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Intrinsic torsion varieties

Georgi Mihaylov
Dipartimento di Matematica, Università di Torino,
Via Carlo Alberto 10, 10123 Torino
g_mihaylov@libero.it

Simon Salamon
Dipartimento di Matematica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino
salamon@calvino.polito.it

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In honour of Oldřich Kowalski

Introduction

At each point of an even-dimensional oriented Riemannian manifold M, the set of compatible complex structures is parametrized by the coset space $Z_n = SO(2n)/U(n)$ that is in fact Hermitian symmetric. The fact that Z_n is a complex manifold is particularly significant. For example, it leads to the definition of twistor spaces [3, 8], and invariant Hermitian structures on a parallelizable manifold are typically described by a *complex* subvariety of Z_n [1].

The fact that Z_n is a symmetric space may be less significant. One can easily conceive of other subgroups H and homogeneous spaces $SO(n)/H$ with which to classify geometrical structures on a Riemannian manifold. However, amongst these, the *coadjoint orbits* appear to have a privileged role, since they are simultaneously complex and symplectic manifolds. The resulting theory retains some features of the almost-Hermitian case, which it is the goal of this essay to generalize when $n = 3$.

In six real dimensions, the coset space Z_3 is isomorphic to complex projective space $\mathbb{CP}^3$, arising from the double covering of Lie groups $SU(4) \to SO(6)$. As a consequence, we consider geometrical structures associated to other coadjoint orbits for $SU(4)$, or equivalently $SO(6)$. In particular, we shall discuss those almost-product structures that arise from the Grassmannians

$$\mathbb{G}\mathrm{r}_2(\mathbb{R}^6) = \frac{SO(6)}{SO(2)\times SO(4)} \cong \frac{U(4)}{U(2)\times U(2)} = \mathbb{G}\mathrm{r}_2(\mathbb{C}^4).$$

We shall also consider 'mixed' structures parametrized by the flag manifold $SO(6)/(U(2)\times SO(2))$, and in essence previously discussed by Blair [6].

One reason for focusing on a coadjoint orbit $\mathscr{O}$ is that such a space can be understood by means of a moment mapping $\tau\colon \mathscr{O}\to\mathbb{R}^3$ induced by the action of the maximal torus T^3 of $SU(4)$ or $SO(6)$. It is well known that the image of τ is a convex polytope, and one of the more symmetric ones is illustrated in Figure 2 in Section 5. A good reference for a study of examples in our situation is [12], and this book also helped to motivate the present study.

This article is based on [19], whose aim is to convert an understanding of the relevant polytopes into various results concerning the underlying differential geometry. One series of indicative results concerns the classification of invariant structures on nilmanifolds by means of varieties representing 'null-torsion' classes. This is explained at the end of both Sections 4 and 5.

1 Riemannian pre-holonomy

In this section, we summarize the preliminary theory that forms the backbone of this article, and motivate the discussion with simple examples.

1.1 Structure groups

Consider a closed subgroup $H\subset SO(n)$. Although we have chosen the letter 'H' to stand for 'holonomy', one can more readily contemplate examples in which there is merely a topological reduction of the structure to H. Algebraically, at each point, such reductions are parametrized by the homogeneous space $R = SO(n)/H$ ('R' for reduction), whose elements are left cosets gH with $g\in SO(n)$.

Now let M be an oriented Riemannian manifold of dimension n. It possesses a principal $SO(n)$-bundle P of oriented orthonormal frames. An H-structure on M is a section s of the associated bundle

$$\begin{array}{c} P/H \\ \big\downarrow R \\ M \end{array} \tag{1}$$

with fibre R. An element of P/H is determined by an equivalence class $[p,H] = [pg, g^{-1}H]$ with $p\in P$ and $g\in SO(n)$; it is unchanged when p is replaced by ph with $h\in H$.

The Levi Civita connection induces a horizontal distribution on this bundle that can be used to interpret geometrically the covariant derivative ∇s. Indeed, its value at a point $m \in M$ can be regarded as a linear map

$$\nabla s : T_m M \to \mathcal{V}_{s(m)},$$

where $\mathcal{V}_{s(m)}$ denotes the tangent space at $s(m)$ to the fibre $\pi^{-1}(m)$. The image of $X \in T_m M$ is denoted $\nabla_X s$ and can be identified with the vertical component of $s_* X$ complementary to the horizontal space $\mathcal{H}_{s(m)}$.

The quantity ∇s represents the *intrinsic torsion tensor* of the H-structure. Fixing a frame $p \in P$ allows us to identify the tangent space $\mathcal{V}_{s(m)}$ with $\mathfrak{h}^\perp$, relative to the orthogonal sum

$$\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp.$$

This in turn enables us to view $(\nabla s)_m$ as an element

$$\xi_m \in \mathbb{R}^n \otimes \mathfrak{h}^\perp.$$

Moving from point to point, the associated tensor ξ measures the failure of the Levi Civita connection to reduce to the H-structure s. This point of view is explained in [22]. A more recent interpretation of conditions defined by the vanishing of components of ξ in terms of 'energy' is given in [10].

1.2 Six dimensions

The focus of this article will be on subgroups H of $SO(6)$. The possible holonomy groups of an (oriented and simply-connected) irreducible Riemannian 6-manifold (excluding $SO(6)$ itself) are

$$U(3), \qquad SU(3), \qquad SO(2)\times SO(3). \tag{2}$$

The subgroup $U(3)$ is the linear holonomy group of the complex projective space

$$\mathbb{CP}^3 = \frac{SU(4)/\mathbb{Z}_4}{U(3)},$$

and also the holonomy group that characterizes a generic complex 3-dimensional Kähler manifold. When the latter has zero Ricci tensor, its holonomy reduces (at least locally) to $SU(3)$ and this is the basis for Calabi-Yau (CY) geometry. In this sense, a real 6-manifold with a $U(3)$ structure is 'pre-Kähler' (traditionally, it is called *almost-Hermitian*), and one with structure $SU(3)$ is 'pre-CY'.

The last subgroup of (2) equals the stabilizer of the Grassmannian

$$\mathbb{Gr}_2(\mathbb{R}^5) = \frac{SO(5)}{SO(2)\times SO(3)}, \tag{3}$$

a symmetric space with the same real homology as (but not homeomorphic to) $\mathbb{CP}^3$. A given tangent space of the manifold (3) can be identified with the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^3$ of the basic representations of the isotropy factors $SO(2), SO(3)$. However, this representation does not appear in Berger's list of irreducible non-symmetric holonomy groups [5], and $SO(2) \times SO(3)$ can only arise as the holonomy of a manifold locally isometric to $\mathbb{Gr}_2(\mathbb{R}^5)$ or its non-compact dual.

For each group H in (2), we are at liberty to consider the homogeneous space $R = SO(6)/H$ featuring in (1). It is remarkable that in the first case, the resulting manifold $SO(6)/U(3)$ can again be identified with $\mathbb{CP}^3$. This follows from the well-known isomorphism $Spin(6) \cong SU(4)$. There is then a double fibration

$$\begin{array}{ccc} \mathbb{RP}^7 = \dfrac{SO(6)}{SU(3)} & & \dfrac{SO(6)}{SO(2)\times SO(3)} \\ & \searrow \quad \swarrow & \\ & \mathbb{CP}^3 = \dfrac{SO(6)}{U(3)} & \end{array} \tag{4}$$

in which the respective fibres have real dimensions 1 and 5.

Of course, the real projective space $\mathbb{RP}^7$ is an S^1-bundle over $\mathbb{CP}^3$; the S^1 (or better, $\mathbb{RP}^1$) parametrizes those real lines contained in a given complex one. The importance of $\mathbb{RP}^7$ in the theory of Riemannian structures derives from the fact that it is also isomorphic to $SO(7)/G_2$ (exploited in [9]). But the latter is an irreducible Riemannian space of the sort classified in [25], whereas $SO(6)/SU(3)$ has *reducible* isotropy. The top right-hand space in (4) also has reducible isotropy; its generic tangent space splits into a 6-dimensional horizontal part duplicating that of $\mathbb{CP}^3$ and a vertical part, tangent to the symmetric space $U(3)/(SO(2)\times SO(3)) \cong SU(3)/SO(3)$.

1.3 An algebraic example

To see that $Z_n = SO(2n)/U(n)$ is a symmetric space, one can proceed as follows.

An element of $\mathfrak{so}(2n)$ is a skew-symmetric matrix. Such a matrix A can be decomposed as

$$A = U + V, \qquad U = \tfrac{1}{2}(A - JAJ), \quad V = \tfrac{1}{2}(A + JAJ),$$

the point being that $JU = UJ$ and $JV = -VJ$. Of these two equations, the former defines $\mathfrak{u}(n)$ as a Lie subalgebra of $\mathfrak{so}(2n)$, and we can write

$$\mathfrak{so}(2n) = \mathfrak{u}(n) \oplus \mathfrak{m}, \tag{5}$$

with $U \in \mathfrak{u}(n)$ and $V \in \mathfrak{m}$.

We need merely observe now that if V_1, V_2 anti-commute with J then their Lie bracket $[V_1, V_2] = V_1V_2 - V_2V_1$ commutes with J. In symbols,

$$V_1, V_2 \in \mathfrak{m} \quad \Rightarrow \quad [V_1, V_2] \in \mathfrak{u}(n),$$

and this is exactly the condition that makes (5) an involutive Lie algebra. For the endomorphism $\sigma\colon \mathfrak{so}(2n) \to \mathfrak{so}(2n)$ acting as $+1$ on $\mathfrak{u}(n)$ and -1 on $\mathfrak{m}$ satisfies

$$[\sigma(A_1), \sigma(A_2)] = [A_1, A_2],$$

and is a Lie algebra automorphism. It integrates to give the involutive symmetry on Z_n.

We remark that if $\Lambda^{1,0} = \mathbb{C}^n$ denotes the standard representation of $U(n)$ then $\mathfrak{m} \cong [\![\Lambda^{2,0}]\!]$, meaning that the complexification of $\mathfrak{m}$ is isomorphic to the second exterior power $\bigwedge^2(\Lambda^{1,0})$ plus complex conjugate. This space (of 2-forms anti-invariant by J) figures largely in Section 4.

2 Coadjoint orbits

Here we summarize some well-known theory that is a meeting point for complex and symplectic geometry. We illustrate it with familiar geometry underlying complex 4-space.

2.1 Symplectic and complex structure

Let G be a compact simple Lie group, and $\mathfrak{g}$ its Lie algebra. The group G acts on $\mathfrak{g}$ via the adjoint representation

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}),$$

and we shall also use Ad to denote the dual action defined by

$$(\mathrm{Ad}(g)(\alpha))(A) = \alpha(\mathrm{Ad}(g^{-1})A), \qquad A \in \mathfrak{g},\ \alpha \in \mathfrak{g}^*.$$

We shall adopt the notation

$$\mathscr{O}_\alpha = \{\mathrm{Ad}(g)(\alpha) : g \in G\}$$

to denote the orbit containing some fixed element $\alpha \in \mathfrak{g}^*$, omitting the subscript α if there is only one orbit under consideration.

Now let $\mathscr{O} = \mathscr{O}_\alpha \subset \mathfrak{g}^*$ denote a coadjoint orbit, and let β be a typical point of $\mathscr{O}$. Any element $A \in \mathfrak{g}$ induces both a function $A\check{}$ on $\mathscr{O}$ (whose value $A\check{}(\beta)$ at β is $\beta(A)$), and a fundamental vector field A^* (whose value is induced by differentiating $\mathrm{Ad}(g(t))\beta$ where $g'(0) = A$). The following result is well known.

1 Proposition. *The prescription $\omega(A^*, B^*) = [A, B]\check{}$ defines a symplectic form on $\mathscr{O}$, called the Kostant-Kirillov-Souriau (KKS) form.*

We can use the inner product defined by (minus) the Killing form to identify $\mathfrak{g}$ and $\mathfrak{g}^*$, and the Lie algebra $\mathfrak{t}$ of a maximal torus T with $\mathfrak{t}^*$. It is well known that any coadjoint orbit intersects $\mathfrak{t}^*$, and we shall see an example of this from first principles in the first part of Section 3. The residual adjoint action preserving $\mathfrak{t}^*$ gives rise to the finite Weyl group $W = N(T)/T$, and so it suffices to take α to lie in a fundamental Weyl chamber (FWC), i.e. a fundamental domain for the action of W on $\mathfrak{t}^*$. Points α in the interior give rise to the 'principal' or generic orbits G/T with stabilizer T, whereas points on the boundary give rise to lower-dimensional orbits.

Now let $G = SU(n)$, and take T to be the subset of diagonal matrices with entries

$$(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n}), \qquad \sum_{i=1}^{n} \theta_i = 0.$$

The above considerations then lead to the fact that any coadjoint orbit for $SU(n)$ is diffeomorphic to one of the complex flag manifolds

$$\frac{SU(n)}{S(U(n_1) \times \cdots \times U(n_k))}, \qquad n = \sum_{i=1}^{k} n_k. \tag{6}$$

We can of course remove the 'S' (the restriction to unit determinant) top and bottom, without changing the spaces.

The manifold (6) represents the set of Hermitian direct sums

$$\mathbb{C}^n = \bigoplus_{i=1}^{k} V_i, \qquad \dim V_i = n_i.$$

There are many ways in which such a splitting defines a flag, for example by selecting

$$\{0\} \subset V_1 \subset V_1 \oplus V_2 \subset V_1 \oplus V_2 \oplus V_3 \subset \cdots \subset \mathbb{C}^n.$$

Making such a choice immediately endows (6) with the structure of a *complex* manifold, but this is not always appropriate because a typical orbit has many inequivalent such structures.

At this juncture, we briefly formulate the Borel-Weil theory that provides a link between coadjoint orbits and representations of a compact group G.

The (isomorphism classes of) complex irreducible representations of G are in one-to-one correspondence with the so-called dominant weights lying in an FWC. Such a weight $\lambda : \mathfrak{t} \to \mathbb{C}$ is effectively a simultaneous eigenvalue for the action of the maximal torus T. For $U(n)$, the dominant weights are n-tuples of the form

$$(a_1,\, a_2,\, \cdots\, , a_n), \qquad a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n.$$

The complex vector space V_λ associated to λ (upon which G acts) contains a eigenvector v_λ with eigenvalue λ.

2 Theorem. *The coadjoint orbit $\mathscr{O}_\lambda$ can be identified with the orbit of $[v_\lambda]$ in the complex projective space $\mathbb{P}(V_\lambda)$, and has the structure of a Kähler manifold.*

The so-called Borel-Weil theorem goes on to asserts that V_λ is isomorphic to the space of holomorphic sections over $\mathscr{O}_\lambda$ of the standard line bundle $\mathcal{O}(1)$ pulled back from $\mathbb{P}(V)$.

2.2 Examples for $SU(4)$

The theory that relates the various coadjoint orbits can be quickly grasped by considering the first case that is not straightforward, namely $n = 4$.

Consider the diagram

$$\begin{array}{ccc} & \dfrac{U(4)}{U(1)\times U(1)\times U(1)\times U(1)} & \\ & \downarrow & \\ & \dfrac{U(4)}{U(2)\times U(1)\times U(1)} & \\ \swarrow & & \searrow \\ \dfrac{U(4)}{U(3)\times U(1)} & & \dfrac{U(4)}{U(2)\times U(2)} \end{array} \tag{7}$$

The top space is the 'full flag manifold' equal to $SU(4)/T^3$, where T^3 is a maximal torus of $SU(4)$. A complex description would be $SL(4,\mathbb{C})/B$, where B is the Borel subgoup of upper triangular matrices. The middle space belongs to the family

$$F_n = \frac{U(n+2)}{U(n)\times U(1)\times U(1)}$$

of flag manifolds studied in [16], itself motivated by an example of distinct complex structures on F_2 [7]. We set $F_2 = \mathscr{F}$. All the coadjoint orbits featuring in (7) have symplectic forms compatible with the partial ordering visible [12]. Moreover, the action of T^3 will provide a moment map of each space to $\mathbb{R}^3$ that we shall study in Section 5.

The spaces on the bottom line of (7) are (left) $\mathbb{CP}^3$ and (right) $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$. The latter can be identified with a complex quadric Q (the so-called *Klein quadric*) in $\mathbb{CP}^5$ as follows. Let $V = \mathbb{C}^4$. A 2-dimensional subspace of V is completely determined by the projective class $[u \wedge v]$ of a nonzero simple (decomposable) 2-form $u \wedge v \in \bigwedge^2 V$. But it is easy to see that an arbitrary element $\sigma \in \bigwedge^2 V$ is simple if and only if $\sigma \wedge \sigma = 0$; this is a non-degenerate quadratic equation in σ and defines the quadric Q in $\mathbb{CP}^5 = \mathbb{P}(\bigwedge^2 V)$. Moreover, one may regard the inclusion $Q \subset \mathbb{P}(\bigwedge^2 V)$ as arising from Borel-Weil theory via the irreducible representation $\bigwedge^2 V$.

The lower triangle of (7) gives rise to the well-known *Klein correspondence* of projective geometry, whereby a point $[\sigma] = [u \wedge v]$ of $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ is associated to the line $\langle u, v\rangle$ in $\mathbb{P}(V) = \mathbb{CP}^3$, and a point $[v]$ of $\mathbb{P}(V)$ determines a (so-called α) plane $\{[v \wedge w] : w \in V\}$ in $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$. A different family of (so-called β) planes in $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ consists of those of the form $\mathbb{P}(\bigwedge^2 U)$ where $\mathbb{P}(U)$ is a projective plane in $\mathbb{P}(V)$, or equivalently a point in the dual space $\mathbb{P}(V)^*$.

To sum up,

3 Lemma. *In the Klein correspondence,*

(i) $Q = \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ *parametrizes* $\mathbb{CP}^1$*'s in* $\mathbb{CP}^3$*,*

(ii) A point $x \in \mathbb{CP}^3$ *determines a plane* $\Pi_\alpha \cong \mathbb{CP}^2$ *in* Q*,*

(iii) A point $y \in (\mathbb{CP}^3)^*$ *determines a plane* $\Pi_\beta \cong \mathbb{CP}^2$ *in* Q*.*

One of our aims is to re-interpret this lemma in terms of structures on $\mathbb{R}^6$, and in turn Riemannian 6-manifolds.

Penrose studied the Klein quadric with $SU(2,2)$ in place of $SU(4)$, and the resulting field theory was generalized to arbitrary flag manifold in [4]. We should also remark that $\mathscr{F}$ is an example of a 3-symmetric space, much used in the theory of harmonic maps [17, 8].

3 Orthogonal geometries in six dimensions

The purpose of this section is to replace the Lie group $SU(4)$ by its finite quotient

$$SU(4)/\mathbb{Z}_2 \cong SO(6), \tag{8}$$

and use this to translate the theory of coadjoint orbits into statements about 2-forms on $\mathbb{R}^6$. This point of view is then perfectly tailored for a discussion of geometrical structures on 6-dimensional Riemannian manifolds.

3.1 Orbits of 2-forms

We shall not need to make the isomorphism (8) explicit, though this can easily be done without resorting to Clifford algebras (see, for example, [1]). It is convenient instead to start from the isomorphism

$$\mathfrak{so}(6) \cong \bigwedge^2(\mathbb{R}^6)^*$$

that identifies (via the metric) skew-symmetric endomorphisms with 2-forms. A coadjoint orbit for $SO(6)$ then becomes an orbit for the natural action of $SO(6)$ on 2-forms. Given $\alpha \in \bigwedge^2(\mathbb{R}^6)^*$, we can use our usual notation

$$\mathscr{O}_\alpha = \{g^*(\alpha) : g \in SO(6)\} \cong \frac{SO(6)}{H}$$

for the orbit containing α. Since we are using forms (rather than vectors) it is customary to regard the group action as a contravariant one (hence the asterisk). The *type* of the orbit is determined by the conjugacy class of the stabilizer H.

Fix an orthonormal basis $(e^1, \dots, e^6)$ of $(\mathbb{R}^6)^*$. We shall indicate the 2-form $e^i \wedge e^j$ by e^{ij}.

4 Lemma. *Any element $\omega \in \bigwedge^2(\mathbb{R}^6)^*$ lies in the $SO(n)$-orbit of*

$$ae^{12} + be^{34} + ce^{56}$$

for suitable $a, b, c \in \mathbb{R}$.

Proof. A parameter count already suggests that this is accurate. The generic orbit $SO(6)$ has 3-dimensional stabilizer T^3, and a, b, c provide the invariants that distinguish the orbits. As for the proof, this is a restatement of a standard diagonalization theorem. Relative to the fixed metric, the form ω defines a skew-symmetric matrix A. Let

$$e^1 + ie^2, \quad e^3 + ie^4, \quad e^5 + ie^6$$

be a unitary basis of eigenvectors of the Hermitian matrix iA (so that the e^k are real vectors belonging to $\mathbb{R}^6$). Then

$$iA(e^1 + ie^2) = \lambda(e^1 + ie^2), \qquad \lambda = \lambda_1 \in \mathbb{R}.$$

Thus $Ae^1 = \lambda e^2$ and $Ae^2 = -\lambda e^1$. In this way, one may identify A with

$$\sum_{k=1}^{3} \lambda_k e^{2k-1} \wedge e^{2k} \in \bigwedge^2(\mathbb{R}^6)^*.$$

We can now apply a suitable element of $SO(6)$ to convert the orthonormal basis (e^k) obtained from the eigenvectors to the one that was pre-assigned. QED

Referring to the lemma, we distinguish four separate cases:
(i) a, b, c are distinct, for example

$$\begin{aligned}\omega_1 &= e^{12} + 2e^{34} + 3e^{56},\\ \omega_2 &= e^{12} + 2e^{34}.\end{aligned}$$

(ii) exactly two of a, b, c coincide up to sign, and at most one is zero, for example

$$\begin{aligned}\omega_3 &= e^{12} + e^{34} + 2e^{56},\\ \omega_4 &= e^{12} + e^{34},\\ \omega_5 &= 2e^{12} + 2e^{34} + e^{56}.\end{aligned} \tag{9}$$

(iii) all three of a, b, c coincide up to sign, for example

$$\begin{aligned}\omega_6 &= e^{12} + e^{34} + e^{56}\\ \omega_7 &= e^{12} - e^{34} - e^{56}.\end{aligned} \tag{10}$$

(iv) two of a, b, c are zero, for example

$$\omega_8 = e^{56}. \tag{11}$$

These four cases give rise to the four types of coadjoint orbits corresponding to (7). Irrespective of the choice of 2-form in each case, the respective stabilizers and orbits are
(i) $H \cong SO(2) \times SO(2) \times SO(2), \quad \mathscr{O} \cong \dfrac{SO(6)}{T^3};$
(ii) $H \cong SO(2) \times U(2), \quad \mathscr{O} \cong \dfrac{SO(6)}{U(2) \times SO(2)};$
(iii) $H \cong U(3), \quad \mathscr{O} \cong \dfrac{SO(6)}{U(3)} = Z_3;$
(iv) $H \cong SO(2) \times SO(4), \quad \mathscr{O} \cong \dfrac{SO(6)}{SO(2) \times SO(4)} \cong \mathbb{G}\mathrm{r}_2(\mathbb{R}^6).$

A comparison with (7) yields the isomorphism

$$Z_3 \cong \mathbb{CP}^3 \tag{12}$$

mentioned in the Introduction, as well as the equivalent descriptions

$$\mathbb{G}\mathrm{r}_2(\mathbb{R}^6) \cong \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$$

of the Klein quadric. An analogous isomorphism

$$\mathbb{G}\mathrm{r}_2(\mathbb{R}^8) = \frac{SO(8)}{SO(2)\times SO(6)} \cong \frac{SO(8)}{U(4)} = Z_4$$

is worth recording, but is not relevant to the present discussion.

Let us justify the stabilizer in (ii), using $\omega_4 = e^{12} + e^{34}$. This determines, in the presence of the fixed metric, a linear transformation $\mathbb{R}^6 \to \mathbb{R}^6$ with kernel $\langle e^5, e^6 \rangle$. But ω_4 also determines a complex structure on the subspace $\langle e^1, e^2, e^3, e^4 \rangle$, and so an orientation on $\langle e^5, e^6 \rangle$. Hence ω_4 determines simple forms e^{56} and e^{1234}, the stabilizer is $SO(2) \times U(2)$ and the resulting coadjoint orbit has real dimension 10.

We can now replace (7) by the following diagram, which is more friendly for the Riemannian geometer working in six dimensions. After displaying that, we discuss the geometrical structures associated to these spaces.

$$\begin{array}{ccccc} & & \dfrac{SO(6)}{T^3} & & \\ & & \downarrow & & \\ & & \dfrac{SO(6)}{SO(2)\times U(2)} & & \\ & \swarrow & & \searrow & \\ \dfrac{SO(6)}{U(3)} & & & & \dfrac{SO(6)}{SO(2)\times SO(4))} \end{array} \tag{13}$$

3.2 Almost-Hermitian and almost-product structures

Consider first the two spaces in the bottom line of (13). We know that (12) parametrizes complex structures J on $\mathbb{R}^6$, orthogonal relative to the standard inner product or metric g . Thus $J^2 = -1$ and

$$g(Ju, Jv) = g(u, v), \qquad u, v \in \mathbb{R}^6. \tag{14}$$

We shall call such a J an *orthogonal complex structure* (OCS), noting that in the present context this is a purely algebraic concept. We shall address questions of integrability later.

From a certain point of view, it is simpler to begin with the Grassmannian $\mathbb{G}\mathrm{r}_2(\mathbb{R}^4)$ that parametrizes oriented 2-dimensional subspaces of $\mathbb{R}^6$. Denote such a subspace by $\mathcal{V}$ and its orthogonal complement by $\mathcal{H} = \mathcal{V}^\perp$. Then $\mathbb{G}\mathrm{r}_2(\mathbb{R}^4)$ effectively parametrizes orthogonal direct sums

$$\mathbb{R}^6 = \mathcal{V} \oplus \mathcal{H}, \qquad \dim \mathcal{V} = 2, \quad \dim \mathcal{H} = 4. \tag{15}$$

We can characterise this splitting by means of the endomorphism P that acts as 1 on $\mathcal{V}$ and as -1 on $\mathcal{H}$. Then $P^2 = 1$ and (14) is valid with P in place of J. We shall call the endomorphism P an *orthogonal product structure* (OPS), though we have one extra input, namely each summand of (15) is oriented. The choice of notation reflects the fact that (soon) the two summands will be regarded as vertical and horizontal spaces respectively.

In practice, we prefer to specify our oriented OPS by means of the simple form

$$\alpha = e \wedge f,$$

formed from an oriented orthonormal basis (e, f) of the defining subspace $\mathcal{V}$. Note that changing the sign of α changes the orientation on $\mathcal{V}$ and so the $SO(2) \times SO(4)$ structure.

3.3 The intermediate case

We now move upwards in (13) to consider the coadjoint orbit in (ii) that we shall continue to denote by $\mathscr{F}$. A point of $\mathscr{F}$ determines both an OCS and an OPS by means of the projections in (13). The resulting endomorphisms commute in the sense that $JP = PJ$, which amounts to saying that J preserves the splitting (and so the summands in) (15). The restriction of J to $\mathcal{H}$ reduces the associated structure group $SO(4)$ to the unitary subgroup $U(2)$.

For the purpose of this essay, let us say that a *mixed structure* on a smooth 6-manifold N is a reduction of structure group from $GL(6,\mathbb{R})$ to $SO(2)\times U(2)$. If we fix the Riemannian metric and overall orientation of N from the outset, a mixed structure is determined by a section s of (1) in which $R = \mathscr{F}$, or by a rank 4 distribution $\mathcal{H}$ equipped with an almost-complex structure. We can then obtain a pair (J, P) and write

$$\mathcal{V} = [\![\mathcal{V}^{1,0}]\!], \quad \mathcal{H} = [\![\mathcal{H}^{1,0}]\!]$$

in standard notation [23]. A mixed structure is closely related to the type of structures studied by Blair in [6].

We are now in a position to reinterpret the Klein correspondence.

One expects a point of $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$, i.e. a splitting (15), to define a $\mathbb{CP}^1$ family of OCS's on $\mathbb{R}^6$. An element J in this family is constructed as follows. The induced metric and orientation of $\mathcal{V}$ determines the restriction $J|\mathcal{V}$, but a choice is needed to extend J to the oriented vector space $\mathcal{H} \cong \mathbb{R}^4$. An OCS on the latter is an element of the space

$$Z_2 \cong \frac{SO(4)}{U(2)} \cong \mathbb{CP}^1,$$

that can also be identified with the 2-sphere

$$S^2 \subset \Lambda^2_+\mathcal{H}$$

of unit self-dual 2-forms on H. Having made the choice, the resulting pair (J, P) is effectively an element of $\mathscr{F}$.

Working backwards, a point $J \in Z_3$ determines a complex projective plane consisting of all complex lines in the complex 3-dimensional vector space $(\mathbb{R}^6, J)$. Given any such complex line, we let $\mathcal{V}$ denote the underlying oriented real 2-dimensional subspace and $\mathcal{H}$ its orthogonal complement; in this way we recover (15). But actually we could have also chosen the space $\overline{\mathcal{V}}$, namely $\mathcal{V}$ with its opposite orientation, and this procedure defines a *different* plane $\mathbb{CP}^2$.

To sum up,

5 Lemma. *Under the Klein correspondence,*

(i) Given an oriented splitting (15), there is a 2-sphere of compatible OCS's, parametrized by $S^2 \subset \Lambda^2_+\mathcal{H}$,

(ii) Given an OCS J*, we have the* J*-invariant 2-planes* $\langle v, Jv\rangle$,

(iii) Given J*, we also have the oppositely-oriented 2-planes* $\langle v, -Jv\rangle$.

In Section 4, we shall see how these correspondences translate into ones between intrinsic torsion tensors.

3.4 Integrability

Having defined various Lie subgroups of $SO(6)$ and orbits $SO(6)/H$, it is easy to cook up corresponding 6-manifolds endowed with such H-structures. The case of almost-Hermitian manifolds and metric almost-product manifolds is well understood. In preparation for the next section, we shall instead say something about the mixed structures à la Blair defined by the intermediate flag manifold $\mathscr{F}$.

To show that the concept of a mixed structure is useful, and to motivate the next section, we propose the

6 Definition. Let us say that a mixed structure on a 6-manifold N is *doubly integrable* if (N, J) is a complex manifold and the rank 2 'vertical' distribution $\mathcal{V}$ is integrable.

Equivalently, (by the Newlander-Nirenberg theorem) the Nijenhius tensor of J is everywhere zero, and (by the Frobenius theorem) $[X, Y] \in \mathcal{V}$ whenever X, Y are themselves sections of $\mathcal{V}$. The local model is then a fibration

$$\begin{array}{c} N \\ \big\downarrow \pi \\ M \end{array} \tag{16}$$

with a 4-dimensional base manifold M, which may or may not be a complex manifold (whereas N is complex, by hypothesis).

This situation captures two very different classes of examples:
(i) N is a *holomorphic* bundle over a complex surface, including elliptic fibrations (fibre T^2) of importance in deformation theory. A key example is the Iwasawa manifold $N = H_c/\Gamma$ over $M = T^4$ as base, H_c being the complex Heisenberg group.
(ii) M^4 has an anti-self-dual conformal structure (so the half Weyl$_+$ of the Weyl tensor vanishes), and $N \subset \Lambda^2_+ T^*M$ is its twistor space [3]. Each S^2 fibre is a rational curve with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$. There are two model examples:

$$\begin{array}{ccc} \mathbb{CP}^3 = \dfrac{SO(5)}{U(2)} & \qquad & \mathbb{F} = \dfrac{SU(3)}{T^2} \\ \downarrow & & \downarrow \\ S^4 = \dfrac{SO(5)}{SO(4)} & & \mathbb{CP}^2. \end{array}$$

We can state two important theorems relating to these two classes. The following stability result was proved in [14] and generalized in [21].

7 Theorem. *Any invariant complex structure $\mathbb{J}$ on the Iwasawa manifold N arises from one, say J, on T^4 and an induced complex structure j on the T^2 fibre that is determined by J.*

This and similar examples typically possess *bi*hermitian structures [2].

Concerning (ii), the following theorem was proved by Hitchin [13].

8 Theorem. *Any Kähler twistor space is either $\mathbb{CP}^3$ or $\mathbb{F}$.*

This result has many generalizations based on the characterization of different types of twistor spaces.

4 Intrinsic torsion

Let H be one of the four isotropy groups considered in (13). In this section, we give an overview of the classification of Riemannian 6-manifolds with an H-structure by means of intrinsic torsion. This leads to the established tactic of classifying H-structures for which one or more of the irreducible components of ξ vanish. The classification of such 'null-torsion' structures on a fixed parallelizable manifold is described by subsets of $R = SO(6)/H$, which are the intended varieties of the essay's title.

4.1 Mixed torsion

To start the ball rolling, we shall explain how to compute the intrinsic torsion for the subgroup $H = SO(2) \times SO(4)$. Relative to this, and (15), we have

$$\begin{aligned}\mathfrak{so}(6) \cong \bigwedge^2(\mathcal{V} \oplus \mathcal{H}) &= \bigwedge^2\mathcal{V} \oplus (\mathcal{V} \wedge \mathcal{H}) \oplus \bigwedge^2\mathcal{H} \\ &\cong \mathfrak{so}(2) \oplus \mathfrak{so}(4) \oplus (\mathcal{V} \otimes \mathcal{H}).\end{aligned}$$

As expected, the last summand can be identified with the tangent space of the coadjoint orbit $R = \mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$.

The intrinsic torsion tensor ξ lies in

$$\begin{aligned}\mathbb{R}^6 \otimes \mathfrak{h}^\perp &\cong (\mathcal{V} \oplus \mathcal{H})\,\mathcal{V}\mathcal{H} \\ &\cong \mathcal{V}\mathcal{V}\mathcal{H} \oplus \mathcal{H}\mathcal{H}\mathcal{V} \\ &\cong \mathcal{V} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}\,S^2_o\mathcal{V} \oplus \mathcal{V}\,S^2_o\mathcal{H} \oplus \mathcal{V}\Lambda^2_+\mathcal{H} \oplus \mathcal{V}\Lambda^2_-\mathcal{H},\end{aligned} \tag{17}$$

omitting some tensor product symbols. We have seven irreducible components, of respective dimensions $2, 4, 4, 8, 18, 6, 6,$ and two are isotypic. This decomposition was essentially carried out by Naveira in his classification of $O(p)\times O(q)$ structures [20]. He lists a total of 36 classes for a general pair (p, q).

The situation for $U(3)$ structures is perhaps better known, and produces the celebrated 16 classes in the Gray-Hervella classification of almost-Hermitian structures [11]. For this case, $\mathfrak{h}^\perp$ is isomorphic to the space $\mathfrak{m}$ mentioned at the end of Section 1, and we obtain the well-known isomorphism

$$\begin{aligned}\mathbb{R}^6 \otimes \mathfrak{h}^\perp &\cong [\![\Lambda^{1,0}]\!] \otimes [\![\Lambda^{2,0}]\!] \\ &\cong [\![\Lambda^{1,0}\otimes\Lambda^{1,0}]\!] \oplus [\![\Lambda^{1,0}\otimes\Lambda^{0,2}]\!] \\ &\cong \mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4,\end{aligned} \tag{18}$$

in which the intrinsic torsion space has four irreducible components (of respective real dimensions $2, 16, 12, 6$).

One can carry out the same operation for each stabilizing group H in (13), and come up with the table in which the third column represents the decomposition of the tangent space of our 6-manifold, and '#' stands for the number of H-irreducible summands in $\mathbb{R}^6 \otimes \mathfrak{h}^\perp$.

H	$\dim R$	$\mathbb{R}^6$	#
$U(3)$	6	$[\![\Lambda^{1,0}]\!]$	4
$SO(2)\times SO(4)$	8	$[\![\mathcal{V}^{1,0}]\!]\oplus\mathcal{H}$	7
$SO(2)\times U(2)$	10	$[\![\mathcal{V}^{1,0}]\!]\oplus[\![\mathcal{H}^{1,0}]\!]$	16
T^3	12	$[\![\mathcal{V}_1^{1,0}]\!]\oplus[\![\mathcal{V}_2^{1,0}]\!]\oplus[\![\mathcal{V}_3^{1,0}]\!]$	36

This uniform approach allows one to interrelate intrinsic torsions for the various structures, and interpret the results geometrically. In particular,

9 Proposition. *The intrinsic torsion tensor of a mixed structure is determined by (knowledge of both) the intrinsic torsion tensors of the associated OCS and OPS.*

Proof. The idea is very simple. The mixed structure is completely determined by the endomorphism $J+P$, acting with eigenvalues $i+1$, $i-1$, $-i+1$, $-i-1$ on the complexified tangent space. Its intrinsic torsion is then computed as $\nabla(J+P)=\nabla J+\nabla P$, and the result follows. *QED*

We can understand the distribution of the various torsion components in terms of the double fibration in (13). Each of the three intrinsic torsion tensors in question lies in the space $\mathbb{R}^6\otimes\mathfrak{h}^\perp$, where $\mathfrak{h}$ is to all intents and purposes the tangent space to (respectively) $\mathbb{CP}^3$, $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$, $\mathscr{F}$. But the double fibration in (13) tells us that, at a mixed structure $(J,P)\in\mathscr{F}$, we have

$$T_{(J,P)}\mathscr{F}=T_J\mathbb{CP}^3+T_P\mathbb{G}\mathrm{r}_2(\mathbb{R}^6),$$

whilst the real 4-dimensional space

$$T_J\mathbb{CP}^3\cap T_P\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)\cong[\![\mathcal{V}^{1,0}\otimes\mathcal{H}^{1,0}]\!]$$

reflects a redundancy in the joint torsion tensors.

4.2 Tensors on the Iwasawa manifold

Following the convention of [1], we may choose a left-invariant basis (e_i) of real vector fields on the complex Heisenberg group H_c so that the dual basis (e^i) of 1-forms satisfies

$$de^i=\begin{cases}0 & i=1,2,3,4,\\ e^{13}+e^{42} & i=5,\\ e^{14}+e^{23} & i=6.\end{cases}\tag{19}$$

These forms pass to the Iwasawa manifold $N = H_c/\Gamma$. The standard bi-invariant complex structure J_0 on H_c or N is given by

$$J_0 e_1 = e_2, \quad J_0 e_3 = e_4, \quad J_0 e_5 = e_6,$$

and we adopt the convention that the dual action is given by $J_0 e^1 = -e^2$ etc.

Extend the action of J_0 to 2-forms by setting

$$J_0(e^{ij}) = (J_0 e^i) \wedge (J_0 e^j),$$

regarding it as an automorphism rather than an endomorphism. The fact that

$$d(e^5 + ie^6) = de^5 + ide^6 = (e^1 + ie^2) \wedge (e^3 + ie^4)$$

has type $(2,0)$ with respect to J_0 tells us that de^5 and de^6 lie in the -1 eigenspace of J_0. Moreover, from (19), we see that

$$d(J_0\alpha) = d\alpha, \quad \text{for all } \alpha \in \bigwedge^2(\mathbb{R}^6)^*; \tag{20}$$

for example $d(J_0 e^{15}) = d(e^{26}) = e^{124} = de^{15}$, whilst $J_0 e^{56} = e^{56}$.

We next fix the Riemannian metric for which (e_i) forms an orthonormal basis; justification for this choice can be found in the work of Lauret [18]. Fixing too the orientation, we have a global action of $SO(6)$. Let T^3 denote the maximal torus of $SO(6)$ that preserves the splitting

$$\mathbb{R}^6 = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle \oplus \langle e_5, e_6 \rangle, \tag{21}$$

consistent with the action of J_0.

According to the fundamental theorem of Riemannian geometry, the associated Levi Civita connection is completely determined by the tensor

$$\begin{aligned} \sum_{i=1}^{6} e_i \otimes de^i &= e_5 \otimes (e^{13} + e^{42}) + e_6 \otimes (e^{14} + e^{23}) \\ &= \mathfrak{Re}\left[(e_5 - ie_6) \otimes (e^1 + ie^2) \wedge (e^3 + ie^4)\right]. \end{aligned} \tag{22}$$

This is the real part of an eigenvector for T^3 whereby $(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ acts as $e^{i(\theta_1+\theta_2+\theta_3)}$. In particular it is unchanged with respect to a standard maximal torus T^2 in $SU(3) \subset SO(6)$. This fact is used in the proof of the

10 Theorem. *Each of the 16 Gray-Hervella subsets of $\mathbb{CP}^3$ is invariant by the action of T^3 induced by (21).*

Proof. The remark above already shows us that the torsion is invariant by a suitable 2-torus. What the theorem asserts is that the *vanishing* of any subset of the four classes (18) is also invariant by the remaining circle action. The latter is generated by the complex structure J_0 that acts on (21) as $(e^{i\pi/2}, e^{i\pi/2}, e^{i\pi/2})$, and separate arguments can be used to show that the vanishing of each class is invariant by the induced action of J_0 on 2-forms. Although a case-by-case account of these arguments can be extracted from [1], they all boil down to the single relation (20). QED

The most natural choice of 2-form on the Iwasawa manifold is

$$\alpha = \omega_8 = e^{56}$$

(see (11)). Its stabilizer in $SO(6)$ is obviously $SO(4) \times SO(4)$, and defines an almost product structure whose 2-dimensional distribution is tangent to the fibres of (16) with $M = T^4$. We shall denote the resulting almost product structure by P_0.

We can test the theory by computing $\nabla\alpha$, that represents the intrinsic torson of the associated OPS. Since e^5 is a 1-form dual to a Killing vector field, its covariant derivative ∇e^5 is totally skew and determined by de^5. It follows that

$$\nabla e^5 = e^1e^3 - e^3e^1 + e^4e^2 - e^2e^4,$$

in which juxtaposition means tensor product. Similarly for ∇e^6, and so

$$\nabla\alpha = (e^1e^3 - e^3e^1 + e^4e^2 - e^2e^4)e^6 + \dot{e}^5(e^1e^4 - e^4e^1 + e^2e^3 - e^3e^2),$$

in which the dot indicates that e^5 needs to be moved into the 'middle'. Strictly speaking, the last two factors then define an element of $\mathcal{V} \wedge \mathcal{H}$, but this is isomorphic to both $\mathcal{V} \otimes \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{V}$, so we can put $\mathcal{V} = \langle e^5, e^6 \rangle$ last with a sign change. Thus

$$\nabla\alpha = (e^{13} + e^{42}) \otimes e^6 - (e^{14} + e^{23}) \otimes e^5.$$

This evidently belongs to the subspace $\mathcal{V}\Lambda^2_+\mathcal{H}$ in (17), and $J_0(\nabla\alpha)$ can be identified with the tensor (22).

Observe that

$$\begin{aligned} \textstyle\bigwedge^3 T^* \;=\; \bigwedge^3(\mathcal{V} \oplus \mathcal{H}) \;&\cong\; \textstyle\bigwedge^2\mathcal{V}\,\mathcal{H} \oplus \mathcal{V}\bigwedge^2\mathcal{H} \oplus \bigwedge^3\mathcal{H} \\ &\cong\; \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{V}\,\Lambda^2_+\mathcal{H} \oplus \mathcal{V}\,\Lambda^2_-\mathcal{H}. \end{aligned} \tag{23}$$

In the above example, the 'complementary' simple 4-form $\beta = e^{1234}$ is closed. For a general OPS,

$$d\beta \;\in\; \textstyle\bigwedge^5 T^* \;\cong\; \mathcal{V} \oplus \mathcal{H},$$

so that between them, $d\alpha$ and $d\beta$ determine *five* of the seven irreducible components of intrinsic torsion. In our example, $d\beta = 0$ confirms that there is no $\mathcal{V}$ component, though ∇ is needed to be certain that there are no components in $\mathcal{V}\,S^2_0\mathcal{H}$ or $S^2_0\mathcal{V}\,\mathcal{H}$.

We can refine the structure determined by α or β by throwing in the standard complex structure J_0 that preserves the rank 2 and rank 4 distributions. The resulting mixed structure (J_0, P_0) on N is determined by the 2-form $e^{12} + e^{34}$. It is well known that the almost-Hermitian manifold (N, g, J_0) belongs to the class $\mathscr{W}_3$, so the previous proposition yields the

11 Corollary. *The intrinsic torsion of the mixed structure* (J_0, P_0) *belongs to the space* $\mathscr{W}_3 \cap \mathcal{V}\Lambda^2_+\mathcal{H}$.

This result is significant because it can be shown that the space in question is a single irreducible component for $SO(2) \times U(2)$ of real dimension 2.

5 Moment mappings

Let $\mathscr{O}$ be a coadjoint orbit for $SO(6)$. It is inherent in the KKS construction mentioned in Section 2 that the inclusion

$$\mu : \mathscr{O} \to \mathfrak{so}(6)^*$$

can be identified with the moment mapping defined by the induced symplectic form on $\mathscr{O}$.

Fix an orthonormal basis $(e_1, \ldots, e_6)$ of $\mathbb{R}^6$, and let (e^i) be the dual basis of $(\mathbb{R}^6)^*$. Adopting the notation of (21), we can identify the natural surjection from $\mathfrak{so}(6)$ to the Lie algebra of T^3 with the orthogonal projection

$$\pi : \textstyle\bigwedge^2(\mathbb{R}^6)^* \longrightarrow \mathbb{R}^3 = \langle e^{12}, e^{34}, e^{56} \rangle. \tag{24}$$

It follows that $\tau = \pi \circ \mu$ is the moment mapping for the action of T^3 on the orbit $\mathscr{O}$. We shall investigate it in more detail.

5.1 Examples

It is well known that if we consider one of the 'minimal' orbits isomorphic to $\mathbb{CP}^3$, the image of τ is a solid tetrahedron in $\mathbb{R}^3$. To clarify this, consider the orbit of the 2-form ω_6 in (10). There are four points in the *intersection* of the orbit with $\mathbb{R}^3$, namely

$$\pm e^{12} \pm e^{34} \pm e^{56}, \tag{25}$$

with the product of the signs positive (to preserve the orientation of the associated OCS). These are the four vertices of the tetrahedron.

This interpretation underlies the use of such a tetrahedron in [1] in describing the torsion of certain nilmanifolds. For each $\mathbf{v} \in \mathrm{Im}(\tau)$,

$$\tau^{-1}(\mathbf{v}) \cong T^k,$$

where k is one of $0, 1, 2, 3$. Vertices, edges and faces correspond to $k = 0, 1, 2$ respectively. In all cases, $\tau^{-1}(\mathbf{v})$ is a single T^3-orbit.

Next consider the Grassmannian $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$, and the orbit $\mathscr{O}$ containing ω_8 in (11). This equals the subset

$$\{e \wedge f : \|e\| = 1 = \|f\|,\ e \cdot f = 0\} \subset \bigwedge^2(\mathbb{R}^6)^*.$$

The question is to find its image after composing the moment mapping with (24). This time, there are six points in the intersection $\mathcal{O} \cap \mathbb{R}^3$, namely $\pm e^{12}$, $\pm e^{34}$, $\pm e^{56}$, and these are the vertices in the following description.

12 Proposition. *The image $\tau(\mathscr{O})$ is a solid octahedron O.*

PROOF. If we take

$$e = \sum a_i e^i, \quad f = \sum b_i e^i, \qquad \sum a_i^2 = 1 = \sum b_i^2, \quad \sum a_i b_i = 0,$$

then $\pi(e \wedge f) = (x, y, z)$, where

$$x = a_1 b_2 - a_2 b_1, \quad y = a_3 b_4 - a_4 b_3, \quad z = a_5 b_6 - a_6 b_5.$$

The Cauchy-Schwartz inequality implies that

$$|x| + |y| + |z| \leqslant 1, \tag{26}$$

which defines O. It is easy to check that each point satisfying (26) is realized by the above construction. QED

Equality in (26) occurs if and only if $f = Je$ where J is one of the eight almost complex structures associated to a 2-form in (25) (without the previous sign restriction). It follows that the external triangular faces of the octahedron represent $\mathbb{CP}^2$'s that parametrize J-invariant 2-planes in $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$.

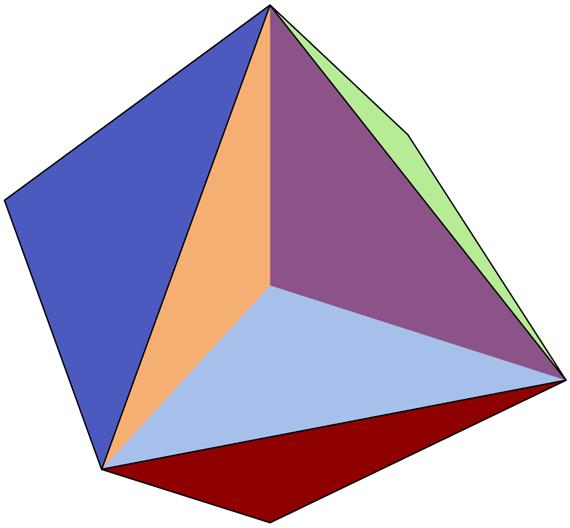

Figure 1: The image of $\tau(\mathscr{O})$ with $e^{56} \in \mathscr{O}$

One can also identify squares, like that arising from the 2-torus

$$\{(e^1 \cos\theta + e^3 \sin\theta) \wedge (e^2 \cos\phi + e^4 \sin\phi) : \theta, \phi \in [0, 2\pi)\},$$

whose image in $\mathbb{R}^3$ is

$$\Diamond = \left\{ xe^{12} + ye^{34} \;:\; x = \cos\theta\cos\phi,\; y = \sin\theta\sin\phi \right\}. \tag{27}$$

Note that $x + y = \cos(\theta - \phi) \leqslant 1$, and that the boundary $x + y = 1$ is achieved by setting $\theta = \phi$. Modifying this observation with other signs shows that $\Diamond$ is a square or diamond in the coordinate plane $\mathbb{R}^2 = \left\langle e^{12}, e^{34} \right\rangle$. Like the external faces, this plane represents images of points where τ is singular. If we consider the rank 4 distribution

$$\mathcal{H}_0 = \langle e_1, e_2, e_3, e_4 \rangle \tag{28}$$

defined by P_0, then $\Diamond$ is also the image of the complex submanifold

$$\mathbb{G}\mathrm{r}_2(\mathbb{R}^4) \cong \mathbb{CP}^1 \times \mathbb{CP}^1 \cong S^2 \times S^2 \subset \Lambda^2_+\mathcal{H}_0 \oplus \Lambda^2_-\mathcal{H}_0, \tag{29}$$

of $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$ arising from the moment mapping for $SO(4)$.

Next consider $\mathscr{F}$. In this case, the image of τ depends crucially on the orbit chosen. If we use the form ω_4 in (9), the image is a symmetrically-truncated cube, as illustrated in Figure 2. However, ω_3 and ω_5 gives rise to two inequivalent polytopes [19]. These polytopes, and the image of the full flag manifold are all illustrated in [12].

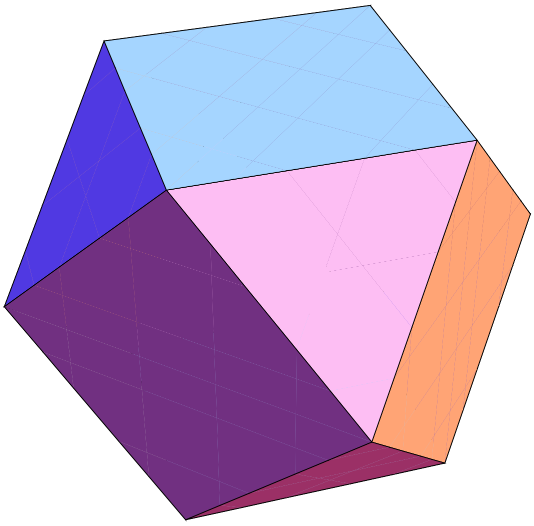

Figure 2: The image of $\tau(\mathscr{O})$ with $e^{12}+e^{34}\in\mathscr{O}$

5.2 Symplectic reduction

We now return to the orbit $\mathscr{O}$ containing e^{56}, isomorphic to the Grassmannian $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$. Since this has real dimension 8, it is no longer possible for T^3 to act transitively on a generic inverse image $\tau^{-1}(\mathbf{v})$ with $\mathbf{v}=(v_1,v_2,v_3)\in O$. Indeed, this observation forces one to consider the *symplectic quotient*

$$\frac{\tau^{-1}(\mathbf{v})}{T^3}. \tag{30}$$

Provided $\mathbf{v}$ is a regular value of τ, we can assert that $\tau^{-1}(\mathbf{v})$ is a smooth manifold of dimension $8-3=5$. Assuming that T^3 acts freely on this inverse image, (30) will be a symplectic manifold of real dimension $5-3=2$.

General methods relate symplectic quotients to Geometric Invariant Theory and can be used to compute their cohomology [15]. In our simple case, we appeal to methods outlined by R. Thomas [24].

13 Theorem. *Given a generic point* $\mathbf{v}$ *in the octahedron* $\tau(\mathbb{G}\mathrm{r}_2(\mathbb{R}^6))$*, the quotient (30) is homeomorphic to a 2-sphere.*

PROOF. An element of $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6) \cong \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ can be regarded as a complex 2-dimensional subspace of $\mathbb{C}^4$ and so the null space of a 2×4 matrix $A \in \mathbb{C}^{2,4}$ of rank 2. Two such matrices A, B have the same null space if and only if $GA = B$ for some $G \in GL(2,\mathbb{C})$. Scalar multiplication by $\mathbb{C}^*$ extends to an action of the group $(\mathbb{C}^*)^4$ by rescaling the individual columns of A. By the Kempf-Ness theorem, the GIT quotient

$$\frac{\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)}{(\mathbb{C}^*)^4} \tag{31}$$

can be identified with (30) provided $\mathbf{v}$ is generic.

To compute (31), we ignore matrices A with one or two zero columns, as these have lower-dimensional orbits. The GIT quotient is now represented by the set of projectivized columns $[A_1], \ldots, [A_4] \in \mathbb{CP}^1$ modulo $SL(2,\mathbb{C})$. This is precisely the set of four distinct points in $\mathbb{CP}^1$ modulo the action of the projective group. Such elements are faithfully parametrized by the cross-ratio, and our quotient can therefore be identified with $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$. QED

In conclusion, $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)/T^3$ is formed from an S^2-bundle over O with degenerations on faces, edges and vertices, although we shall see in the remainder of this subsection that an explicit presentation is far from straightforward.

Whilst τ is the composition of the moment mapping and a *linear* map on $\mathfrak{so}(6)$, we are at liberty to consider quadratic invariants under the action of T^3. We shall exploit these to parametrize the set (30) of T^3 orbits in $\tau^{-1}(\mathbf{v})$ up to finite ambiguity.

The decomposition of the second symmetric power of $\mathfrak{so}(6)^* \cong \bigwedge^2(\mathbb{R}^6)^*$ is known from the theory of curvature tensors R_{ijkl} in Riemannian geometry. There are two contractions

$$\begin{aligned} \mathfrak{b}: &\quad S^2(\textstyle\bigwedge^2(\mathbb{R}^6)^*) \longrightarrow \textstyle\bigwedge^4(\mathbb{R}^6)^* \\ \mathfrak{r}: &\quad S^2(\textstyle\bigwedge^2(\mathbb{R}^6)^*) \longrightarrow S^2(\mathbb{R}^6)^*. \end{aligned}$$

The first is defined by wedging and (if we already take for granted the symmetry $R_{ijkl} = R_{klij}$) its kernel defines those tensors satisfying the first Bianchi identity $R_{ijkl} + R_{iklj} + R_{iljk} = 0$. The second, when applied to R_{ijkl}, defines the Ricci tensor

$$R_{ik} = \sum_{j,l} g^{jl} R_{ijkl},$$

whose trace is the scalar curvature.

If we consider a 2-form in the $SO(6)$-orbit of ω_6 in (10), then $\mathfrak{b}(\alpha \wedge \alpha)$ can be identified with the image $*\alpha$ under the Hodge star operator. The latter is independent of the choice of α, and gives us an $SO(6)$ isomorphism

$$\bigwedge^4(\mathbb{R}^6)^* \cong \bigwedge^2(\mathbb{R}^6)^*.$$

Therefore, use of $\mathfrak{b}$ will only duplicate the moment mapping τ, and not give us anything new. What is worse, in the case of the Grassmannian, α is a simple 2-form and $\mathfrak{b}(\alpha\wedge\alpha)=0$. Accordingly, we turn our attention to $\mathfrak{r}$.

Now

$$S^2(\mathbb{R}^6)^* \supset S^2\langle e^1,e^2\rangle \oplus S^2\langle e^3,e^4\rangle \oplus S^2\langle e^5,e^6\rangle,$$

and each of the right-hand summands contains a 1-dimensional subspace invariant by T^3. Putting them together gives a T^3-equivariant linear mapping

$$\sigma : S^2(\bigwedge^2(\mathbb{R}^6)^*) \longrightarrow \mathbb{R}^3,$$

that refines the induced inner product. If we set

$$\sigma(\alpha\otimes\alpha) = (s_1,s_2,s_3), \tag{32}$$

then s_1 (for example) represents the Hermitian norm squared of the projection of the isotropic vector $e+if$ to the 2-dimensional subspace $\langle e^1,e^2\rangle$. With appropriate constants in the definition of $\mathfrak{r}$, we can assert that the orbit $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$ is mapped onto a filled triangle

$$\{(s_1,s_2,s_3)\in\mathbb{R}^3 : s_1+s_2+s_3=2,\ s_i\geqslant 0\}. \tag{33}$$

We shall make this mapping explicit in the following proof.

14 Theorem. *Let $\mathscr{O}=\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$ be the $SO(6)$-orbit of e^{56}. Let $\mathbf{u},\mathbf{v}$ be generic points of $\sigma(\mathscr{O})$ and $\tau(\mathscr{O})$ respectively. Then $\sigma^{-1}(\mathbf{u})\cap\tau^{-1}(\mathbf{v})$ consists of at most finitely many T^3 orbits.*

Proof. Let $e\wedge f\in\mathscr{O}$ with $\|e\|=1=\|f\|$ and $\langle e,f\rangle=0$. We are only interested in the T^3 orbit containing $e\wedge f$. Using the T^3 action, we may suppose that

$$e=(a_1,0,a_2,0,a_3,0),\qquad a_i>0,\quad \sum a_i^2=1,$$

and that

$$f=(b_1,c_1,b_2,c_2,b_3,c_3),\qquad \sum(b_i^2+c_i^2)=1,\quad \sum a_ib_i=0.$$

To some extent, the orbit is parametrized by the coefficients a_i,c_j,b_k, though we were at liberty to replace our original choice of e by any unit 1-form inside $\langle e,f\rangle$, and this already gives rise to an S^1 ambiguity.

If $\mathbf{v} = (v_1, v_2, v_3)$ then $a_i c_i = v_i$, so that $c_i = v_i/a_i$ is determined by a_i, assuming that $\mathbf{v}$ is fixed. Define σ explicitly by setting

$$\sigma(e \wedge f) = (s_1, s_2, s_3), \qquad s_i = a_i^2 + b_i^2 + c_i^2,$$

which is consistent with (32) and (33). Now set

$$x_i = (a_i b_i)^2 = -a_i^4 + s_i a_i^2 - v_i^2. \tag{34}$$

Since $\sum a_i b_i = 0$, we can use elementary symmetric functions to deduce that

$$\sum x_i^2 - 2(x_2 x_3 + x_3 x_1 + x_1 x_2) = 0. \tag{35}$$

The quadric (35) is a circular cone, half of which lies symmetrically in the first octant of $\mathbb{R}^3$, touching each of the three coordinate planes in a diagonal line.

Now rearrange (34) as

$$(a_i^2 - \tfrac{1}{2} s_i)^2 = \lambda_i - x_i, \tag{36}$$

where $\lambda_i = \frac{1}{4} s_i^2 - v_i^2 \geqslant 0$. Because $\sum (a_i^2 - \frac{1}{2} s_i) = 0$, the identity (35) remains valid when we replace the coordinates x_i by $x_i - \lambda_i$, and we obtain a plane

$$2 \sum_{i=1}^{3} \kappa_i x_i + \sum \lambda_i^2 - 2(\lambda_2 \lambda_3 + \lambda_3 \lambda_2 + \lambda_1 \lambda_2) = 0,$$

where $\kappa_i = 2\lambda_i - \sum \lambda_j$. This will intersect the cone in a conic that (if non-empty) represents the expected S^1 ambiguity. Having chosen a point of this conic, the 2-form $e \wedge f$ is determined by a consistent choice of square roots in (34), (36) and values for a_i, b_j. QED

5.3 Interpreting null-torsion classes

Let N again be the Iwasawa manifold with the standard metric for which the 1-forms in (19) are orthonormal. Each point $p \in \mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$ defines an $SO(2) \times SO(4)$-structure on N, and we may compute its intrinsic torion. The associated OPS is given by a splitting

$$\mathfrak{n} = \mathcal{V} \oplus \mathcal{H}, \qquad \dim \mathcal{V} = 2 \tag{37}$$

of the real Lie algebra $\mathfrak{n}$ underlying H_c. Integrability can be interpreted purely in terms of the Lie algebra structure (i.e. bracket).

Consider a generic simple unit 2-form

$$\alpha = e \wedge f = f^{12} + f^3 \wedge e^5 + f^4 \wedge e^6 + a e^{56},$$

where $f^i \in \mathcal{H}_0$ (see (28)). We are assuming that $\alpha \wedge \alpha = 0$, so

$$e^{56} \wedge (af^{12} - f^{34}) + f^{123} \wedge e^5 + f^{124} \wedge e^6 = 0,$$

which forces

$$f^{34} = af^{12}, \qquad f^{123} = 0 = f^{124}.$$

The second equation follows from the first if $a \neq 0$, and implies that f^3, f^4 are each a linear combination of f^1, f^2.

If $a \neq 0$, we may re-name the f^i's so that

$$\alpha = a(f^{12} + f^1 \wedge e^5 + f^2 \wedge e^6 + e^{56}) = a(f^1 - e^6) \wedge (f^2 + e^5). \tag{38}$$

If $a = 0$, we may re-name f^1, f^2 so that

$$\alpha = f^{12} + f^1 \wedge e = f^1 \wedge (f^2 + e), \tag{39}$$

where $e \in \langle e^5, e^6 \rangle$. We can use these equations to establish

15 Theorem. *The $SO(2) \times SO(4)$ structures on N satisfying*

$$[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H} \tag{40}$$

form the submanifold (29) of $\mathbb{G}r_2(\mathbb{R}^6)$ whose image in O is (27).

Proof. Since $\mathcal{H}$ is the annihilator of $\langle e, f \rangle$, the horizontal integrability condition (40) is satisfied if and only if $d\alpha = \alpha \wedge \gamma$ for some 1-form γ. Given (38),

$$-\frac{1}{a} d\alpha = (f^1 - e^6) \wedge (e^{13} + e^{42}) + (f^2 + e^5) \wedge (e^{14} + e^{23}),$$

It follows that $\gamma \in \langle e^5, e^6 \rangle$, but this forces a contradiction anyway. We may now assume (39) with $de = 0$ and so $e = 0$. Hence $\alpha = f^{12}$ belongs to (29), whose image in O we have already identified. QED

By contrast to (40), one can show that the set of OPS's on the Iwasawa manifold N satisfying

$$[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V} \tag{41}$$

forms a 6-dimensional T^3-invariant subset of $\mathbb{G}r_2(\mathbb{R}^6)$, characterized by those 2-forms in (38) and (39) for which the real 2-plane $\langle f^1, f^2 \rangle$ is J_0-invariant. This is a 'big' intrinsic torsion variety, in the sense that its image by τ is the entire octahedron.

Although $SO(2) \times SO(4)$ cannot be a holonomy group on N, there do exist points of $\mathbb{G}r_2(\mathbb{R}^6)$ for which both (40) and (41) are satisfied, giving rise to transverse foliations. The set of such points consists of J_0-invariant 2-planes in $\mathcal{H}_0$ represented by the 2-forms

$$\alpha = \pm f^1 \wedge (J_0 f^1);$$

in other words, J_0-holomorphic or anti-holomorphic planes in $\mathbb{G}r_2(\mathbb{R}^4)$. It is the disjoint union of two 2-spheres that map to opposite edges of the octahedron. This description emphasizes once again the importance of J_0 and the fact that the Iwasawa manifold N is the quotient of a *complex* Lie group, but calls for a more detailed investigation.

References

[1] E. Abbena, S. Garbiero, S. Salamon: *Almost Hermitian geometry on six dimensional nilmanifolds*, Ann. Scuola Norm. Sup. Pisa **30** (2001), 147–170.

[2] V. Apostolov, P. Gauduchon, G. Grantcharov: *Bi-Hermitian structures on complex surfaces.*, Proc. London Math. Soc. **79** (1999), 414–428.

[3] M. F. Atiyah, N. J. Hitchin, I. M. Singer: *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London **A 362** (1978), 425–461.

[4] R. Baston, A. Eastwood: *The Borel-Weil theorem*, pp 233–291 in Invitations to Geometry and Topology, Oxf. Grad. Texts Math. 7, Oxford University Press, 2002.

[5] A. L. Besse: *Einstein Manifolds*, Ergeb. Math. Grenzgeb. vol. 10, Springer-Verlag, Berlin, 1987.

[6] D. E. Blair: *Geometry of manifolds with structural group* $\mathcal{U}(n) \times \mathcal{O}(s)$, J. Differ. Geom. **4** (1970), 155–167.

[7] A. Borel, F. Hirzebruch: *Characteristic classes and homogeneous spaces, II*, Amer. J. Math. **81** (1959), 315–382.

[8] F. E. Burstall, J. H. Rawnsley: *Twistor Theory for Riemannian Symmetric Spaces*, Lecture Notes Math. 1424, Springer-Verlag, Berlin, 1990.

[9] S. Chiossi, S. Salamon: *The intrinsic torsion of* SU(3) *and* G_2 *structures*, pp 115–133 in Differential Geometry, Valencia 2001, World Scientific, 2002

[10] J.C. Gonzalez Davila, F. Martin Cabrera: *Harmonic G-structures*, arXiv:0706.0116.

[11] A. Gray, L. M. Hervella: *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[12] V. Guillemin, E. Lerman, S. Sternberg: *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge University Press, 1996.

[13] N. J. Hitchin: *Kählerian twistor spaces*, Proc. London Math. Soc. **43** (1981), 133–150.

[14] G. Ketsetzis, S. Salamon: *Complex structures on the Iwasawa manifold*, Adv. Geom. **4** (2004), 165–179.

[15] F. C. Kirwan: *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton University Press, 1982.

[16] D. Kotschick, S. Terzizć: *Chern numbers and the geometry of partial flag manifolds*, arXiv:0709.3026, to appear in Comment. Math. Helv.

[17] O. Kowalski: *Generalized Symmetric Spaces*, Lect. Notes Math. 805, Springer-Verlag, Berlin, 1990.

[18] J. Lauret: *A canonical compatible metric for geometric structures on nilmanifolds*, Ann. Global Anal. Geom. **30** (2006), 107–138.

[19] G. Mihaylov: *Special Riemannian Structures in Six Dimensions*, Doctoral thesis, University of Milan, 2008.

[20] A. M. Naveira: *A classification of Riemannian almost-product manifolds*, Rend. Mat. **3** (1983), 577–592.

[21] S. Rollenske: *Nilmanifolds: complex structures, geometry and deformations*, arXiv:0709.0467

[22] S. Salamon: *Almost parallel structures*, Contemp. Math. **288** (2001), 162–181.

[23] S. M. Salamon: *Hermitian geometry*, pp 233–291 in Invitations to Geometry and Topology, Oxf. Grad. Texts Math. 7, Oxford University Press, 2002.

[24] R. Thomas: *Notes on GIT and symplectic reduction for bundles and varieties*, pp 221–273 in Surveys in Differential Geometry, vol. 10: A tribute to Professor S-S. Chern, International Press, Somerville, 2006.

[25] J. A. Wolf: *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math. **120** (1968), 59–148.