeq^s B$ via $A = \bigcup_{i=1}^{\infty} A_i$, $B = \bigcup_{i=1}^{\infty} B_i$, as in Definition 6. Let $B \simeq^s C$ via $B = \bigcup_{i=1}^{\infty} B'_i$, $C = \bigcup_{i=1}^{\infty} C_i$, as in Definition 6. Since B is finite, there exists i_0 such that $i \geq i_0$ implies $B_i = B = B'_i$. Since homotopy of continuous functions and homotopy type of digital images are transitive relations, it follows easily from Definition 6 that $A \simeq^s C$. $\boxed{QED}$

4 Long homotopy type

In this section, we introduce *long homotopy type*. We obtain for this notion several properties analogous to those discussed in Section 3 for homotopic similarity.

The following definition is a step in the direction of the idea that a long homotopy is a homotopy over an infinite time interval. The following is essentially an EC version of Definition 3.

Definition 7. Let (X, κ) and (Y, λ) be digital images. Let $f, g : X \to Y$ be continuous. Let $F : X \times \mathbf{N}^* \to Y$ be a function such that

- for all $x \in X$, $F(x, 0) = f(x)$ and there exists $n \in \mathbf{N}^*$ such that $t \geq n$ implies $F(x, t) = g(x)$.
- For all $x \in X$, the induced function $F_x : \mathbf{N}^* \to Y$ defined by
$$F_x(t) = F(x, t) \text{ for all } t \in [0, \infty]_{\mathbf{Z}}$$
is an EC-path in Y.
- For all $t \in \mathbf{N}^*$, the induced function $F_t : X \to Y$ defined by
$$F_t(x) = F(x, t) \text{ for all } x \in X$$
is (κ, λ)–continuous.

Then F is an l-homotopy from f to g. If for some $x_0 \in X$ and $y_0 \in Y$ we have $F(x_0, t) = y_0$ for all $t \in \mathbf{N}^*$, we say F is a pointed l-homotopy. We write $f \simeq^l_{\kappa,\lambda} g$, or $f \simeq^l g$ when the adjacencies κ and λ are understood, to indicate that f and g are l-homotopic functions. $\square$

Note that the definition above generalizes EC homotopy of paths: if two EC paths $f, g : [0, \infty]_{\mathbf{Z}} \to Y$ are EC homotopic, then the EC homotopy from f to g is an l-homotopy of f to g.

Proposition 5. *Let $f, g : X \to Y$ be (unpointed or pointed) continuous functions between digital images. If f and g are (unpointed or pointed) homotopic in Y, then f and g are (unpointed or pointed, respectively) l-homotopic in Y. The converse is true if X is finite.*

Proof. We give a proof for the unpointed assertions. The pointed assertions are proven similarly.

If $f \simeq g$, there is a homotopy $h : X \times [0, m]_{\mathbf{Z}} \to Y$ such that $h(x, 0) = f(x)$ and $h(x, m) = g(x)$. Then the function $H : X \times [0, \infty]_{\mathbf{Z}} \to Y$ defined by $H(x, t) = h(x, \min\{m, t\})$ is an l-homotopy from f to g.

Suppose X is finite and $f \simeq^l g$. Then there is an l-homotopy $H : X \times [0,\infty]_{\mathbf{Z}} \to Y$ such that $H(x,0) = f(x)$ and, for all $x \in X$, there exists

$$t_x = \min\{t \in \mathbf{N}^* \mid s \geq t \text{ implies } H(x,s) = H(x,t)\}.$$

Let $m = \max\{t_x \mid x \in X\}$. Then the function $h : X \times [0,m]_{\mathbf{Z}} \to Y$ defined by $h(x,t) = H(x,t)$ is a homotopy from f to g. $\boxed{QED}$

Remark 2. At the current writing, we have not found answers to the following.

(Unpointed and pointed versions:) Is l-homotopy a symmetric relation among continuous functions between digital images?

Remark 3. (Unpointed and pointed versions:) Is l-homotopy a transitive relation among continuous functions between digital images?

Positive answers to the questions of Remarks 2 and 3 are necessary for (unpointed or pointed) l-homotopy to be an equivalence relation. In the absence of such results, we proceed with the following definition.

Definition 8. Let (X,κ) and (Y,λ) be digital images. Let $f,g : X \to Y$ be continuous. Let $F : X \times \mathbf{Z} \to Y$ be a function such that

- for all $x \in X$, there exists $N_{F,x} \in \mathbf{N}$ such that $t \leq -N_{F,x}$ implies $F(x,t) = f(x)$ and $t \geq N_{F,x}$ implies $F(x,t) = g(x)$.
- For all $x \in X$, the induced function $F_x : \mathbf{Z} \to Y$ defined by

  $$F_x(t) = F(x,t) \text{ for all } t \in \mathbf{Z}$$

  is (c_1,λ)-continuous.
- For all $t \in \mathbf{Z}$, the induced function $F_t : X \to Y$ defined by

  $$F_t(x) = F(x,t) \text{ for all } x \in X$$

  is (κ,λ)–continuous.

Then F is a long homotopy from f to g. If for some $x_0 \in X$ and $y_0 \in Y$ we have $F(x_0,t) = y_0$ for all $t \in \mathbf{N}^*$, we say F is a pointed long homotopy. We write $f \simeq^L_{\kappa,\lambda} g$, or $f \simeq^L g$ when the adjacencies κ and λ are understood, to indicate that f and g are long homotopic functions. □

It is easy to show that the existence of an l-homotopy implies a long homotopy:

Proposition 6. *Let $f,g : (X,\kappa) \to (Y,\lambda)$ be continuous functions between digital images. If $f \simeq^l g$ then $f \simeq^L g$. If the l-homotopy between f and g is pointed, then the long homotopy between f and g is pointed.*

Proof. Clearly, if $F : X \times \mathbf{N}^* \to Y$ is a (pointed) l-homotopy from f to g, then the function $F' : X \times \mathbf{Z} \to Y$ defined by

$$F'(x,t) = \begin{cases} F(x,t) & \text{if } t \geq 0; \\ F(x,0) & \text{if } t < 0, \end{cases}$$

is a (pointed) long homotopy from f to g. $\boxed{QED}$

The same argument used in Proposition 5 proves the corresponding assertion for long homotopy:

Proposition 7. *Let $f, g : X \to Y$ be (unpointed or pointed) continuous functions between digital images. If f and g are (unpointed or pointed) homotopic in Y, then f and g are (unpointed or pointed, respectively) long homotopic in Y. The converse is true if X is finite.* □

Unlike with l-homotopy, it is easy to see that long homotopy is symmetric.

Theorem 4. *Long homotopy and pointed long homotopy are reflexive and symmetric relations.*

Proof. We state a proof for the unpointed assertion. The same argument works for the pointed assertion.

For the reflexive property, we note the following. Given a continuous function $f : (X, \kappa) \to (Y, \lambda)$, it is clear that the function $F : X \times \mathbf{Z} \to Y$ given by $F(x,t) = f(x)$ is a long homotopy from f to f.

For the symmetric property, we note that if $F : X \times \mathbf{Z} \to Y$ is a long homotopy from f to g, where $f, g : (X, \kappa) \to (Y, \lambda)$ are continuous, then $F' : X \times \mathbf{Z} \to Y$, defined by $F'(x,t) = F(x,-t)$, is a long homotopy from g to f. $\boxed{QED}$

Remark 4. At the current writing, we lack answers to the following: Is long homotopy (pointed or unpointed) between continuous functions a transitive relationship?

These seem to be difficult problems. If X is finite and $f, g, h : X \to Y$ with $f \simeq^L g \simeq^L h$, then $f \simeq^L h$ since it follows from Proposition 7 that in this case, long homotopy coincides with homotopy, which is transitive. To demonstrate the difficulty involved in the general case, we will prove transitivity for another special case.

It is easy to see that if $c, d : X \to Y$ are two different constant maps whose constant values $c = c(x)$ and $d = d(x)$ are in the same component of X, then c and d are homotopic, and thus long homotopic. Thus the following theorem is a very special case of transitivity, but the proof is already somewhat involved.

Say that a digital image X is *locally finite* when each point $x \in X$ is adjacent to only finitely many other points of X. E.g., if X is finite, or if X has a c_u-adjacency, then X is locally finite.

Theorem 5. *Let X be locally finite, and let $f : X \to Y$ be a continuous function, and let $c, d : X \to Y$ be two constant functions with constant values c and d in the same component of Y. If $f \simeq^L c$, then $f \simeq^L d$.*

Proof. Let $\sigma : [0, k]_{\mathbf{Z}} \to Y$ be a path from c to d. Our proof is by induction on k. If $k = 0$, then $c = d$ and there is nothing to prove. Letting $c' = \sigma(k-1)$, for our induction case we may assume that $f \simeq^L c'$, and we will show that $f \simeq^L d$. (Note that c' and d are adjacent.)

Let $H : X \times \mathbf{Z} \to Y$ be a long homotopy of f to c'. Then for each $x \in X$, there is a number N_x such that, whenever $t \geq N_x$, we have $H(x, t) = c'$. Since X is locally finite, there is a number $N'_x \geq N_x$ such that, whenever $t \geq N'_x$, we have $H(x', t) = c'$ for every x' adjacent to x.

Then we define $G : X \times \mathbf{Z} \to Y$ as:

$$G(x, t) = \begin{cases} H(x, t) & \text{if } t \leq N'_x; \\ d & \text{if } t > N'_x. \end{cases}$$

We claim that G is a long homotopy of f to d. It is clear that for all $x \in X$ there exists $n_x \in \mathbf{N}$ such that $t \leq -n_x$ implies $G(x, t) = H(x, t) = f(x)$ and $t \geq n_x$ implies $G(x, t) = d$. Furthermore, the induced function $G_x(t)$ is given by:

$$G_x(t) = \begin{cases} H_x(t) & \text{if } t \leq N'_x; \\ d & \text{if } t > N'_x. \end{cases},$$

which is continuous since $H(x, N'_x) = c'$ is adjacent to d.

Lastly we show that the induced function $G_t(x)$ is continuous: take any point y adjacent to x, and we will show that $G_t(x)$ is adjacent or equal to $G_t(y)$.

- When $t \leq N'_x$, we have $G_t(x) = H_t(x)$, which is adjacent or equal to $H_t(y)$ because H is a homotopy.
  - If $H_t(y) = G_t(y)$, we have the desired conclusion that $G_t(x)$ is adjacent or equal to $G_t(y)$.
  - Otherwise, $G_t(y) = d \neq H_t(y)$, so $t > N'_y$, which implies $H(x, t) = c'$. Thus, $G_t(x) \in \{c', d\}$, so $G_t(x)$ is adjacent or equal to $G_t(y)$.
- If $t > N'_x$ then $G_t(x) = d$. For $G_t(y)$ there are two cases. If $t \geq N'_y$ then $G_t(y) = d = G_t(x)$. If $t < N'_y$ we still must have $t \geq N_y$ since $t > N'_x$ and y is adjacent to x. Thus in this case $G_t(y) = H_t(y) = c'$, and thus $G_t(x) = d$ and $G_t(y) = c'$ are adjacent as desired.

$\boxed{QED}$

Definition 9. Let $f : (X, \kappa) \to (Y, \lambda)$ and $g : (Y, \lambda) \to (X, \kappa)$ be continuous functions. Suppose $g \circ f \simeq^L 1_X$ and $f \circ g \simeq^L 1_Y$. Then we say (X, κ) and (Y, λ) have the same long homotopy type, denoted $X \simeq^L_{\kappa,\lambda} Y$ or simply $X \simeq^L Y$. If there exist $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = y_0$, $g(y_0) = x_0$, the long homotopy $g \circ f \simeq^L 1_X$ holds x_0 fixed, and the long homotopy $f \circ g \simeq^L 1_Y$ holds y_0 fixed, then (X, x_0, κ) and (Y, y_0, λ) have the same pointed long homotopy type, denoted $(X, x_0) \simeq^L_{\kappa,\lambda} (Y, y_0)$ or $(X, x_0) \simeq^L (Y, y_0)$. □

Proposition 8. *If $X \simeq_{\kappa,\lambda} Y$, then $X \simeq^L_{\kappa,\lambda} Y$. If $(X, x_0) \simeq_{\kappa,\lambda} (Y, y_0)$, then $(X, x_0) \simeq^L_{\kappa,\lambda} (Y, y_0)$. The converses of both statements hold when X and Y are both finite.*

Proof: The assertions follow from Definition 9 and Proposition 7. □

Theorem 6. *Long homotopy type, and pointed long homotopy type, are reflexive and symmetric relations among digital images.*

Proof. The assertions follow easily from Definition 9. $\boxed{QED}$

Remark 5. At the current writing, we lack answers to the following: Is long homotopy type (unpointed or pointed) a transitive relation among digital images?

These appear to be difficult problems. A positive resolution is necessary in order for us to conclude that long homotopy type is an equivalence relation. Since homotopy equivalence is an equivalence relation, Proposition 8 implies (for both the pointed and unpointed questions) that if there exists an example of non-transitivity for long homotopy type, i.e., images A, B, C such that $A \simeq^L B \simeq^L C$ with A and C not long homotopically equivalent, then at least one of A, B, C must be infinite.

In the next result, we prove transitivity for a special case. The following resembles Theorem 3 for homotopic similarity, but requires that the intermediate image be a single point. It does not seem easy to generalize to finite sets as in Theorem 3.

Theorem 7. *Let $X \simeq^L \{a\} \simeq^L Y$. Then $X \simeq^L Y$. If $(X, x_0) \simeq^L (\{a\}, a) \simeq^L (Y, y_0)$, then $(X, x_0) \simeq^L (Y, y_0)$.*

Proof. We state a proof for the pointed assertion; the unpointed assertion is handled similarly.

By hypothesis, there are pointed functions $f : (X, x_0) \to (\{a\}, a)$ and $g : (\{a\}, a) \to (X, x_0)$ and a pointed long homotopy $H : X \times \mathbf{Z} \to X$ from 1_X to $g \circ f = \overline{x_0}$. Similarly, there are pointed functions $h : (Y, y_0) \to (\{a\}, a)$ and

$k : (\{a\}, a) \to (Y, y_0)$ and a pointed long homotopy $K : Y \times \mathbf{Z} \to Y$ from 1_Y to $k \circ h = \overline{y_0}$.

Let $\overline{y_0}'$ be the constant function from X to Y. Let $\overline{x_0}'$ be the constant function from Y to X. Then H is a pointed long homotopy from 1_X to $\overline{x_0} = \overline{x_0}' \circ \overline{y_0}'$, and K is a pointed long homotopy from 1_Y to $\overline{y_0} = \overline{y_0}' \circ \overline{x_0}'$. The assertion follows. $\boxed{QED}$

A digital image with bounded diameter and an image with infinite diameter cannot have the same homotopy type, but Example 1 shows that such a pair of images can have the same long homotopy type. Example 3 gives two digital images with unbounded diameters that have the same long homotopy type but not the same homotopy type.

5 Real homotopy type

In this section we present another generalization of digital homotopy that we call *real homotopy.* As in the case of long homotopy, we will allow the time interval to be infinite, this time using the real interval $[0, 1]$. Though nondiscrete sets are not typically used in digital topology, we will see as in the other sections that real homotopy and digital homotopy are equivalent when the images under consideration are finite. The advantage in using the real interval rather than the integer interval $[0, \infty)$ is that two copies of $[0, 1]$ can be concatenated in a natural way, which allows us to prove that real homotopy is transitive.

It also turns out that long homotopy can tell us a lot about real homotopy.

We begin with a preliminary definition that is a kind of continuity property for a function from a real interval into a digital image. Informally we want to require that such a function be locally constant with jump discontinuities only between adjacent points.

Definition 10. Let (X, κ) be a digital image, and $[0, 1] \subset \mathbf{R}$ be the unit interval. A function $f : [0, 1] \to X$ is a *real [digital] [κ-]path in X* if:

- there exists $\epsilon_0 > 0$ such that f is constant on $(0, \epsilon_0)$ with constant value equal or κ-adjacent to $f(0)$,
- there exists $\epsilon_1 > 0$ such that f is constant on $(1 - \epsilon_1, 1)$ with constant value equal or κ-adjacent to $f(1)$,
- for each $t \in (0, 1)$ there exists $\epsilon_t > 0$ such that f is constant on each of the intervals $(t - \epsilon_t, t)$ and $(t, t + \epsilon_t)$, and these two constant values are equal or κ-adjacent, with at least one of them equal to $f(t)$.

If $t = 0$ and $f(0) \neq f((0, \epsilon_0))$, or $0 < t < 1$ and the two constant values $f((t - \epsilon_t, t))$ and $f((t, t + \epsilon_t))$ are not equal, or $t = 1$ and $f(1) \neq f((1 - \epsilon_1, 1))$, we say t is a *jump* of f. $\square$

In fact such a real path can only have finitely many jumps, as the following Proposition shows.

Proposition 9. *Let $p, q \in (X, \kappa)$. Let $f : [a, b] \to X$ be a real κ-path from p to q. Then the number of jumps of f is finite.*

Proof. Suppose f has an infinite set of jumps in the domain $[a, b]$. By the Bolzano-Weierstrass Theorem this set of jumps has an accumulation point. Thus there exists $t_0 \in [a, b]$ and a sequence of distinct jumps $\{t_j\}_{j=1}^{\infty}$ such that $\lim_{j \to \infty} t_j = t_0$. Then for every $\varepsilon > 0$, at least one of the intervals $(t_0 - \varepsilon, t_0)$ and $(t_0, t_0 + \varepsilon)$ has infinitely many members of $\{t_j\}_{j=1}^{\infty}$, contrary to the requirement of Definition 10 that there be $\varepsilon > 0$ such that f is constant on each of the intervals $(t_0 - \varepsilon, t_0)$ and $(t_0, t_0 + \varepsilon)$. $\boxed{QED}$

Now we can define real digital homotopy of functions. The following is a "real" version of Definitions 3 and 7.

Definition 11. Let (X, κ) and (Y, κ') be digital images, and let $f, g : X \to Y$ be (κ, κ') continuous. Then a *real [digital] homotopy* of f and g is a function $F : X \times [0, 1] \to Y$ such that:

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$
- for all $x \in X$, the induced function $F_x : [0, 1] \to Y$ defined by

$$F_x(t) = F(x, t) \text{ for all } t \in [0, 1]$$

  is a real κ-path in X.
- for all $t \in [0, 1]$, the induced function $F_t : X \to Y$ defined by

$$F_t(x) = F(x, t) \text{ for all } x \in X$$

  is (κ, κ')–continuous.

If such a function exists we say f and g are *real homotopic* and write $f \simeq^{\mathbf{R}} g$. If there are points $x_0 \in X$ and $y_0 \in Y$ such that $F(x_0, t) = y_0$ for all $t \in [0, 1]$, we say f and g are *pointed real homotopic.* $\square$

Unlike long homotopy, real homotopy is easily shown to be an equivalence relation.

Theorem 8. *Real homotopy and pointed real homotopy are equivalence relations among continuous functions between digital images.*

Proof. We give the proof for the unpointed assertion. A similar argument can be used to establish the pointed assertion.

Reflexivity is clear: for any digitally continuous function $f : X \to Y$, the function $F(x,t) = f(x)$ is a real homotopy from f to f.

For symmetry, let $f, g : X \to Y$ be digitally continuous with $f \simeq^{\mathbf{R}} g$, and let $F : X \times [0,1] \to Y$ be a real homotopy from f to g. Then define $G : X \times [0,1] \to Y$ by $G(x,t) = F(x, 1-t)$. It is easy to verify that G is a real homotopy from g to f, and so $g \simeq^{\mathbf{R}} f$.

For transitivity, let $f, g, h : X \to Y$ with $f \simeq^{\mathbf{R}} g$ and $g \simeq^{\mathbf{R}} h$. Let $F, G : X \times [0,1] \to Y$ be real homotopies from f to g and g to h, respectively. Then define $H : X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } t \le 1/2, \\ G(x,2t-1) & \text{if } t \ge 1/2. \end{cases}$$

Again it is routine to check that H is a real homotopy from f to h, and so $f \simeq^{\mathbf{R}} h$ as desired. $\boxed{QED}$

Next we show that long homotopy of functions implies real homotopy.

Theorem 9. *Let X and Y be digital images, and let $f, g : X \to Y$ be continuous. If $f \simeq^{L} g$ then $f \simeq^{\mathbf{R}} g$. If f and g are pointed long homotopic, then they are pointed real homotopic.*

Proof. Let $h : (0,1) \to \mathbf{R}$ be a homeomorphism with $\lim_{x \to 0} h(x) = -\infty$ and $\lim_{x \to 1} h(x) = \infty$. For example h can be taken to be a rescaled version of the tangent function. Let $H : X \times \mathbf{Z} \to Y$ be a long homotopy from f to g, and define $F : X \times [0,1] \to Y$ by:

$$F(x,t) = \begin{cases} f(x) & \text{if } t = 0, \\ H(x, \lfloor h(t) \rfloor) & \text{if } t \in (0,1), \\ g(x) & \text{if } t = 1. \end{cases}$$

We claim that F is a real homotopy from f to g. We have defined F so that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all x. Observe also that each induced function $F_t(x)$ is continuous - for $t \in \{0,1\}$ this is true because f and g are continuous, and for other t because H_s is continuous for any $s \in \mathbf{Z}$.

It remains to show that the induced function $F_x : [0,1] \to Y$ is a real path for every x. Note that the value of $H_x(t) = H(x, \lfloor h(t) \rfloor)$ changes only when $h(t)$ is an integer. When $h(t)$ is an integer, the value of H_x can only change from one point of Y to an adjacent point. Thus for any $t \in (0,1)$, there is some ϵ_t such that H_x is constant on $(t-\epsilon_t, t)$ and $(t, t+\epsilon_t)$, and these constant values are adjacent, and one of them equals $F_x(t)$.

It remains to show the existence of ϵ_0 and ϵ_1 as in Definition 11. Because H is a long homotopy, there is a natural number N_x such that $H(x,t) = g(x)$ whenever $t > N_x$ and $H(x,t) = f(x)$ whenever $t < -N_x$. Then choose ϵ_0 with $0 < \epsilon_0 < h^{-1}(-N_x)$, and then F_x will be constant on $[0, \epsilon_0)$ as required. Choose ϵ_1 such that $h^{-1}(N_x) < \epsilon_1 < 1$, and F_x will be constant on $(1-\epsilon_1, 1]$, as required. Thus F_x is a real path and so F is a real homotopy. $\boxed{QED}$

Next we show that real homotopy is weaker than digital homotopy, and that the two notions are equivalent when the domain is finite. This result is analogous to Proposition 7 for long homotopy.

Theorem 10. *Let (X,κ) and (Y,κ') be digital images, and let $f, g : X \to Y$ be (κ,κ')–continuous. Then $f \simeq g$ implies $f \simeq^{\mathbf{R}} g$, and the converse is true when X is finite. If f and g are pointed homotopic, then they are pointed real homotopic, and the converse is true when X is finite.*

Proof. We give the proof for the unpointed assertion. A similar argument can be used to establish the pointed assertion.

First we assume that $f \simeq g$. This implies, by Proposition 5, that $f \simeq^L g$. Hence by Theorem 9, $f \simeq^{\mathbf{R}} g$.

Now for the converse we assume that $f \simeq^{\mathbf{R}} g$ and X is finite, and show that $f \simeq g$. Let F be a real homotopy from f to g. Since X is finite and each real path $F_x(t)$ has finitely many jumps, there are only finitely many values of $t \in [0,1]$ which can be jumps for any of the paths $F_x(t)$. Let $j_0 < \cdots < j_k$ be these jump points, and choose $t_1, \ldots, t_{k-1}$ so that $j_i < t_i < j_{i+1}$ for each i. Also let $t_0 = 0$ and $t_{k+1} = 1$. Since jumps in real paths only move to adjacent points, $F_x(t_i)$ is adjacent or equal to $F_x(t_{i+1})$ for each i.

Now define $G : X \times [0, k+1]_{\mathbf{Z}} \to Y$ by $G(x,i) = F(x,t_i)$. We will show that G is a homotopy from f to g. We have $G(x,0) = F(x,t_0) = F(x,0) = f(x)$ and $G(x,k+1) = F(x,t_{k+1}) = F(x,1) = g(x)$. Since $F_{t_i}(x)$ is (κ,κ')–continuous for each $t_i \in [0,1]$, we have $G_i(x)$ (κ,κ')-continuous for each $i \in [0,k+1]_{\mathbf{Z}}$ as required.

It remains to show that the induced function $G_x : [0,k+1]_{\mathbf{Z}} \to Y$ is $(2,\kappa')$–continuous for each $x \in X$. Equivalently, we must show that $G_x(i)$ is κ'–adjacent or equal to $G_x(i+1)$ for each i. But we have already stated that $F_x(t_i) = G_x(i)$ is adjacent or equal to $F_x(t_{i+1}) = G_x(i+1)$ for each i, as desired. $\boxed{QED}$

In the case when X is finite, the converse above shows that a real homotopy implies an ordinary (finite) homotopy, which by Proposition 7 implies a long homotopy. Combining all these results gives:

Corollary 1. *Let X be a finite digital image, Y be any digital image, and $f, g : X \to Y$ be continuous. Then the following three statements are equivalent: $f \simeq g$, $f \simeq^L g$, and $f \simeq^{\mathbf{R}} g$. Similar equivalences hold for pointed relations.* □

Remark 6. The following simple question seems hard to answer: Is Corollary 1 true without the finiteness assumption?

With our real homotopy relation we can make the obvious definition for real homotopy type of digital images.

Definition 12. We say digital images (X, κ) and (Y, κ') have the *same real homotopy type*, denoted $X \simeq^{\mathbf{R}}_{\kappa,\kappa'} Y$ or $X \simeq^{\mathbf{R}} Y$ when κ and κ' are understood, if there are continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq^{\mathbf{R}} 1_X$ and $f \circ g \simeq^{\mathbf{R}} 1_Y$. If there exist $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = y_0$, $g(y_0) = x_0$, and the real homotopies implicit above are pointed with respect to x_0 and y_0, we say X and Y have the *same pointed real homotopy type*, denoted $(X, x_0) \simeq^{\mathbf{R}}_{\kappa,\kappa'} (Y, y_0)$ or $(X, x_0) \simeq^{\mathbf{R}} (Y, y_0)$. □

Having the same real homotopy type is also easily seen to be an equivalence relation.

Theorem 11. *Having the same real homotopy type or pointed real homotopy type is an equivalence relation among digital images.*

Proof. We prove the unpointed assertion. Simple modifications to the argument give the pointed assertion.

For the reflexive property it is easy to see that an identity map 1_X shows that $X \simeq^{\mathbf{R}} X$. The symmetric property follows from the symmetry in Definition 12. It remains to prove transitivity.

Suppose $X \simeq^{\mathbf{R}} Y \simeq^{\mathbf{R}} W$. Then there are continuous functions $f : X \to Y$, $f' : Y \to X$, $g : Y \to W$, and $g' : W \to Y$, and real homotopies $F : X \times [0, 1] \to X$ from $f' \circ f$ to 1_X, $F' : Y \times [0, 1] \to Y$ from $f \circ f'$ to 1_Y, $G : Y \times [0, 1] \to Y$ from $g' \circ g$ to 1_Y, and $G' : W \times [0, 1] \to W$ from $g \circ g'$ to 1_W. We will show that $X \simeq^{\mathbf{R}} W$ using the functions $g \circ f : X \to W$ and $f' \circ g' : W \to X$.

Consider the function $H : X \times [0, 1] \to X$ defined by $H(x, t) = f'(G(f(x), t))$. We will show that H is a real homotopy from $f' \circ g' \circ g \circ f$ to $f' \circ f$. First observe that

$$H(x, 0) = f'(G(f(x), 0)) = f'(g'(g(f(x)))) = f' \circ g' \circ g \circ f(x)$$

and

$$H(x, 1) = f'(G(f(x), 1)) = f'(f(x)) = f' \circ f(x).$$

Also observe that $H_t = f' \circ G_t \circ f$, and thus H_t is continuous by Theorem 2. For H_x, we have $H_x = f' \circ G_{f(x)}$. Since $G_{f(x)}$ is a real path and f' is continuous, it is easy to see that H_x is a real path.

Thus we have shown that $f' \circ g' \circ g \circ f \simeq^{\mathbf{R}} f' \circ f$. By our assumption we have $f' \circ f \simeq^{\mathbf{R}} 1_X$, and thus by transitivity of real homotopy (Theorem 8) we have $f' \circ g' \circ g \circ f \simeq^{\mathbf{R}} 1_X$. A similar argument shows that $g \circ f \circ f' \circ g' \simeq 1_W$, and thus $X \simeq^{\mathbf{R}} W$ as desired. $\boxed{QED}$

Because of the theorem above, when $X \simeq^{\mathbf{R}} Y$, we say X and Y are *real homotopy equivalent*.

Corollary 2. *If $X \simeq Y$ then $X \simeq^{\mathbf{R}} Y$. If $(X, x_0) \simeq (Y, y_0)$ then $(X, x_0) \simeq^{\mathbf{R}} (Y, y_0)$. If both X and Y are finite, then the converses hold.*

Proof. This follows easily from Theorem 10. $\boxed{QED}$

Corollary 3. *If $X \simeq^L Y$, then $X \simeq^{\mathbf{R}} Y$. If $(X, x_0) \simeq^L (Y, y_0)$, then $(X, x_0) \simeq^{\mathbf{R}} (Y, y_0)$.*

Proof. This follows from Definitions 9 and 12 and Theorem 9. $\boxed{QED}$

Note that in light of Corollary 3, Example 3 below shows it is possible for two digital images to be pointed real homotopy equivalent without being pointed homotopy equivalent.

6 Examples

A finite set and an infinite set cannot have the same digital homotopy type. In particular the set $\mathbf{Z}^n$ is not of the same digital homotopy type as a single point, even though the continuous objects they represent ($\mathbf{R}^n$ and a point) are classically homotopy equivalent. In the following example we show that $\mathbf{Z}^n$ is homotopically similar to a point, and also of the same long homotopy type (and thus the same real homotopy type).

Example 1. Let $x = (x_0, x_1, \ldots, x_{n-1}) \in \mathbf{Z}^n$ and let $u, v \in [1, n]_{\mathbf{Z}}$. Then $(\{x\}, x)$ and $(\mathbf{Z}^n, x)$ are (c_u, c_v)-pointed homotopically similar and have the same (c_u, c_v)-pointed long homotopy type.

Proof. Corresponding to Definition 6, let $X_j = \{x\}$, let $Y_j = \Pi_{i=0}^{n-1}[x_i - j, x_i + j]_{\mathbf{Z}}$, let f_j be the inclusion map, and let g_j be the constant map with image $\{x\}$. Since a digital cube is pointed (c_u, c_v)-contractible with respect to any of its points [1], the assertion of pointed homotopic similarity follows.

Let $H : \mathbf{Z}^n \times \mathbf{N}^* \to \mathbf{Z}^n$ be the map defined as follows. Let $H(y, 0) = y$. For $t > 0$, let q be the reduction of t modulo n, and we bring the q-th coordinate of

$H(y,t)$ one unit closer to the q-th coordinate of x, i.e., if $H((y_0,\ldots,y_{n-1}),t-1)=(z_0,\ldots,z_{n-1})$ then

$$H((y_0,\ldots,y_{n-1}),t)=\begin{cases}(z_0,\ldots,z_{q-1},z_q-1,z_{q+1},\ldots,z_{n-1}) & \text{if } z_q>x_q;\\ (z_0,\ldots,z_{n-1}) & \text{if } z_q=x_q;\\ (z_0,\ldots,z_{q-1},z_q+1,z_{q+1},\ldots,z_{n-1}) & \text{if } z_q<x_q.\end{cases}$$

Since, in a single time step, H changes only one coordinate, it is easily seen that this map holds x fixed and, for all $u,v\in[1,n]_{\mathbf{Z}}$, is a (c_u,c_v)-l-homotopy from $1_{\mathbf{Z}^n}$ to the map $i_x\circ\overline{x}$, where $\overline{x}:\mathbf{Z}^n\to\{x\}$ is a constant map and $i_x:\{x\}\to\mathbf{Z}^n$ is the inclusion function. From Proposition 6, we have that $1_{\mathbf{Z}^n}$ and $i_x\circ\overline{x}$ are pointed long homotopic. Since $1_{\{x\}}=\overline{x}\circ i_x$, the assertion of the same pointed long homotopy type follows. $\boxed{QED}$

Recall that a tree is an acyclic graph in which every pair of points is connected by a unique injective path. We consider both finite and infinite trees.

Example 2. Let (X,κ) be a digital image that is a tree. Let $x_0\in X$. Then (X,x_0) is pointed homotopically similar to, and has the same pointed long homotopy type, as $(\{x_0\},x_0)$.

Proof. In the following, for $x\neq x_0$ we use $parent(x)$ to denote the unique vertex in X adjacent to x along the unique shortest path in X from x to x_0; and, by $dist(x,y)$ (the *distance* between vertices x and y) we mean the length of the shortest path in X from x to y. Note if $dist(x,x_0)=n>0$, then $dist(parent(x),x_0)=n-1$.

Corresponding to the notation of Definition 6, let $X_j=\{x\in X\mid dist(x,x_0)\leq j\}$. Let $Y=Y_j=\{x_0\}$. Let $f_j:X_j\to Y_j$ be the function $f_j(x)=x_0$. Let $g_j:Y_j\to X_j$ be the function $g(x_0)=x_0$. Let $H_j:X_j\times[0,j]_{\mathbf{Z}}\to X_j$ be the function

$$H_j(x,t)=\begin{cases}x & \text{if } t=0;\\ x_0 & \text{if } t>0 \text{ and } H(x,t-1)=x_0;\\ parent(H(x,t-1)) & \text{if } t>0 \text{ and } H(x,t-1)\neq x_0.\end{cases}$$

Then H_j is a pointed homotopy from 1_{X_j} to $g_j\circ f_j$. Further, $f_j\circ g_j=1_{Y_j}$. It follows easily that (X,x_0) and $(\{x_0\},x_0)$ are pointed homotopically similar.

Let $H:X\times\mathbf{N}^*\to X$ be the function

$$H(x,t)=\begin{cases}x & \text{if } t=0;\\ x_0 & \text{if } t>0 \text{ and } H(x,t-1)=x_0;\\ parent(H(x,t-1)) & \text{if } t>0 \text{ and } H(x,t-1)\neq x_0.\end{cases}$$

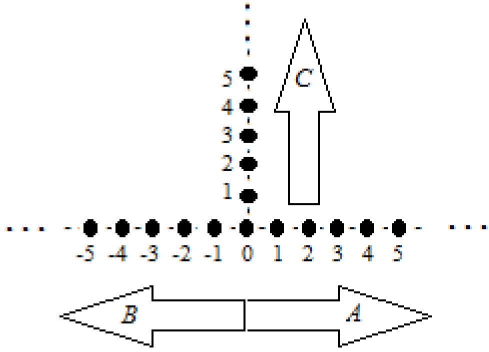

Figure 1. The image Y of Example 3, with the subsets A, B, C of its proof marked by arrows

Then H is a pointed l-homotopy from 1_X to $\overline{x_0}$. From Proposition 6, 1_X and $i_{\{x_0\}} \circ \overline{x_0}$ are pointed long homotopic, where $i_{\{x_0\}}$ is the inclusion function of $\{x_0\}$ into $\mathbf{Z}^n$. Since $\overline{x_0} \circ i_{\{x_0\}} = 1_{\{x_0\}}$, it follows that (X, x_0) and $(\{x_0\}, x_0)$ have the same pointed long homotopy type. $\boxed{QED}$

Example 3. Let $X = \mathbf{Z} \times \{0\} \subset \mathbf{Z}^2$ and let $Y = X \cup (\{0\} \times \mathbf{N}) \subset \mathbf{Z}^2$ (see Figure 1). Then $(X, (0,0)) \simeq^s_{c_1,c_1} (Y, (0,0))$, and $(X, (0,0)) \simeq^L_{c_1,c_1} (Y, (0,0))$, but $(X, (0,0))$ and $(Y, (0,0))$ do not have the same (c_1, c_1)-pointed homotopy type.

Proof. It follows from Example 2 and Theorems 3 and 7 that $(X, (0,0)) \simeq^s_{c_1,c_1} (Y, (0,0))$, and $(X, (0,0)) \simeq^L_{c_1,c_1} (Y, (0,0))$.

Suppose $(X, (0,0)) \simeq_{c_1,c_1} (Y, (0,0))$. Then there are continuous functions $f : X \to Y$, $g : Y \to X$, with, and pointed homotopies $H : X \times [0,k]_{\mathbf{Z}} \to X$ and $H' : Y \times [0,m]_{\mathbf{Z}} \to Y$ such that $H(x,0) = x$ and $H(x,k) = g \circ f(x)$ for all $x \in X$, and $H'(y,0) = y$ and $H'(y,m) = f \circ g(y)$ for all $y \in Y$.

We show f is a surjection. Note if $A = \mathbf{N}^* \times \{0\}$, $B = \{(n,0) \,|\, n \in \mathbf{Z},\ (-n,0) \in A\}$, and $C = \{0\} \times \mathbf{N}^*$, we have $Y = A \cup B \cup C$. Suppose there exists $p = (u,v) \in Y \setminus f(X)$.

- If $p \in A$ then, since $f(X)$ is c_1-connected and contains $(0,0)$, $A_1 = \{(x,0) \,|\, x \geq u\} \subset Y \setminus f(X)$. In particular,

$$A_2 = [u, u+2m]_{\mathbf{Z}} \times \{0\} \subset Y \setminus f(X). \quad (1)$$

  Since $u > 0$, A_2 is the set of all points in Y within m steps of $(u+m, 0)$, so we have a contradiction of the assumption that 1_Y and $f \circ g$ are homotopic in m steps, as statement (1) implies no point of $(f \circ g)(X)$ is within m steps of $(u+m, 0)$. Therefore, we cannot have $p \in A$.

- The cases $p \in B$ and $p \in C$ yield contradictions similarly.

Thus, we must have that f is a surjection, since assuming otherwise yields a contradiction.

Since $Y \setminus \{(0,0)\}$ is disconnected and each of A, B, and C is infinite, the fact that f is a continuous surjection implies there exist infinitely many $x \in X$ such that $f(x) = (0,0)$. Therefore, there exist $p_0 = (a,0), p_1 = (b,0) \in X$ with $b > a+2k$ such that $f(p_0) = f(p_1) = (0,0)$. Then $g \circ f(p_0) = g \circ f(p_1) = g(0,0)$. Therefore, at least one of p_0 or p_1 is carried by $g \circ f$ more than k steps away from itself, contrary to the assumption that $g \circ f$ and 1_X are homotopic within k steps. The assertion that $(X,(0,0))$ and $(Y,(0,0))$ do not have the same pointed homotopy type follows from this contradiction. $\boxed{QED}$

Example 4. There exist digital images (X,κ) and (Y,λ) that are homotopically similar but not pointed homotopically similar, and that have the same long homotopy type but not the same pointed long homotopy type, and that have the same real homotopy type but not the same pointed real homotopy type.

Proof. By [7, 4], there exist finite digital images X and Y that are homotopically equivalent but not pointed homotopically equivalent. By Theorem 1, X and Y are homotopically similar but not pointed homotopically similar. By Proposition 8, X and Y have the same long homotopy type but not the same pointed long homotopy type. By Theorem 10, X and Y have the same real homotopy type but not the same pointed real homotopy type. $\boxed{QED}$

7 Fundamental groups

In this section, we show that digital images that are pointed homotopically similar, or that have the same pointed real homotopy type, or that have the same pointed long homotopy type, have isomorphic fundamental groups.

Theorem 12. *Let $(X,x_1) \simeq^s (Y,y_1)$. Let $\{X_j, Y_j, f_j, g_j\}_{j=1}^{\infty}$ be as in Definition 6. Then there is an isomorphism $F : \Pi_1^{\kappa}(X,x_1) \to \Pi_1^{\lambda}(Y,y_1)$.*

Proof. By hypothesis, for all indices j we have that $g_j \circ f_j$ is pointed homotopic in X_j, hence in X, to 1_{X_j} and $f_j \circ g_j$ is pointed homotopic in Y_j, hence in Y, to 1_{Y_j}.

We define a function $F : \Pi_1^{\kappa}(X,x_1) \to \Pi_1^{\lambda}(Y,y_1)$ as follows. Let f be an x_1-based EC loop in X. There is a smallest positive integer j such that the image of f is contained in X_j. Then $f_j \circ f$ is a y_1-based EC loop in Y_j. Define $F([f]) = [f_j \circ f]$.

Suppose $f' \in [f]_X$. For some smallest indices j, j', the images of f, f' lie in $X_j, X_{j'}$, respectively. Further, there is some $a > j, j'$ such that f and f' are EC-homotopic in X_a. We have:

$$\begin{aligned} F([f]) = [f_j \circ f] = [(f_a|X_j) \circ f] = [f_a \circ f] = [f_a \circ f'] \\ = [(f_a|X_{j'}) \circ f'] = [f_{j'} \circ f'] = F([f']). \quad (2) \end{aligned}$$

Therefore, F is well defined.

Suppose f is an x_1-based EC loop in X such that $F([f]) = [\overline{y_1}]$, the identity element of $\Pi_1^{\lambda}(Y, y_1)$. If j is the minimal index such that the image of f is contained in Y_j, then

$$F([f]) = [\overline{y_1}] = [f_j \circ f],$$

so

$$[f] = [(g_j \circ f_j) \circ f] = [g_j \circ (f_j \circ f)] = [g_j \circ \overline{y_1}] = [\overline{x_1}],$$

the identity element of $\Pi_1^{\kappa}(X, x_1)$. Therefore, F is one-to-one.

Given a y_1-based EC loop g in Y, the image of g is contained in some Y_j for some smallest j. If $j' \leq j$ is the minimal index such that the image of $g_j \circ g$ is contained in $X_{j'}$, then

$$[g] = [f_j \circ g_j \circ g] = [f_{j'} \circ (g_j \circ g)] = F([(g_j \circ g)]).$$

Thus, F is onto.

Let L_i be x_1-based EC loops in X, $i \in \{0, 1\}$. Suppose the minimal indices for the X_j containing the images of the L_i are j_0, j_1 respectively, where, without loss of generality, $j_0 \leq j_1$. Then j_1 is the minimal index of the X_j containing $L_0 * L_1$. Then

$$\begin{aligned} F([L_0 * L_1]) = [f_{j_1} \circ (L_0 * L_1)] = [f_{j_1}(L_0) * f_{j_1}(L_1)] = [(f_{j_1}|X_{j_0})(L_0) * f_{j_1}(L_1)] \\ = [f_{j_0} \circ (L_0)] \cdot [f_{j_1} \circ (L_1)] = F(L_0) \cdot F(L_1). \end{aligned}$$

Therefore, F is a homomorphism. This completes the proof. $\boxed{QED}$

Theorem 13. *Let* $(X, x_0) \simeq_{\kappa,\lambda}^{\mathbf{R}} (Y, y_0)$. *Then* $\Pi_1^{\kappa}(X, x_0)$ *and* $\Pi_1^{\lambda}(Y, y_0)$ *are isomorphic.*

Proof. The hypothesis implies that there exist pointed continuous functions $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (X, x_0)$, and pointed long homotopies $H : X \times [0, 1] \to X$ from $g \circ f$ to 1_X and $G : Y \times [0, 1] \to Y$ from $f \circ g$ to 1_Y.

Given an x_0-based EC loop L in X, define $f_* : \Pi_1^\kappa(X, x_0) \to \Pi_1^\lambda(Y, y_0)$ by $f_*([L]) = [f \circ L]$. This function f_* is the homomorphism induced by $f : X \to Y$ (see [4] for a proof that f_* is a well-defined homomorphism). It remains to show that f_* is one-to-one and onto.

Suppose $f_*([L]) = [f \circ L] = [\overline{y_0}]$, the identity element of $\Pi_1^\lambda(Y, y_0)$. Since $g \circ f \simeq^{\mathbf{R}} 1_X$, we have $g \circ f \circ L \simeq^{\mathbf{R}} L$. Since the image of L is finite, an argument similar to that used in the proof of Theorem 10 shows that $g \circ f \circ L \simeq L$ by a pointed homotopy, and thus by a pointed EC homotopy. Thus,

$$[L] = [g \circ f \circ L] = [g \circ (f \circ L)] = [g \circ \overline{y_0}] = [\overline{x_0}],$$

the identity element of $\Pi_1^\kappa(X, x_0)$. Hence, f_* is one-to-one.

Suppose M is a y_0-based EC loop in Y. Then there is a real homotopy in Y between M and $f \circ g \circ M$ that holds the endpoints fixed. As above, since the image of M is finite, the argument from Theorem 10 gives a homotopy between M and $f \circ g \circ M$ that holds the endpoints fixed. Therefore,

$$[M] = [f \circ g \circ M] = f_*([g \circ M]).$$

Thus, f_* is onto. $\boxed{QED}$

Theorem 14. *Let* $(X, x_1) \simeq_{\kappa,\lambda}^L (Y, y_1)$*. Then* $\Pi_1^\kappa(X, x_0)$ *and* $\Pi_1^\lambda(Y, y_0)$ *are isomorphic.*

Proof. By Theorem 9, (X, κ, x_0) and (Y, λ, y_0) have the same real homotopy type. Then Theorem 13 gives the result. $\boxed{QED}$

8 Wedges and Cartesian products

In this section we show that the wedge and Cartesian product operations preserve pointed homotopic similarity, pointed long homotopy type, and pointed real homotopy type.

Recall that if X_1, X_2 are digital images in $\mathbf{Z}^m$ with the same adjacency relation κ such that $X_1 \cap X_2 = \{x_0\}$ for some point x_0, and such that x_0 is the only point of X_1 adjacent to any point of X_2 and is also the only point of X_2 adjacent to any point of X_1, then $X = X_1 \cup X_2$, with the κ adjacency, is the *wedge* of X_1 and X_2, denoted $X = X_1 \wedge X_2$, and x_0 is the *wedge point* of X. Also, if $X = X_1 \wedge X_2$, $Y = Y_1 \wedge Y_2$, x_0 is the wedge point of X, y_0 is the wedge point of Y, and $f_i : (X_i, x_0) \to (Y_i, y_0)$ are (κ, λ)-pointed continuous for $i \in \{1, 2\}$, then the function $f_1 \wedge f_2 : X \to Y$ defined by

$$(f_1 \wedge f_2)(x) = \begin{cases} f_1(x) & \text{if } x \in X_1; \\ f_2(x) & \text{if } x \in X_2, \end{cases}$$

is easily seen to be (κ, λ)-continuous.

Theorem 15. *Suppose X_1, X_2 are digital images in $\mathbf{Z}^m$, and Y_1 and Y_2 are digital images in $\mathbf{Z}^n$. If $(X_1, x_0) \simeq^s_{\kappa,\lambda} (Y_1, y_0)$ and $(X_2, x_0) \simeq^s_{\kappa,\lambda} (Y_2, y_0)$, and $X = X_1 \wedge X_2$ has wedge point x_0 and $Y = Y_1 \wedge Y_2$ has wedge point y_0, then $(X, x_0) \simeq^s_{\kappa,\lambda} (Y, y_0)$.*

Proof. By hypothesis, there are subsets $X_{j,n}$ of X_j and $Y_{j,n}$ of Y_j, such that $X_{j,n} \subset X_{j+1,n}$ and $Y_{j,n} \subset Y_{j+1,n}$, for $j \in \{1,2\}$, $n \in \mathbf{N}$, $X_j = \bigcup_{n=1}^{\infty} X_{j,n}$, $Y_j = \bigcup_{n=1}^{\infty} Y_{j,n}$, and pointed continuous functions $f_n : (X_{1,n}, x_0) \to (Y_{1,n}, y_0)$, $g_n : (Y_{1,n}, y_0) \to (X_{1,n}, x_0)$, $f'_n : (X_{2,n}, x_0) \to (Y_{2,n}, y_0)$, $g'_n : (Y_{2,n}, y_0) \to (X_{2,n}, x_0)$, such that $g_n \circ f_n$ is pointed homotopic in $X_{1,n}$ to $1_{X_{1,n}}$, $f_n \circ g_n$ is pointed homotopic in $Y_{1,n}$ to $1_{Y_{1,n}}$, $g'_n \circ f'_n$ is pointed homotopic in $X_{2,n}$ to $1_{X_{2,n}}$, and $f'_n \circ g'_n$ is pointed homotopic in $Y_{2,n}$ to $1_{Y_{2,n}}$. Also, x_0 is the wedge point for each $X_{1,n} \wedge X_{2,n}$, and y_0 is the wedge point for each $Y_{1,n} \wedge Y_{2,n}$.

Then it is easily seen that $(g_n \wedge g'_n) \circ (f_n \wedge f'_n)$ is pointed homotopic in $X_1 \wedge X_2$ to $1_{X_1 \wedge X_2}$ and $(f_n \wedge f'_n) \circ (g_n \wedge g'_n)$ is pointed homotopic in $Y_1 \wedge Y_2$ to $1_{Y_1 \wedge Y_2}$. The assertion follows. $\boxed{QED}$

Theorem 16. *Suppose X_1, X_2 are digital images in $\mathbf{Z}^m$, and Y_1 and Y_2 are digital images in $\mathbf{Z}^n$. If $(X_1, x_0) \simeq^L_{\kappa,\lambda} (Y_1, y_0)$, and $(X_2, x_0) \simeq^L_{\kappa,\lambda} (Y_2, y_0)$, and $X = X_1 \wedge X_2$ has wedge point x_0 and $Y = Y_1 \wedge Y_2$ has wedge point y_0, then $(X, x_0) \simeq^L_{\kappa,\lambda} (Y, y_0)$.*

Proof. By hypothesis, for $i \in \{1,2\}$ there exist pointed continuous functions $f_i : (X_i, x_0) \to (Y_i, y_0)$ and $g_i : (Y_i, y_0) \to (X_i, x_0)$, long pointed homotopies $H_i : (X_i, x_0) \times \mathbf{Z} \to (X_i, x_0)$ from 1_{X_i} to $g_i \circ f_i$ in X_i, and long pointed homotopies $K_i : (Y_i, y_0) \times \mathbf{Z} \to (Y_i, y_0)$ from 1_{Y_i} to $f_i \circ g_i$ in Y_i.

Then the function $H : (X_1 \wedge X_2, x_0) \times \mathbf{Z} \to (X_1 \wedge X_2, x_0)$ defined by

$$H(x,t) = \begin{cases} H_1(x,t) & \text{if } x \in X_1; \\ H_2(x,t) & \text{if } x \in X_2, \end{cases}$$

is a pointed long homotopy in X from $1_{X_1 \wedge X_2}$ to $(g_1 \wedge g_2) \circ (f_1 \wedge f_2)$. Similarly, the function $K : (Y_1 \wedge Y_2, x_0) \times \mathbf{Z} \to (Y_1 \wedge Y_2, x_0)$ defined by

$$K(y,t) = \begin{cases} K_1(y,t) & \text{if } y \in Y_1; \\ K_2(y,t) & \text{if } y \in Y_2, \end{cases}$$

is a pointed long homotopy in Y from $1_{Y_1 \wedge Y_2}$ to $(f_1 \wedge f_2) \circ (g_1 \wedge g_2)$. The assertion follows. $\boxed{QED}$

Arguments similar to those above demonstrate the following.

Theorem 17. *Suppose X_1, X_2 are digital images in $\mathbf{Z}^m$, and Y_1 and Y_2 are digital images in $\mathbf{Z}^n$. If $(X_1, \kappa, x_0) \simeq^{\mathbf{R}}_{\kappa,\lambda} (Y_1, \lambda, y_0)$ and $(X_2, x_0) \simeq^{\mathbf{R}}_{\kappa,\lambda} (Y_2, y_0)$, and $X = X_1 \wedge X_2$ has wedge point x_0 and $Y = Y_1 \wedge Y_2$ has wedge point y_0, then $(X, x_0) \simeq^{\mathbf{R}}_{\kappa,\lambda} (Y, y_0)$.* $\square$

Now we consider Cartesian products. For our pointed assertions in the following, we assume

$$x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n_i}) \in X_i \subset \mathbf{Z}^{n_i}, \qquad y_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i}) \in Y_i \subset \mathbf{Z}^{n_i},$$

$$x_0 = (x_1, x_2, \ldots, x_{n_i}) =$$
$$(x_{1,1}, x_{1,2}, \ldots, x_{1,n_1}, x_{2,1}, x_{2,2}, \ldots, x_{2,n_2}, \ldots, x_{k,1}, x_{k,2}, \ldots, x_{k,n_k}) \in \Pi_{i=1}^k X_i \subset \mathbf{Z}^D,$$

$$y_0 = (y_1, y_2, \ldots, y_{n_i}) =$$
$$(y_{1,1}, y_{1,2}, \ldots, y_{1,n_1}, y_{2,1}, y_{2,2}, \ldots, y_{2,n_2}, \ldots, y_{k,1}, y_{k,2}, \ldots, y_{k,n_k}) \in \Pi_{i=1}^k Y_i \subset \mathbf{Z}^D,$$

where $D = \sum_{i=1}^k n_i$.

Theorem 18. *Let X_i and Y_i be digital images in $(\mathbf{Z}^{n_i}, c_{n_i})$, $i \in \{1, 2, \ldots, k\}$. Let $x_i \in X_i$, $y_i \in Y_i$. Let $D = \sum_{i=1}^k n_i$.*

- *If $X_i \simeq_{c_{n_i}, c_{n_i}} Y_i$ for $i \in \{1, 2, \ldots, k\}$, then $\Pi_{i=1}^k X_i \simeq_{c_D, c_D} \Pi_{i=1}^k Y_i$. If $(X_i, x_i) \simeq_{c_{n_i}, c_{n_i}} (Y_i, y_i)$ for $i \in \{1, 2, \ldots, k\}$, then $(\Pi_{i=1}^k X_i, x_0) \simeq_{c_D, c_D} (\Pi_{i=1}^k Y_i, y_0)$.*
- *If $X_i \simeq^s_{c_{n_i}, c_{n_i}} Y_i$ for $i \in \{1, 2, \ldots, k\}$, then $\Pi_{i=1}^k X_i \simeq^s_{c_D, c_D} \Pi_{i=1}^k Y_i$. If $(X_i, x_i) \simeq^s_{c_{n_i}, c_{n_i}} (Y_i, y_i)$ for $i \in \{1, 2, \ldots, k\}$, then $(\Pi_{i=1}^k X_i, x_0) \simeq^s_{c_D, c_D} (\Pi_{i=1}^k Y_i, y_0)$.*
- *If $X_i \simeq^L_{c_{n_i}, c_{n_i}} Y_i$ for $i \in \{1, 2, \ldots, k\}$, then $\Pi_{i=1}^k X_i \simeq^L_{c_D, c_D} \Pi_{i=1}^k Y_i$. If $(X_i, x_i) \simeq^L_{c_{n_i}, c_{n_i}} (Y_i, y_i)$ for $i \in \{1, 2, \ldots, k\}$, then $(\Pi_{i=1}^k X_i, x_0) \simeq^L_{c_D, c_D} (\Pi_{i=1}^k Y_i, y_0)$.*
- *If $X_i \simeq^{\mathbf{R}}_{c_{n_i}, c_{n_i}} Y_i$ for $i \in \{1, 2, \ldots, k\}$, then $\Pi_{i=1}^k X_i \simeq^{\mathbf{R}}_{c_D, c_D} \Pi_{i=1}^k Y_i$. If $(X_i, x_i) \simeq^{\mathbf{R}}_{c_{n_i}, c_{n_i}} (Y_i, y_i)$ for $i \in \{1, 2, \ldots, k\}$, then $(\Pi_{i=1}^k X_i, x_0) \simeq^{\mathbf{R}}_{c_D, c_D} (\Pi_{i=1}^k Y_i, y_0)$.*

Proof. We give proofs for the unpointed assertions. In all cases, the proof of the pointed assertion is virtually identical to that for its unpointed analog. We let $X = \Pi_{i=1}^k X_i \subset \mathbf{Z}^D$, $Y = \Pi_{i=1}^k Y_i \subset \mathbf{Z}^D$.

First we prove the statement about ordinary homotopy equivalence. Suppose $X_i \simeq_{c_{n_i}, c_{n_i}} Y_i$ for $i \in \{1, 2, \ldots, k\}$. Then there exist (c_{n_i}, c_{n_i})-continuous functions $f_i : X_i \to Y_i$ and $g_i : Y_i \to X_i$, and homotopies $H_i : X_i \times [0, u_i]_{\mathbf{Z}} \to X_i$ from 1_{X_i} to $f_i \circ g_i$ and $K_i : Y_i \times [0, v_i]_{\mathbf{Z}} \to Y_i$ from 1_{Y_i} to $g_i \circ f_i$. Without loss of generality, we can replace each u_i and each v_i with $U = \max\{u_1, v_1, \ldots, u_k, v_k\}$, since if $u_i < U$ then we can extend H_i by defining $H_i(x, t) = H_i(x, u_i) = g_i \circ f_i(x)$ for $u_i \leq t \leq U$, and similarly for K_i.

For $a_i \in X_i$, let $f : X \to Y$ be defined by

$$f(a_1, a_2, \ldots, a_k) = (f_1(a_1), f_2(a_2), \ldots, f_k(a_k)).$$

For $b_i \in Y_i$, let $g : Y \to X$ be defined by

$$g(b_1, b_2, \ldots, b_k) = (g_1(b_1), g_2(b_2), \ldots, g_k(b_k)).$$

Let $H : X \times [0, U]_{\mathbf{Z}} \to X$ be defined by

$$H(a_1, a_2, \ldots, a_k, t) = (H_1(a_1, t), H_2(a_2, t), \ldots, H_k(a_k, t)).$$

Let $K : Y \times [0, U]_{\mathbf{Z}} \to Y$ be defined by

$$K(b_1, b_2, \ldots, b_k, t) = (K_1(b_1, t), K_2(b_2, t), \ldots, K_k(b_k, t)).$$

It is easy to see that H is a (c_D, c_D)-homotopy from 1_X to $g \circ f$, and K is a (c_D, c_D)-homotopy from 1_Y to $f \circ g$.

Minor modifications in the argument given above allow us to demonstrate the claims for long homotopy type and real homotopy type.

It remains to prove the statement about homotopic similarity. Suppose $X_i \simeq^s_{c_{n_i}, c_{n_i}} Y_i$ for $i \in \{1, 2, \ldots, k\}$. Then there exist $\{X_{i,j}\}_{j=1}^{\infty} \subset X_i$ such that $X_{i,j} \subset X_{i,j+1}$ and $\bigcup_{j=1}^{\infty} X_{i,j} = X_i$, $\{Y_{i,j}\}_{j=1}^{\infty} \subset Y_i$ such that $Y_{i,j} \subset Y_{i,j+1}$ and $\bigcup_{j=1}^{\infty} Y_{i,j} = Y_i$, continuous functions $f_{i,j} : X_{i,j} \to Y_{i,j}$ and $g_{i,j} : Y_{i,j} \to X_{i,j}$, and homotopies $H_{i,j} : X_{i,j} \times [0, u_{i,j}]_{\mathbf{Z}} \to X_{i,j}$ from $g_{i,j} \circ f_{i,j}$ to $1_{X_{i,j}}$ and $K_{i,j} : Y_{i,j} \times [0, v_{i,j}]_{\mathbf{Z}} \to Y_{i,j}$ from $f_{i,j} \circ g_{i,j}$ to $1_{Y_{i,j}}$. As above, for each j we can replace each $u_{i,j}$ and each $v_{i,j}$ by $U_j = \max\{u_{i,j}, v_{i,j}\}_{i=1}^{k}$. Further, these functions satisfy the homotopy restrictions required by Definition 6.

Notice that for all j, $\Pi_{i=1}^{k} X_{i,j} \subset \Pi_{i=1}^{k} X_{i,j+1}$ and $\Pi_{i=1}^{k} Y_{i,j} \subset \Pi_{i=1}^{k} Y_{i,j+1}$. Also, $\bigcup_{j=1}^{\infty} \Pi_{i=1}^{k} X_{i,j} = X$ and $\bigcup_{j=1}^{\infty} \Pi_{i=1}^{k} Y_{i,j} = Y$.

Let $f_j : \Pi_{i=1}^{k} X_{i,j} \to \Pi_{i=1}^{k} Y_{i,j}$ be defined by

$$f_j(a_1, \ldots, a_k) = (f_{1,j}(a_1), \ldots, f_{k,j}(a_k))$$

for $a_i \in X_{i,j}$. Let $g_j : \Pi_{i=1}^{k} Y_{i,j} \to \Pi_{i=1}^{k} X_{i,j}$ be defined by

$$g_j(b_1, \ldots, b_k) = (g_{1,j}(b_1), \ldots, g_{k,j}(b_k))$$

for $b_i \in Y_{i,j}$. Let $H_j : \Pi_{i=1}^k X_{i,j} \times [0, U_j]_{\mathbf{Z}} \to \Pi_{i=1}^k X_{i,j}$ be defined by

$$H_j(a_1, \ldots, a_k, t) = (H_{1,j}(a_1, t), \ldots, H_{k,j}(a_k, t)),$$

for $a_i \in X_{i,j}$. Let $K_j : \Pi_{i=1}^k Y_{i,j} \times [0, U_j]_{\mathbf{Z}} \to \Pi_{i=1}^k Y_{i,j}$ be defined by

$$K_j(b_1, \ldots, b_k, t) = (K_{1,j}(b_1, t), \ldots, K_{k,j}(b_k, t)),$$

for $b_i \in Y_{i,j}$. Then it is easily seen that H_j is a homotopy from $f_j \circ g_j$ to 1_{X_j}, and K_j is a homotopy from $g_j \circ f_j$ to 1_{Y_j}. Also, it is easily shown that these homotopies satisfy the restrictions required by Definition 6. Therefore, $X \simeq^s Y$. $\boxed{QED}$

9 Further remarks and open questions

We have introduced three notions of digital images having homotopic resemblance - homotopic similarity, having the same long homotopy type, and having the same real homotopy type - in both unpointed and pointed versions. Unlike the usual definition of digital homotopy equivalence, these let us consider two digital images X and Y as similar with respect to homotopy properties even if one of them has a component with infinite diameter and the other does not. We have shown that two digital images that are homotopy equivalent are homotopically similar, have the same long homotopy type, and have the same real homotopy type, and that the converses hold when both images are finite; however, we have shown the converses to be false if one of the images has infinite diameter. We have shown that two digital images that share any of these three pointed homotopy resemblances have isomorphic fundamental groups. We have also shown that wedges preserve pointed homotopy similarity, pointed long homotopy type, and pointed real homotopy type; as do finite Cartesian products when we use relaxed adjacencies.

Remark 7. In addition to several questions stated earlier that we have not answered at this writing, we have the following. (Unpointed and pointed versions:) Which, if any, of homotopic similarity, having the same long homotopy type, and having the same real homotopy type, implies either of the others?

Corollary 3 is our only result concerning the question of Remark 7, that having the same long homotopy type implies having the same real homotopy type.

Remark 8. Also currently unanswered (unpointed and pointed versions): Which, if any, of these relations are equivalent?

As above, a negative example for the question of Remark 8 would require a pair of digital images (X, Y) in which at least one of the members is infinite.

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