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# On Riemannian geometry of orthonormal frame bundles

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**Abstract.** O. Kowalski and the present author have studied systematically Riemannian geometry of the metric on the orthonormal frame bundle induced by the Sasaki-Mok metric on the linear frame bundle. We have also studied more general metrics which rescale the horizontal part by a nonzero constant. In particular, we have obtained nonstandard Einstein metrics and also metrics with zero scalar curvature on the orthonormal frame bundles over any space of constant sectional curvature. In the present paper, we shall survey our results without detailed proofs.

**Keywords:** Riemannian metric, linear frame bundle, orthonormal frame bundle, Riemannian curvature, Ricci curvature, scalar curvature

**MSC 2000 classification:** primary 53C15, 53C20, 53C30, secondary 53C21, 53C25

## Introduction

The linear frame bundle $LM$ over an $n$-dimensional manifold $M$ is a principal fiber bundle with the structure group $\mathrm{GL}(n, \mathbb{R})$. If we fix a Riemannian metric $g$ of $M$, then many natural lifts of $g$ from $M$ to $LM$ are Riemannian metrics on $LM$. O. Kowalski and the present author have given in [11] full classification of naturally lifted (possibly degenerate) pseudo-Riemannian metrics on $LM$ which come from a second order natural transformation of a Riemannian metric $g$ on $M$. (See [10, 15, 16, 17, 18] for the concept of naturality.) The diagonal lift $g^d$ (which is denoted by $\bar{g}$ in this paper) has been defined by K. P. Mok in [20] as the simplest Riemannian metric among them. It is called sometimes the Sasaki-Mok metric because it resembles the Sasaki metric of the tangent bundle over a Riemannian manifold. After K. P. Mok, the Sasaki-Mok metric on $LM$ has been studied systematically by L. A. Cordero and M. de León in [3] and [4]. See also the monograph [5] written jointly with C. T. J. Dodson. O. Kowalski and the present author have investigated in [12] a more general family of natural metrics on $LM$.

The orthonormal frame bundle $OM$ over an $n$-dimensional Riemannian manifold $(M, g)$ is a subbundle of $LM$ with the structure group $O(n)$. Because

$OM$ is a submanifold of $LM$, we can restrict the diagonal lift $\bar{g}$ to $OM$. We denote this induced metric by $\tilde{g}$, and call it the diagonal lift of $g$ to $OM$. We shall see in Section 1 that the metric $\tilde{g}$ on $OM$ coincides with the metric treated by B. O'Neill in [21] as an example of the Riemannian submersions. O'Neill has also calculated the sectional curvatures of this metric in a very compact and elegant form, but his formulas are not too convenient for geometric applications. We have presented an alternative in [13] (see Proposition 2). The initial properties of $\tilde{g}$ on $OM$ have been studied also (in a few pages) by Mok in [20] and presented later in the survey [5]. To the author's knowledge, the paper [13] is *the first systematic study of curvature* of this metric.

O. Kowalski and the present author have studied in [14] more general metrics $\tilde{g}_c$ on $OM$ defined depending on a nonzero constant $c$, which are Riemannian for $c > 0$ and pseudo-Riemannian for $c < 0$. The metric $\tilde{g}_1$ for $c = 1$ is the diagonal lift $\tilde{g}$ itself. We are mainly interested in the geometry of $OM$ in the case when the base manifold has constant sectional curvature. We have given Einstein metrics on $OM$ over any space of constant sectional curvatures.

**Acknowledgements.** This article is a survey of part of results obtained by work collaborated with Oldřich Kowalski over a quarter of the century. I would like to send him my great thanks for his hospitalities at every time when I visited Prague. I also would like express my sincere gratitude to him for variable discussions and hints during preparing this article.

# 1 The diagonal lift of a metric on the base manifold

Let $M$ be an $n$-dimensional smooth manifold, $n \geq 2$. Then the *linear frame bundle* $LM$ over $M$ consists of all pairs $(x, u)$, where $x$ is a point of $M$ and $u$ is a basis for the tangent space $M_x$ of $M$ at $x$. We denote by $p$ the natural projection of $LM$ to $M$ defined by $p(x, u) = x$. If $(\mathcal{U};\ x^1, x^2, \ldots, x^n)$ is a system of local coordinates in $M$, then a basis $u = (u_1, u_2, \ldots, u_n)$ for $M_x$ can be expressed uniquely in the form $u_\lambda = \sum_{i=1}^n u_\lambda^i (\partial/\partial x^i)_x$ for $\lambda = 1, 2, \ldots, n$, and hence $(p^{-1}(\mathcal{U});\ x^1, x^2, \ldots, x^n, u_1^1, u_1^2, \ldots, u_n^n)$ is a system of local coordinates in $LM$.

Let $\nabla$ be a linear connection on $M$. Then the tangent space $(LM)_{(x,u)}$ of $LM$ at $(x, u) \in LM$ splits into the horizontal and vertical subspace $H_{(x,u)}$ and $V_{(x,u)}$ with respect to $\nabla$:

$$(LM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

If a point $(x, u) \in LM$ and a vector $X \in M_x$ are given, then the *horizontal lift* of $X$ to $LM$ at $(x, u)$ is a unique vector $X^h \in H_{(x,u)}$ such that $p_*(X^h) = X$. We define naturally $n$ different vertical lifts of $X \in M_x$. If $\omega$ is a one-form on $M$, then $\iota_\mu \omega$, $\mu = 1, 2, \ldots, n$, are functions on $LM$ defined by $(\iota_\mu \omega)(x, u) = \omega(u_\mu)$ for all $(x, u) = (x, u_1, u_2, \ldots, u_n) \in LM$. The *vertical lifts* $X^{v,\lambda}$, $\lambda = 1, 2, \ldots, n$, of $X \in M_x$ to $LM$ at $(x, u)$ are the $n$ vectors such that $X^{v,\lambda}(\iota_\mu \omega) = \omega(X)\delta^\lambda_\mu$, $\lambda, \mu = 1, 2, \ldots, n$, holds for all one-forms $\omega$ on $M$, where $\delta^\lambda_\mu$ denotes the Kronecker's delta. The $n$ vertical lifts are always uniquely determined, and they are linearly independent if $X \neq 0$.

The *diagonal lift* $\bar{g}$ to $LM$ of a Riemannian metric $g$ of $M$ is determined by the formulas

$$\bar{g}(X^h, Y^h) = g(X, Y) \circ p,$$

$$\bar{g}(X^h, Y^{v,\mu}) = 0,$$

$$\bar{g}(X^{v,\lambda}, Y^{v,\mu}) = \delta^{\lambda\mu} g(X, Y) \circ p$$

for all $X, Y \in \mathfrak{X}(M)$ and $\lambda, \mu = 1, 2, \ldots, n$.

The *canonical vertical vector fields* of $LM$ are vector fields $\boldsymbol{U}^\mu_\lambda$, $\lambda, \mu = 1, 2, \ldots, n$, defined, in terms of local coordinates, by $\boldsymbol{U}^\mu_\lambda = \sum_i u^i_\lambda \partial/\partial u^i_\mu$. Here $\boldsymbol{U}^\mu_\lambda$'s do not depend on the choice of local coordinates and they are defined globally on $LM$. For a vector $u_\lambda = \sum_i u^i_\lambda (\partial/\partial x^i)_x \in M_x$, $\lambda = 1, 2, \ldots, n$, we see that ${u_\lambda}^h = \sum_i u^i_\lambda (\partial/\partial x^i)^h_{(x,u)}$ and ${u_\lambda}^{v,\mu} = \sum_i u^i_\lambda (\partial/\partial x^i)^{v,\mu}_{(x,u)} = \boldsymbol{U}^\mu_\lambda(x, u)$.

The *orthonormal frame bundle*

$$OM = \{(x, u) \in LM \mid g_x(u_\lambda, u_\mu) = \delta_{\lambda\mu}, \lambda, \mu = 1, 2, \ldots, n\}$$

over a Riemannian manifold $(M, g)$ is an $n(n+1)/2$-dimensional subbundle of $LM$ with the structure group $O(n)$. The vector fields $T_{\lambda\mu}$, $\lambda, \mu = 1, 2, \ldots, n$, defined on $LM$ by

$$T_{\lambda\mu} = \frac{1}{\sqrt{2}}(\boldsymbol{U}^\mu_\lambda - \boldsymbol{U}^\lambda_\mu)$$

are tangent to $OM$ at every point $(x, u) \in OM$. Here $T_{\lambda\mu}$ is skew-symmetric with respect to $\lambda$ and $\mu$. In particular $T_{\lambda\lambda} = 0$ for all $\lambda = 1, 2, \ldots, n$.

We endow $OM$ with the Riemannian metric $\tilde{g}$ induced from the diagonal lift $\bar{g}$ of $g$. This metric $\tilde{g}$ is uniquely determined by the formulas

$$\tilde{g}(X^h, Y^h) = g(X, Y) \circ p,$$

$$\tilde{g}(X^h, T_{\lambda\mu}) = 0,$$

$$\tilde{g}(T_{\lambda\mu}, T_{\nu\omega}) = \delta_{\lambda\nu}\delta_{\mu\omega} - \delta_{\lambda\omega}\delta_{\mu\nu}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\lambda, \mu, \nu, \omega = 1, 2, \ldots, n$. We have at every point $(x, u) \in OM$, a collection

$$\{{u_\lambda}^h,\ T_{\lambda\mu}(x, u) \mid 1 \leq \lambda < \mu \leq n\}$$

is an orthonormal basis for the tangent space $(OM)_{(x,u)}$ of the Riemannian manifold $(OM, \tilde{g})$.

Let $\boldsymbol{X} = \sum_{\lambda<\mu} \boldsymbol{X}_{\lambda\mu} T_{\lambda\mu}$ and $\boldsymbol{Y} = \sum_{\lambda<\mu} \boldsymbol{Y}_{\lambda\mu} T_{\lambda\mu}$ be *vertical* vector fields tangent to $OM$. Then we can assume that the matrices $[\boldsymbol{X}_{\lambda\mu}]$ and $[\boldsymbol{Y}_{\lambda\mu}]$ consist of the components of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, are skew-symmetric because $T_{\lambda\mu}$'s are skew-symmetric with respect to $\lambda$ and $\mu$. We see easily that

$$\tilde{g}_{(x,u)}(\boldsymbol{X}, \boldsymbol{Y}) = -\operatorname{trace}([\boldsymbol{X}_{\lambda\mu}][\boldsymbol{Y}_{\lambda\mu}]).$$

This implies that $\tilde{g}$ coincides with the metric on $OM$ treated by B. O'Neill in [21] as an example of the Riemannian submersions.

## 2 Geometry viewed from downstairs

We deal with the case that the sectional curvature function $K$ of $(M, g)$ satisfies some specific kind of pinching.

**1 Theorem** ([13]). *Let $(M, g)$ be an $n$-dimensional Riemannian manifold, $n \geq 2$, such that its sectional curvature $K$ satisfies*

$$0 < \delta < K < \frac{2}{3\sqrt{3n(n-1)}}\sqrt{\delta}$$

*for some positive number $\delta$. Then the orthonormal frame bundle $(OM, \tilde{g})$ has nonnegative sectional curvature. If, in particular, $n = 2$, then the sectional curvature of $(OM, \tilde{g})$ is positive.*

For the *proof* of Theorem 1 we use

**2 Proposition** ([13]). *The sectional curvature $\tilde{K}_{(x,u)}$ at $(x, u)$ of $(OM, \tilde{g})$ is given by*

$$\tilde{K}_{(x,u)}(X^h, Y^h) = K_x(X, Y) - \frac{3}{4}\sum_{\alpha} \|R_x(X, Y)u_\alpha\|^2,$$
$$X, Y \in M_x, \ \|X\| = \|Y\| = 1, \ \langle X, Y \rangle = 0,$$
$$\tilde{K}_{(x,u)}(X^h, T_{\lambda\mu}) = \frac{1}{2}\|R_x(u_\lambda, u_\mu)X\|^2, \quad X \in M_x, \ \|X\| = 1,$$
$$\tilde{K}_{(x,u)}(T_{\lambda\mu}, T_{\lambda\omega}) = \frac{1}{8}, \quad \lambda < \mu < \omega,$$
$$\tilde{K}_{(x,u)}(T_{\lambda\mu}, T_{\nu\omega}) = 0, \quad \lambda < \mu < \nu < \omega,$$

*where $K_x$ is the sectional curvature at $x$ of $(M, g)$, and $\|\cdot\|$ is the norm with respect to $g_x$.*

Sketch of the proof of Theorem 2. Suppose that there is a constant $k$ such that $|R(e_\lambda, e_\mu, e_\lambda, e_\mu)| < k$ for each orthonormal basis $(e_1, e_2, \dots, e_n)$ for $M_x$ and every two indices $\lambda, \mu = 1, 2, \dots, n$. Then we have $|R(e_\lambda, e_\mu, e_\nu, e_\omega)| < 2k$ when just three indices are distinct and $|R(e_\lambda, e_\mu, e_\nu, e_\omega)| < 3k$ when all four indices are distinct. Choose an orthonormal basis $(u_1, u_2, \dots, u_n)$ for $M_x$ such that $u_1 = X$ and $u_2 = Y$. Put $k = 2\sqrt{\delta}/\big(3\sqrt{3n(n-1)}\,\big)$. Because $K(U,V) < k$ for every orthonormal pair $(U,V)$, then $|R(u_1, u_2, u_\alpha, u_\beta)| < 3k$ holds for all $\alpha, \beta = 1, 2, \dots, n$. Thus we obtain that $3\sum_{\alpha=1}^n \|R(u_1,u_2)u_\alpha\|^2/4 < K(u_1,u_2)$, which means $3\sum_{\alpha=1}^n \|R(X,Y)u_\alpha\|^2/4 < K(X,Y)$. Hence $\tilde{K}(X^h, Y^h) > 0$.

The particular case $n = 2$ is now obvious. QED

**3 Theorem** ([13]). *Let $(M,g)$ be an $n$-dimensional Riemannian manifold, $n \geq 2$, such that its sectional curvature $K$ satisfies*

$$0 < \delta < K < \frac{2}{3\sqrt{n(n-1)}}\sqrt{\delta}$$

*for some positive number $\delta$. Then the orthonormal frame bundle $(OM, \tilde{g})$ has positive scalar curvature.*

For the *proof* of Theorem 3 we use

**4 Proposition.** *The scalar curvature $\widetilde{\mathrm{Sc}}(\tilde{g})_{(x,u)}$ at $(x,u)$ of $(OM, \tilde{g})$ is given by*

$$\widetilde{\mathrm{Sc}}(\tilde{g})_{(x,u)} = \mathrm{Sc}(g)_x - \frac{1}{4}\sum_{\alpha,\beta,\gamma=1}^{n} \|R_x(u_\alpha, u_\beta)u_\gamma\|^2 + \frac{1}{8}n(n-1)(n-2),$$

*where $\mathrm{Sc}(g)_x$ is the scalar curvature at $x$ of $(M,g)$.*

Sketch of the proof of Theorem 3. For all indices $\lambda$, $\mu$, $\nu$, $\omega$, and all orthonormal basis $(u_1, u_2, \dots, u_n)$ for $M_x$ we see that $|R(u_\lambda, u_\mu, u_\nu, u_\omega)| < 2\sqrt{\delta}/\sqrt{n(n-1)}$. On the other hand, we see that $\mathrm{Sc}(g)_x > n(n-1)\delta$ and $\|R_x\|^2 < 4\,\mathrm{Sc}(g)_x$. Hence the result follows. QED

**5 Corollary** ([13]). *Let $(M,g)$ be an $n$-dimensional compact Riemannian manifold with positive sectional curvature, $n \geq 2$. Then for each sufficiently small $c > 0$ the Riemannian manifold $(M,h)$, $h = c^2 g$, homothetic to $(M,g)$ has the property that the corresponding orthonormal frame bundle $(OM, \tilde{h})$ has nonnegative sectional curvature and positive scalar curvature. For $n = 2$, it has positive sectional curvature.*

Proof of Corollary 5. For the sectional curvature $K$ of $(M,g)$, there are constants $A$ and $B$ such that $0 < A < K < B$. Now, choose a positive number $c < 2\sqrt{A}/\big(3\sqrt{3n(n-1)B}\,\big)$. The sectional curvature $K_h = c^2 K$ of

$(M,h)$ satisfies $0 < Ac^2 < K_h < Bc^2$. Next, put $\delta = Ac^2$. According to our choice of $c$, we have

$$Bc^2 < \frac{4A}{27n(n-1)B} = \frac{4\delta}{27n(n-1)Bc^2},$$

which shows that $Bc^2 < 2\sqrt{\delta}/\big(3\sqrt{3n(n-1)}\,\big)$. Here we use Theorem 1 to complete the proof of the first part of the Corollary. The proof of the second part follows analogously from Theorem 3, and the last statement (for $n = 2$) can be easily checked from Proposition 2. $\boxed{QED}$

## 3 Einstein metrics on orthonormal frame bundles

We have seen in [13] that, if $(M,g)$ is a space of constant sectional curvature $1/2$ or $(n-2)/(2n)$, then the diagonal lift $\tilde{g}$ of $g$ to $OM$ is Einstein. We study a bit more general metric $\tilde{g}_c$ defined on $OM$ depending on nonzero constant $c$ to get Einstein metrics over every space of constant sectional curvature.

We endow $OM$ with the metric $\tilde{g}_c$, $c \neq 0$, uniquely determined by the formulas

$$\tilde{g}_c(X^h, Y^h) = c\, g(X,Y) \circ p,$$
$$\tilde{g}_c(X^h, T_{\lambda\mu}) = 0,$$
$$\tilde{g}_c(T_{\lambda\mu}, T_{\nu\omega}) = \delta_{\lambda\nu}\delta_{\mu\omega} - \delta_{\lambda\omega}\delta_{\mu\nu}$$

for all $X, Y \in \mathfrak{X}(M)$ and all indices $\lambda, \mu, \nu, \omega = 1, 2, \ldots, n$. The metric $\tilde{g}_c$ is Riemannian if $c > 0$ and pseudo-Riemannian with signature $(n,\, n(n-1)/2)$ if $c < 0$. Now we have at every point $(x,u) \in OM$, a collection

$$\{\frac{1}{\sqrt{|c|}}\, {u_\lambda}^h,\ T_{\lambda\mu}(x,u) \mid 1 \leq \lambda < \mu \leq n\}$$

is a (pseudo-)orthonormal basis for the tangent space $(OM)_{(x,u)}$, and the collection $\{T_{\lambda\mu}(x,u) \mid 1 \leq \lambda < \mu \leq n\}$ forms an orthonormal basis for the vertical subspace $V_{(x,u)} \cap (OM)_{(x,u)}$.

Let $\phi$ be a local isometry of the base manifold $(M,g)$. Then we define a transformation $L\phi$ of $OM$ by

$$(L\phi)(x,u) = (\phi(x), \phi_{*x}u_1, \phi_{*x}u_2, \ldots, \phi_{*x}u_n).$$

Then, by direct calculations, we have

**6 Proposition** ([14])**.** *If $\phi$ is a local isometry of the base manifold $(M, g)$, then, for every constant $c \neq 0$, the metric $\tilde{g}_c$ on $OM$ is invariant by the transformation $L\phi$.*

The orthogonal group $O(n)$ acts on $OM$ on the right by

$$\varphi_a(x, u) = (x, ua) = \Big(x, \sum_{\alpha=1}^{n} a_1^{\alpha} u_{\alpha}, \sum_{\alpha=1}^{n} a_2^{\alpha} u_{\alpha}, \dots, \sum_{\alpha=1}^{n} a_n^{\alpha} u_{\alpha}\Big), \qquad (1)$$

where $a = [a_{\lambda}^{\mu}] \in O(n)$.

**7 Proposition** ([14])**.** *For every constant $c \neq 0$, the metric $\tilde{g}_c$ on $OM$ is invariant by the action* (1) *of the orthogonal group $O(n)$ on $OM$.*

The metric $\tilde{g}_c$, $c \neq 0$, on $OM$ is invariant in the following sense:

**8 Definition** ([14])**.** A pseudo-Riemannian metric $G$ on the orthonormal frame bundle $OM$ over a Riemannian manifold $(M, g)$ is said to be *strongly invariant* if

(1) the metric $G$ is invariant by the transformation $L\phi$ for every local isometry $\phi$ of $(M, g)$,

(2) the metric $G$ is invariant by the action (1) of the orthogonal group $O(n)$.

The following Proposition is obvious:

**9 Proposition** ([14])**.** *Let $(M, g)$ be a homogeneous Riemannian manifold and $(OM, G)$ its orthonormal frame bundle equipped with a strongly invariant pseudo-Riemannian metric $G$. Let $\mathrm{I}(M, g)$ denote the full group of isometries. Then the group $\mathrm{I}(M, g) \times O(n)$ acts transitively on $(OM, G)$ through the actions given in Definition 8. Hence $(OM, G)$ is a homogeneous pseudo-Riemannian manifold.*

We assume in the rest of this section that the base manifold $(M, g)$ is a space of constant sectional curvature $K$. Then, calculating directly the covariant derivatives of the Riemannian curvature tensor field of $\tilde{g}_c$, we have

**10 Theorem** ([14])**.** *Let $(M, g)$ be an $n$-dimensional non-flat space of constant sectional curvature $K$, $n \geq 2$. If $c \neq 2K$, then the orthonormal frame bundle $(OM, \tilde{g}_c)$ is never locally symmetric.*

One can strongly expect that this Theorem should still hold in the general case of the base manifold. But the corresponding proof seems to be rather complicated. In the case that $n = 2$, the metric $\tilde{g}_c$ is locally symmetric for all $c \neq 0$ if $K = 0$ and for $c = 2K$ if $K \neq 0$. In fact, they have a constant sectional curvature:

**11 Theorem** ([14]). *Let $(M, g)$ be a two-dimensional space of constant sectional curvature $K$. Then the orthonormal frame bundle $(OM, \tilde{g}_c)$ is flat if $K = 0$, and a space of constant sectional curvature $1/8$ if $c = 2K \neq 0$. In the latter case, $\tilde{g}_c$ is Riemannian for $K > 0$ and pseudo-Riemannian for $K < 0$.*

We have seen in [14] the sign of the sectional curvatures of $\tilde{g}_c$.

**12 Theorem** ([14]). *Let $(M, g)$ be a space of constant sectional curvature $K$. If $K$ satisfies*

$$0 \leq K \leq \frac{2c}{3}, \quad c > 0,$$

*or*

$$\frac{2c}{3} \leq K \leq 0, \quad c < 0,$$

*then the orthonormal frame bundle $(OM, \tilde{g}_c)$ has nonnegative sectional curvature.*

For the Ricci tensor $\widetilde{\mathrm{Ric}}$ of $\tilde{g}_c$ we have

**13 Proposition** ([14]). *Let $(M, g)$ be an $n$-dimensional space of constant sectional curvature $K$, $n \geq 2$. The Ricci tensor $\widetilde{\mathrm{Ric}}$ of $(OM, \tilde{g}_c)$ is given, at each fixed point $(x, u) \in OM$, by*

$$\widetilde{\mathrm{Ric}}(X^h, Y^h) = \frac{(n-1)K(c-K)}{c} g(X, Y) \circ p,$$

$$\widetilde{\mathrm{Ric}}(X^h, T_{\lambda\mu}) = 0,$$

$$\widetilde{\mathrm{Ric}}(T_{\lambda\mu}, T_{\lambda\mu}) = \frac{K^2}{c^2} + \frac{n-2}{4},$$

$$\widetilde{\mathrm{Ric}}(T_{\lambda\mu}, T_{\nu\omega}) = 0$$

*for all $X, Y \in M_x$ and distinct indices $\lambda, \mu, \nu, \omega = 1, 2, \ldots, n$, $(\lambda, \mu) \neq (\nu, \omega)$.*

Now, solving the equation for the Einstein spaces, we have our main result in [14]:

**14 Theorem** ([14]). *Let $(M, g)$ be an $n$-dimensional space of constant sectional curvature $K$, $n \geq 2$. Then the orthonormal frame bundle $(OM, \tilde{g}_c)$ is an Einstein space if and only if*

*(1)* $n > 2$; $\quad c = 2K \neq 0 \quad$ *or* $\quad c = \dfrac{2nK}{n-2} \neq 0$,

*(2)* $n = 2$; $\quad K = 0 \quad$ *for all* $\quad c \neq 0$, $\quad$ *or* $\quad c = 2K \neq 0$.

*We have*

$$\widetilde{\mathrm{Ric}} = \frac{(n-1)}{4} \tilde{g}_c$$

*for $c = 2K \neq 0$, and*

$$\widetilde{\mathrm{Ric}} = \frac{(n-1)(n^2-4)}{4n^2}\tilde{g}_c$$

*for $c = 2nK/(n-2)$, $n > 2$. The metric $\tilde{g}_c$ is Riemannian for $K > 0$ and pseudo-Riemannian for $K < 0$.*

In the case (2) of Theorem 14, $(OM, \tilde{g}_c)$ is a space of constant sectional curvature. (See Theorem 11.)

**15 Corollary** ([14]). *Let $(M, g)$ be an $n$-dimensional space of constant sectional curvature $K$, $n > 2$. Then there exist at least two strongly invariant Einstein metrics on the orthonormal frame bundle $OM$, at least one of which is not locally symmetric if $K \neq 0$. These metrics are Riemannian for $K > 0$ and pseudo-Riemannian for $K < 0$.*

If $(M, g)$ is a non-flat space of constant sectional curvature, then the Ricci tensor of $\tilde{g}_c$ is never parallel for every nonzero constant $c$ excluded in Theorem 14:

**16 Proposition** ([14]). *Let $(M, g)$ be an $n$-dimensional non-flat space of constant sectional curvature $K$, $n \geq 2$. If $c \neq 2K$ for $n \geq 2$ and $c \neq 2nK/(n-2)$ for $n > 2$, then the Ricci tensor $\widetilde{\mathrm{Ric}}$ of $(OM, \tilde{g}_c)$ is never parallel.*

Finally we have

**17 Theorem** ([14]). *Let $(M, g)$ be an $n$-dimensional space of constant sectional curvature $K$, $n \geq 2$. Then the scalar curvature $\widetilde{\mathrm{Sc}}(\tilde{g}_c)$ of the orthonormal frame bundle $(OM, \tilde{g}_c)$ is zero if and only if*

*(1)* $n > 2; \quad c = \dfrac{(-4 \pm 2\sqrt{n+2}\,)K}{n-2} \neq 0,$

*(2)* $n = 2; \quad K = 0$ *for all* $c \neq 0$, *or* $c = \dfrac{K}{2} \neq 0.$

*In the case (1), one of the metrics is Riemannian and the other is pseudo-Riemannian. In the case (2), the metric is Riemannian for $K > 0$ and pseudo-Riemannian for $K < 0$.*

Consider the standard sphere $S^n = O(n+1)/O(n)$, $n > 2$. The orthogonal group $O(n+1)$ acts simply transitively (by the tangent transformations) on each $(OS^n, \tilde{g}_c)$, $c \neq 0$, as a group of isometries. Indeed, the isotropy group $O(n)$ acts simply transitively on each fiber of $(OS^n, \tilde{g}_c)$. Hence, we can identify each $(OS^n, \tilde{g}_c)$ with $O(n+1)$ endowed with some invariant metric $G_c$. With this identification, we see that the group $O(n+1)$ admits at least two invariant Riemannian Einstein metrics, one of which is not locally symmetric. Also $O(n+1)$ admits one invariant Riemannian metric with zero scalar curvature. The authors do not know if the locally non-symmetric Einstein metric is naturally reductive. See [6, p.62] for an open problem.

**18 Remark.** X. Zou has rescaled in [24] the Sasaki-Mok metric on both horizontal and vertical part, and calculated the sectional curvature, the Ricci curvature and the scalar curvature as a continuation of the work by G. Jensen [7, 8]. Yet, some of the formulas in [24] seem to be incorrect. (See [14] for more details.)

## 4 Geometry viewed from upstairs

We shall back to the diagonal metric $\tilde{g} = \tilde{g}_1$ and view the geometry from the upstairs. The following result is a partial counterpart to Theorem 14.

**19 Theorem** ([13]). *Let $(M, g)$ be an $n$-dimensional curvature homogeneous Riemannian manifold, $n \leq 4$, such that the orthonormal frame bundle $(OM, \tilde{g})$ is Einstein. Then $(M, g)$ has constant sectional curvature.*

The *proof* of Theorem 19 based on the following Lemma.

**20 Lemma** ([13]). *Let $(M, g)$ be an $n$-dimensional Riemannian manifold, $n \geq 2$, such that the orthonormal frame bundle $(OM, \tilde{g})$ is an Einstein manifold. Then*

*(1) $\nabla \operatorname{Ric}$ is a Codazzi tensor.*

*(2) If $n \geq 3$, then $(M, g)$ is irreducible.*

Proof of Lemma 20. If the orthonormal frame bundle $(OM, \tilde{g})$ is Einstein, then we derive $\sum_{\alpha=1}^{n} \langle (\nabla_{u_\alpha} R)_x(u_\lambda, u_\mu)u_\alpha,\, X\rangle = 0$ for each orthonormal basis $(u_1, u_2, \ldots, u_n)$ and each vector $X$ tangent to $M$ at $x \in M$. Now, from the second Bianchi identity applied to the first three arguments we see that $\nabla \operatorname{Ric}$ is a Codazzi tensor.

Next, let $(M, g)$ with $n \geq 3$ be a direct product $M_1 \times M_2$, where $\dim M_1 \geq 2$ and $\dim M_2 \geq 1$. Here $(M, g)$ cannot be flat because otherwise we have a contradiction with the Einstein property. Suppose now $M_1$ not to be flat. Choose an orthonormal frame $(u_1, u_2, \ldots, u_\kappa, u_{\kappa+1}, \ldots, u_n)$ at $x$ such that $u_1, u_2, \ldots, u_\kappa$ are tangent to the $M_1$-direction and $u_{\kappa+1}, \ldots, u_n$ are tangent to the $M_2$-direction. Our calculations give the inequality $\widetilde{\operatorname{Ric}}(T_{12}, T_{12}) > \widetilde{\operatorname{Ric}}(T_{\lambda\mu}, T_{\lambda\mu})$ for $\lambda$ and $\mu$ such that $1 \leq \lambda \leq \kappa < \mu \leq n$, which is a contradiction to the Einstein property. $\boxed{QED}$

Proof of Theorem 19. For $n = 2$ the Theorem is trivial (and the Einstein structure on $(OM, \tilde{g})$ is not needed). Let us have $n = 3$ or $n = 4$. According to [23] and [2], a curvature homogeneous space in dimension 3, or 4, for which

$\nabla$ Ric is a Codazzi tensor, is locally symmetric. According to Lemma 20, this is just our case. Moreover, the space $(M, g)$ must be irreducible.

For dimension $n = 3$ it follows that $(M, g)$ is of constant sectional curvature.

For dimension $n = 4$ all irreducible symmetric spaces are Einstein spaces (see *e.g.* [1]). We choose at $x \in M$ a Singer–Thorpe basis $(e_1, e_2, e_3, e_4)$, *i.e.,* such that the components $R_{\lambda\mu\nu\omega} = R(e_\lambda, e_\mu, e_\nu, e_\omega)$ of the Riemannian curvature $R$ satisfy the condition $R_{1212} = R_{3434}$, $R_{1313} = R_{2424}$, $R_{1414} = R_{2323}$ and $R_{\lambda\mu\nu\omega} = 0$ whenever just three indices are distinct. We get, after some calculations, that $R_{1234} = R_{3124} = R_{2314} = 0$ and $R_{1212} = R_{1313} = R_{1414}$. Hence $(M, g)$ has constant sectional curvature at each point $x$. $\boxed{QED}$

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