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Jacobi fields and osculating rank of the Jacobi operator in some special classes of homogeneous Riemannian spaces

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Abstract. The geometry of Riemannian symmetric spaces is really richer than that of Riemannian homogeneous spaces. Nevertheless, there exists a large literature of special classes of homogeneous Riemannian manifolds with an important list of features which are typical for a Riemannian symmetric space. Normal homogeneous spaces, naturally reductive homogeneous spaces or g. o. spaces are some interesting examples of these classes of spaces where, in particular, the Jacobi equation can be also written as a differential equation with constant coefficients and the osculating rank of the Jacobi operator is constant.

Compact rank one symmetric spaces are among the very few manifolds that are known to admit metrics with positive sectional curvature. In fact, there exist only three non-symmetric (simply-connected) normal homogeneous spaces with positive curvature: $V_1 = Sp(2)/SU(2)$, $V_2 = SU(5)/(Sp(2) \times S^1)$, given by M. Berger [6] and the Wilking's example $V_3 = (SU(3) \times SO(3))/U^{\bullet}(2)$, [37]. Here, we show some geometric properties of all these spaces, properties related with the existence of isotropic Jacobi fields and the determination of the constant osculating rank of the Jacobi operator. It provides different way to "measure" of how they are so close or not to the class of compact rank one symmetric spaces.

Keywords: normal homogeneous space, naturally reductive homogeneous space, g.o. space, Jacobi fields, Jacobi operator, constant osculating rank

MSC 2000 classification: primary 53C30, secondary 53C25

1 Introduction

In Berger [6], a classification of simply connected normal homogeneous spaces that have strictly positive curvature, that is, the Riemannian structure comes from a bi-invariant metric on a group and all sectional curvatures are bounded

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from below by a positive constant, was given. In the course of his classification he found two new examples of topological types which admit strictly positive curvature. These two examples are the manifolds $V_1 = Sp(2)/SU(2)$ and $V_2 = SU(5)/(Sp(2)\times S^1)$ of dimensions 7 and 13, respectively. In [37] the author proves that the Berger's classification is not totally correct. In fact, he shows that there is a third example $V_3 = (SU(3) \times SO(3))/U^{\bullet}(2)$, equipped with a one-parameter family of bi-invariant metrics, where $U^{\bullet}(2)$ is the image under the embedding $(\iota, \pi) : U(2) \to SU(3) \times SO(3)$ given by the natural inclusion

$$\iota(A) = \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}, \quad A \in U(2),$$

and π is the projection $\pi : U(2) \to U(2)/S^1$. As Wilking noted, the algebraic structure corresponding to V_3 already appears in the papers of Bérard Bergery [4], [5]. Nevertheless, it seems interesting to remark that it was Wilking who saw the point that the manifold V_3 was contained in the Aloff - Wallach's classification [1].

The resolution of the Jacobi equation for geodesics on a general Riemannian manifold (M, g) can be quite a difficult task. In the Euclidean space the solution is trivial. For the symmetric spaces, the problem is reduced to a system of differential equations with constant coefficients and one can directly show that every Jacobi field vanishing at two points is the restriction of a Killing vector field along the geodesic. A Jacobi field V which is the restriction of a Killing vector field along a geodesic is called *isotropic*. It means that V is the restriction of a fundamental vector field X^* with X belonging to the Lie algebra of the isometry group $I(M, g)$ of (M, g). For homogeneous Riemannian manifolds, if V vanishes at a point o of the geodesic then it is obtained as restriction of X^*, where X belongs to the isotropy algebra of $I(M, g)$ at o in M. This particular situation was what originally motivated the term 'isotropic'. If q is another zero of an isotropic Jacobi field V, then q is said to be isotropically conjugate to o along the geodesic. In [9], [10] is proved the following:

1 Theorem. *Let G/H be a simply connected normal homogeneous space of rank* 1 *such that every point q conjugate to $o = [H]$ is isotropically conjugate to o. Then G/H is diffeomorphic to a Riemannian symmetric space of rank* 1.

The proof of Chavel is based on the fact that on the manifolds V_1 and V_2 there exist geodesics with conjugate points which are not isotropically conjugate. In [13], the same result is proved for V_3.

Ziller in [39] proposed to examine conjectures like: *A naturally reductive homogeneous space with the property that all Jacobi fields vanishing at two points are isotropic is locally symmetric.*

Since the Riemannian symmetric spaces verify $\nabla R = 0$, where R denotes the Riemannian curvature tensor, we can expect that those Riemannian homogeneous manifolds nearest to the symmetric spaces must verify some polynomial relations between $\nabla^i R$. In this direction it seems interesting the following

2 Proposition. *For any Riemannian manifold if* $\nabla R \neq 0$, *then* $\nabla^i R \neq 0$, *for all* $i \geq 2$.

The proof was given by Lichnerowicz for the compact case [23] and by Nomizu and Ozeki for the non-compact case [28].

In [33], Tsukada gives a criterion for the existence of totally geodesic submanifolds of naturally reductive homogeneous spaces (n.r.h.s.). This criterion is based on the curvature tensor and on a finite number of its derivatives with respect to the Levi-Civita connection. In particular, to prove this result he uses two basic formulae proved exclusively for n.r.h.s. by K. Tojo [32]. From these formulae he obtained that the curvature tensor along a geodesic can be considered, using parallel translation, as a curve in the space of curvature tensors on the tangent space at the origin. Later, using the general theory, he concluded that over n.r.h.s., the Jacobi tensor field has constant osculating rank $r \in \mathbb{N}$. In fact, in [33] the curves of constant osculating rank in the Euclidean space are defined and this concept is applied to n.r.h. s. Working with the Levi-Civita connection and using the interesting geometric result of Tsukada, mentioned above in [26], it is proved that on the manifold V_1, which appears in the Berger's classification as an example of n.r.h.s., but non symmetric, has osculating constant rank 2. Given the generality of the method used in [26], the authors conjectured that it could also be applied to solving problems related to the Jacobi equation in several other examples of n.r.h.s.In fact, in [24] has been proved that the Wilking's manifold V_3 is also of constant osculating rank 2. In [3], the authors had successfully applied this method on the manifold $F^6 = U(3)/(U(1) \times U(1) \times U(1))$ and they have proved that this manifold is of constant osculating rank 4.

By definition, a Riemannian g.o. manifold is a homogeneous Riemannian manifold on which every geodesic is an orbit of a one-parameter group of isometries. The first counter-example of a Riemannian g. o. manifold which is not naturally reductive is the six-dimensional Kaplan's example. Following [26], in [2] is proved that the Kaplan's example is also of constant osculating rank 4.

2 A formula for the covariant derivative of the Jacobi operator

Let M be an n-dimensional, connected, real analytic Riemannian manifold,

$g = <,>$ its Riemannian metric, $m \in M$, $v \in T_m M$ a unit tangent vector and let $\gamma : J \to M$ be a geodesic in M defined on some open interval J of $I\!R$ with $0 \in J$, $m = \gamma(0)$. For a geodesic $\gamma(t)$ in M the associated Jacobi operator R_t is the self-adjoint tensor field along γ defined by $R_t(\ .\ ) = R(\ .\ ,\gamma')\gamma'$. In the following for the curvature tensor we follow the notations of [21]. The covariant derivative $R_t^{i)}$ of the Jacobi operator R_t along γ is the self-adjoint tensor field defined by

$$R_t^{i)}(\ .\ ) = (\nabla_{\gamma'} \overset{i)}{\dots} \nabla_{\gamma'} R)(\ .\ ,\gamma')\gamma',$$

where ∇ is the Levi-Civita connection associated to the metric. Its value at $\gamma(0)$ will be denoted by

$$R_0^{i)}(\ .\ ) = (\nabla_{\gamma'} \overset{i)}{\dots} \nabla_{\gamma'} R)(\ .\ ,\gamma')\gamma'(0)$$

and we also denote $R_t^{0)}$ by R_t. Using the induction method it is possible to prove

3 Theorem. [26] *Let X be a vector field along the geodesic γ. For $n \geq 1$, we have*

$$\nabla_{\gamma'} \overset{n)}{\dots} \nabla_{\gamma'} R(X,\gamma')\gamma' = \sum_{i=0}^{n} \binom{n}{i} (R_t^{n-i)})(\nabla_{\gamma'} \overset{i)}{\dots} \nabla_{\gamma'} X)(\ .\ ,\gamma')\gamma'(0).$$

3 An algebraic expression for the covariant derivative of the Jacobi operator on a n.r.h.s.

Let G be a Lie group, H a closed subgroup, $M = G/H$ the space of left cosets of H and $\pi : G \to G/H$ the natural projection. For $x \in G$ we denote by τ the induced action of G on G/H given by $\tau(x)(sH) = xsH$, $x, s \in G$. The Lie algebras of G and H will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

4 Definition. [9], [21, Vol. II, p. 202] $M = G/H$ is said to be

a) Reductive homogeneous space if the Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathfrak{m}$ is an $Ad(H)$-invariant subspace.

   In this case, $\mathfrak{m}$ is identified with the tangent space at the origin $o = [H]$ and we get $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$.

b) Riemannian homogeneous space if G/H is a Riemannian manifold such that the metric is preserved by $\tau(x)$, for all $x \in G$. Then it corresponds with an $Ad(H)$-invariant inner product $< , >$ on $\mathfrak{m}$.

c) Naturally reductive homogeneous space *if there exists a reductive decomposition* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ *satisfying*

$$< [u,v]_{\mathfrak{m}}, w > + < [u,w]_{\mathfrak{m}}, v >= 0,$$

for all $u, v, w \in \mathfrak{m}$, *where* $[u,v]_{\mathfrak{m}}$ *denotes the* $\mathfrak{m}$*-component of* $[u,v]$ *and* $< , >$ *is the inner product induced on* $\mathfrak{m}$.

d) Normal Riemannian homogeneous space when there exists a bi-invariant metric on $\mathfrak{g}$ whose restriction to $\mathfrak{m} = \mathfrak{h}^{\perp}$ coincides with $< , >$.

An affine connection on M is said to be invariant if it is invariant under $\tau(x)$, for all $x \in G$. It is well known ([4, Ch X, p. 186]) that there exists an invariant affine connection D on G/H, the *canonical connection*, whose torsion T and curvature B are also invariant. Because D is G-invariant T and B are given by

$$T_o(u,v) = -[u,v]_{\mathfrak{m}}, \quad B_o(u,v)w = [[u,v]_{\mathfrak{h}}, w],$$

for $u, v, w \in \mathfrak{m}$, where $[u,v]_{\mathfrak{h}}$ denotes the $\mathfrak{h}$-component of $[u,v]$.

From now on we will assume that $M = G/H$ is a n.r.h.s. We know (see [38], [21, p.197]) that the Levi Civita connection ∇ is given by $\nabla_u v = (1/2)[u,v]_{\mathfrak{m}}$. Evidently, ∇_u is a skew symmetric linear endomorphism of $(\mathfrak{m}, <, >)$. Therefore $\exp \nabla_u$ is a linear isometry of $(m, <, >)$. Since the Riemannian connection is a natural free torsion connection on M, we have [32], [21, Ch. X]

5 Proposition. *The following properties hold:*

(i) *For each* $v \in \mathfrak{m}$, *the curve* $\gamma(t) = \tau(\exp tv)o$ *is a geodesic with* $\gamma(0) = o$, *and* $\gamma'(t) = v$.

(ii) *The parallel translation along* γ *is given as follows*

$$\tau(\exp tv)_* e^{-t\nabla_v} : T_oM \to T_oM.$$

(iii) *The* $(1,3)$*-tensor* R_t *on* $\mathfrak{m}$ *obtained by parallel translation of the Jacobi operator along* γ *is given as follows*

$$R_t = e^{t\nabla_v} \cdot R_0.$$

Above, R_0 denotes the Jacobi operator at the origin o and $e^{t\nabla_v}$ denotes the action on the space $\mathcal{R}(\mathfrak{m})$ of curvature tensors on $\mathfrak{m}$.

6 Proposition. [21, Ch.1, p. 202] *Let* $\gamma(t)$ *be a geodesic as above. If* X *is a differentiable vector field along* γ, *then*

$$R_0(X) = -[[X,v]_{\mathfrak{h}}, v] - (1/4)[[X,v]_{\mathfrak{m}}, v]_{\mathfrak{m}}.$$

7 Proposition. [26] *Under the same hypothesis that in the Proposition 2, we have for* $n > 0$

$$(-1)^{n-1}2^n(R_0^{n)})(X) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i [\dots[[X,v]_{\mathfrak{m}},\dots,v]_{\mathfrak{h}}^{i+1)},\dots,v]_{\mathfrak{m}},$$

where for each term of the sum we have $n+2$ *brackets and the exponent* $i+1)$ *means the position of the bracket valued on* $\mathfrak{h}$.

We know that for each $Z \in \mathfrak{g}$, the mapping $(t,p) \in I\!R \times M$ to $(\exp tZ)p$ is a one-parameter group of isometries and consequently, it induces a Killing vector field Z^* given by

$$Z_o^* = \frac{d}{dt}_{|t=0}(\exp tZ)o.$$

Z^* is called the *fundamental vector field* or the *infinitesimal* G-*motion* corresponding to Z on M.

On n.r.h.s., ∇ and D have the same geodesics and, consequently, the same Jacobi fields ([9], [39]). Such geodesics are orbits of one-parameter subgroups of G of type $\exp tu$, where $u \in \mathfrak{m}$. Then, taking into account that $DT = DB = 0$ and the parallel translation with respect to D of tangent vectors at the origin o along the geodesic $\gamma(t) = (\exp tu)o$, $u \in \mathfrak{m}$, $\|u\| = 1$, coincides with the differential of $\exp tu \in G$ action on M, it follows that the Jacobi equation can be expressed as the differential equation

$$X'' - T_uX' + B_uX = 0$$

in the vector space $\mathfrak{m}$, where

$$T_uX = T(u,X) = -[u,X]_{\mathfrak{m}}$$

and

$$B_uX = B(u,X)u = [[u,X]_{\mathfrak{h}},u].$$

The operator T_u is skew-symmetric with respect to $<,>$, B_u is self-adjoint and they satisfy $R_u = B_u - \frac{1}{4}T_u^2$ [14].

8 Proposition. [14] *A Jacobi field* V *along* $\gamma(t)$ *with* $V(0) = 0$ *is* G*-isotropic if and only if there exists an* $A \in \mathfrak{h}$ *such that*

$$V'(0) = [A,u].$$

Then $V = A^* \circ \gamma$. Obviously, any G-isotropic Jacobi field is isotropic, because G is in $I(M,g)$.

The following characterization for G-isotropic Jacobi fields on normal homogeneous spaces is useful.

9 Lemma. [14] *A Jacobi field V along $\gamma(t) = (\exp tu)o$ on a normal homogeneous space with $V(0) = 0$ is G-isotropic if and only if $V'(0) \in (Ker\ B_u)^{\perp}$.*

The following result can be considered interesting to obtain Jacobi vector fields along $\gamma(t)$ vanishing at two points which are G-isotropic or not.

10 Proposition. [13] *Let $(M = G/K, g)$ be a normal homogeneous space and let u, v be orthonormal vectors in $\mathfrak{m}$ such that*

(i) $[u, v] \in \mathfrak{m} \setminus \{0\}$,

(ii) $[[u, v], u]_{\mathfrak{m}} = \lambda v$, *for some* $\lambda > 0$,

(iii) $[[[u, v], u]_{\mathfrak{k}}, u] = \mu[u, v]$, *for some* $\mu \leq 0$.

Then , we have:

(A) *If $\mu = 0$, the vector fields $V(t)$ along $\gamma(t) = (\exp tu)o$ given by*

$$\begin{aligned} V(t) &= (\exp tu)_{*o}\Big(\Big(- A\sin\sqrt{\lambda}t + B(1 - \cos\sqrt{\lambda}t)\Big)v \\ &\quad + \Big(A(1 - \cos\sqrt{\lambda}t) + B\sin\sqrt{\lambda}t\Big)w\Big), \end{aligned}$$

for A, B constants with $w = \frac{1}{\sqrt{\lambda}}[u, v]$, are Jacobi fields such that $V(\frac{2p\pi}{\sqrt{\lambda}}) = 0$, for all $p \in \mathbb{Z}$, which are not G-isotropic.

(B) *If $\mu < 0$, the vector fields $V(t)$ along $\gamma(t) = (\exp tu)o$ given by*

$$\begin{aligned} V(t) &= (\exp tu)_{*o}\Big(\sqrt{\tfrac{\lambda}{\lambda-\mu}}\Big(\tfrac{\mu}{\lambda}s(t) - \sin s(t) - \tfrac{\mu}{2\lambda}s_o(1 - \cos s(t))\Big)v \\ &\quad + (1 - \cos s(t) - \tfrac{\mu}{2\lambda}s_o \sin s(t))w\Big), \end{aligned}$$

where $s(t) = \sqrt{\lambda - \mu}t$ and $s_o \in]0, 2\pi[$ verifying $\frac{\mu}{\lambda}s_o \sin s_o = 2(1 - \cos s_o)$, is a Jacobi field such that $V(\frac{s_o}{\sqrt{\lambda-\mu}}) = 0$ and it is not G-isotropic.

4 Some examples

4.1 The Berger's manifold $V_1 = Sp(2)/SU(2)$

The Lie algebra $\mathfrak{sp}(2)$ of the symplectic group $Sp(2)$ can be view as a subalgebra of $\mathfrak{su}(4)$ and so, as a subalgebra of $\mathfrak{su}(5)$. A basis for $\mathfrak{sp}(2)$ is given by (see [6])

$$\begin{array}{lllllllll} S_1 &=& P_1 - P_2, & S_2 &=& P_3 - P_4, & S_3 &=& Q_{12}, \\ S_4 &=& R_{12}, & S_5 &=& Q_{34}, & S_6 &=& R_{34}, \\ S_7 &=& Q_{13} - Q_{24}, & S_8 &=& R_{13} + R_{24}, & S_9 &=& Q_{14} + Q_{23}, \\ S_{10} &=& R_{14} - R_{23}, & & & & & & \end{array}$$

where $Q_{kl} = E_{kl} - E_{lk}$, $R_{kl} = \sqrt{-1}(E_{kl} + E_{lk})$ and $P_k = \sqrt{-1}(E_{kk} - E_{55})$ and E_{kl} denote the (5×5) matrices $(\delta_{ak}\delta_{bl})_{1\leq a,b\leq 5}$, for $1 \leq k, l \leq 5$. Then an orthonormal basis of $\mathfrak{sp}(2)$ with respect to the inner product $< X, Y >= -\frac{1}{5}\,\mathrm{trace}\, XY$ and adapted to the reductive decomposition $\mathfrak{sp}(2) = \mathfrak{su}(2) \oplus \mathfrak{m}$ is $\{K_1, K_2, K_3; M_1, \ldots M_7\}$, where

$$K_1 = \frac{1}{2}(3S_1 + S_2), \quad K_2 = S_5 + \frac{\sqrt{3}}{2}S_7, \quad K_3 = S_6 + \frac{\sqrt{3}}{2}S_8$$

generate $\mathfrak{su}(2)$ and

$$\begin{array}{lllllllll} M_1 & = & \frac{1}{2}(S_1 - 3S_2), & M_2 & = & \sqrt{\frac{5}{2}}S_3, & M_3 & = & \sqrt{\frac{5}{2}}S_4, \\ M_4 & = & \frac{1}{\sqrt{2}}(\sqrt{3}S_5 - S_7), & M_5 & = & \frac{1}{\sqrt{2}}(\sqrt{3}S_6 - S_8), & M_6 & = & \frac{\sqrt{5}}{2}S_9, \\ M_7 & = & \frac{\sqrt{5}}{2}S_{10} \end{array}$$

is basis for $\mathfrak{m}$. The brackets $[M_i, M_j]$ $1 \leq i < j \leq 7$, are given by

$$\begin{array}{ll} [M_1, M_2] = M_3, & [M_1, M_3] = -M_2, \\ [M_1, M_4] = -(M_5 + \sqrt{6}K_3), & [M_1, M_5] = M_4 + \sqrt{6}K_2, \\ [M_1, M_6] = -M_7, & [M_1, M_7] = M_6, \\ [M_2, M_3] = M_1 + 3K_1, & [M_2, M_4] = M_6, \\ [M_2, M_5] = -M_7, & [M_2, M_6] = -M_4 + \sqrt{\frac{3}{2}}K_2, \\ [M_2, M_7] = M_5 - \sqrt{\frac{3}{2}}K_3, & [M_3, M_4] = M_7 \\ [M_3, M_5] = M_6, & [M_3, M_6] = -M_5 + \sqrt{\frac{3}{2}}K_3, \end{array}$$

$$\begin{array}{ll} [M_3, M_7] = -M_4 + \sqrt{\frac{3}{2}}K_2, & [M_4, M_5] = -M_1 + K_1, \\ [M_4, M_6] = M_2 + \sqrt{\frac{5}{2}}K_2, & [M_4, M_7] = M_3 + \sqrt{\frac{5}{2}}K_3, \\ [M_5, M_6] = M_3 - \sqrt{\frac{5}{2}}K_3, & [M_5, M_7] = -M_2 + \sqrt{\frac{5}{2}}K_2, \\ [M_6, M_7] = -M_1 + 2K_1 & . \end{array}$$

Hence, using Proposition 10, we can prove

11 Proposition. [13] *On V_1 there exist 2-dimensional subspaces $\mathfrak{m}'$ of $\mathfrak{m}$ such that for all $u \in \mathfrak{m}'$, the geodesic $\gamma(t) = (\exp tu)o$ admits Jacobi fields vanishing at two points which are not isotropic.*

In the classical literature, curves of constant osculating rank in the Euclidean space are defined and this concept is applied to n.r.h.s. Let $c : I \to \mathbb{R}^n$ be a curve defined on an open interval I of $I\!R$ into $I\!R^n$. We say that c has *constant osculating rank* r if for all $t \in I$, its higher order derivatives $c'(t), \dots, c^{r)}(t)$ are linearly independent and $c'(t), \dots, c^{r+1)}(t)$ are linearly dependent in $I\!R^n$. It is a fundamental fact that if c has constant osculating rank r, there exist smooth real functions $a_1, \dots, a_r : I \to I\!R$ such that

$$c(t) = c(0) + a_1(t)c'(0) + \cdots + a_r(t)c^{r)}(0).$$

We return to a n.r.h.s. M. For a unit vector $v \in \mathfrak{m}$ determining the geodesic γ, $R_t = e^{t\nabla_v} R_o$ is a curve in $\mathcal{R}(\mathfrak{m})$. Since $e^{t\nabla_v}$ is a one-parameter subgroup of the group of linear isometries of $\mathcal{R}(\mathfrak{m})$, the curve R_t has constant osculating rank r, [33]. Therefore, for the Jacobi operator we have

$$R_t = R_0 + a_1(t)R_0^{1)} + \cdots + a_r(t)R_0^{r)}.$$

Working with the Levi-Civita connection and using the interesting geometric Tsukada's result mentioned above, in [26], it is proved that the manifold V_1 has constant osculating rank 2. In fact, we prove that

$$R_0^{2n)} = (-1)^{n-1}R_0^{2)}, \qquad R_0^{2n+1)} = (-1)^n R_0^{1)}.$$

In order to be able to determine the explicit form of the Jacobi operator along an arbitrary geodesic γ with initial vector v at the origin, it is useful to determine the values of $R_0^{i)}$, $i = 0, 1, 2, 3, 4$. We obtain them by using Proposition 7. In the following we always suppose that $v \in \mathfrak{m}$ is an arbitrary unit vector, that is, $v = \sum_{i=1}^7 x^i M_i$, $\sum_{i=1}^7 (x^i)^2 = 1$. Further, we denote by $\{E_i,\ i = 1, \dots, 7\}$ the orthonormal frame field along γ obtained by parallel translation of the basis $\{M_i\}$. Evidently, the operators $R_0^{i)}$, $i = 0, 1, 2, 3, 4$, are given by

$$R_0^{i)} = R_{kl}^{i)}(0), \quad R_{kl}^{i)}(0) = < R^{i)}(E_k), E_l > (0).$$

In [26] it is proved

12 Lemma. *At $\gamma(0)$ we have:*

(i) $R_0^{3)} = -\|\gamma'\|^2 R_0^{1)} = -R_0^{1)}$,

(ii) $R_0^{4)} = -\|\gamma'\|^2 R_0^{2)} = -R_0^{2)}$.

13 Proposition. *At $\gamma(0)$ we have:*

(i) $R_0^{2n)} = (-1)^{n-1} R_0^{2)}$,

(ii) $R_0^{2n+1)} = (-1)^n R_0^{1)}$.

14 Corollary. *V_1 has constant osculating rank* 2.

15 Corollary. *Along the geodesic γ the Jacobi operator can be written as*

$$R_t = R_0 + R_0^{2)} + R_0^{1)} \sin t - R_0^{2)} \cos t.$$

4.2 The Wilking's manifold $V_3 = (SU(3) \times SO(3))/U^{\bullet}(2)$

We recall that $V_3 := (SO(3) \times SU(3))/U^{\bullet}(2)$ is equipped with a one-parameter family of biinvariant metrics, where $U^{\bullet}(2)$ is the image of $U(2)$ under the embedding $(\pi, \iota) : U(2) \hookrightarrow SO(3) \times SU(3)$, where π is the projection $\pi : U(2) \to U(2)/S^1 \cong SO(3)$, being $S^1 \subset U(2)$ the center of $U(2)$, and ι is the natural inclusion

$$\iota(A) := \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix} \quad \text{for } A \in U(2).$$

Using the natural isomorphism between $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$, we can consider the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{su}(3)$ of $SO(3) \times SU(3)$ as the subalgebra of $\mathfrak{su}(5)$ of matrices of the form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \quad X_1 \in \mathfrak{su}(2),\ X_2 \in \mathfrak{s}u(3).$$

Then the Lie algebra $\mathfrak{u}^{\bullet}(2)$ of $U^{\bullet}(2)$ may be expressed as

$$\mathfrak{u}^{\bullet}(2) = \left\{ \begin{pmatrix} \pi_* A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -\operatorname{trace} A \end{pmatrix} \mid A \in \mathfrak{u}(2) \right\},$$

where π_* denotes the differential map of π. On V_3 it is considered the one-parameter family of metrics g_λ induced by the bi-invariant metrics $<,>_\lambda$, for $\lambda > 0$, on $SO(3) \times SU(3)$ given by

$$< X, Y >_\lambda = -\frac{1}{2}(\lambda \operatorname{trace} X_1 Y_1 + \operatorname{trace} X_2 Y_2),$$

for $X, Y \in \mathfrak{so}(3) \oplus \mathfrak{su}(3)$. Then (V_3, g_λ) is isometric to the Aloff-Wallach spaces $(M^7_{11}, \tilde{g}_t)$, for $t = -\frac{3}{2\lambda+3}$, [37].

Next, we choose an orthonormal basis of $\mathfrak{so}(3) \oplus \mathfrak{su}(3)$ adapted to the reductive decomposition $\mathfrak{u}^{\bullet}(2) \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal space to $\mathfrak{u}^{\bullet}(2)$ in $\mathfrak{so}(3) \oplus \mathfrak{su}(3)$. An orthonormal basis for $\mathfrak{u}^{\bullet}(2)$ is given by

$$\begin{aligned} K_1 &= \tfrac{1}{\sqrt{1+\lambda}}(P_1 - P_2 + P_3 - P_4), & K_2 &= \tfrac{1}{\sqrt{1+\lambda}}(R_{12} + R_{34}), \\ K_3 &= \tfrac{1}{\sqrt{1+\lambda}}(Q_{12} + Q_{34}), & K_4 &= \tfrac{1}{\sqrt{3}}(P_3 + P_4) \end{aligned}$$

and $\{M_1, \dots, M_7\}$, given by

$$\begin{aligned} M_1 &= \tfrac{1}{\sqrt{\lambda(1+\lambda)}}(P_1 - P_2 - \lambda(P_3 - P_4)), \qquad M_2 = \tfrac{1}{\sqrt{\lambda(1+\lambda)}}(R_{12} - \lambda R_{34}), \\ M_3 &= \tfrac{1}{\sqrt{\lambda(1+\lambda)}}(Q_{12} - \lambda Q_{34}), \end{aligned}$$

$$M_4 = Q_{35}, \quad M_5 = Q_{45}, \quad M_6 = R_{35}, \quad M_7 = R_{45},$$

constitute an orthonormal basis for $\mathfrak{m}$. The brackets $[M_i, M_j]$ $1 \leq i < j \leq 7$, are given by

$$[M_1, M_2] = -\tfrac{2}{\sqrt{\lambda(1+\lambda)}}((1-\lambda)M_3 + \sqrt{\lambda}K_3),$$
$$[M_1, M_3] = \tfrac{2}{\sqrt{\lambda(1+\lambda)}}((1-\lambda)M_2 + \sqrt{\lambda}K_2),$$
$$[M_1, M_4] = -\sqrt{\tfrac{\lambda}{1+\lambda}}M_6, \qquad [M_1, M_5] = \sqrt{\tfrac{\lambda}{1+\lambda}}M_7,$$
$$[M_1, M_6] = \sqrt{\tfrac{\lambda}{1+\lambda}}M_4, \qquad [M_1, M_7] = -\sqrt{\tfrac{\lambda}{1+\lambda}}M_5,$$
$$[M_2, M_3] = -\tfrac{2}{\sqrt{\lambda(1+\lambda)}}((1-\lambda)M_1 + \sqrt{\lambda}K_1),$$
$$[M_2, M_4] = -\sqrt{\tfrac{\lambda}{1+\lambda}}M_7, \qquad [M_2, M_5] = -\sqrt{\tfrac{\lambda}{1+\lambda}}M_6,$$
$$[M_2, M_6] = \sqrt{\tfrac{\lambda}{1+\lambda}}M_5, \qquad [M_2, M_7] = \sqrt{\tfrac{\lambda}{1+\lambda}}M_4,$$
$$[M_3, M_4] = \sqrt{\tfrac{\lambda}{1+\lambda}}M_5, \qquad [M_3, M_5] = -\sqrt{\tfrac{\lambda}{1+\lambda}}M_4,$$
$$[M_3, M_6] = \sqrt{\tfrac{\lambda}{1+\lambda}}M_7, \qquad [M_3, M_7] = -\sqrt{\tfrac{\lambda}{1+\lambda}}M_6,$$
$$[M_4, M_5] = \tfrac{1}{\sqrt{1+\lambda}}(\sqrt{\lambda}M_3 - K_3),$$
$$[M_4, M_6] = -\tfrac{1}{\sqrt{1+\lambda}}(\sqrt{\lambda}M_1 - K_1) + \sqrt{3}K_4,$$

$$[M_4, M_7] = \tfrac{-1}{\sqrt{1+\lambda}}(\sqrt{\lambda}M_2 - K_2), \qquad [M_5, M_6] = \tfrac{-1}{\sqrt{1+\lambda}}(\sqrt{\lambda}M_2 - K_2),$$
$$[M_5, M_7] = \tfrac{1}{\sqrt{1+\lambda}}(\sqrt{\lambda}M_1 - K_1) + \sqrt{3}K_4, \qquad [M_6, M_7] = \tfrac{1}{\sqrt{1+\lambda}}(\sqrt{\lambda}M_3 - K_3).$$

Taking into account that one obtains

$$[K_1, M_1] = 0, \qquad [K_1, M_2] = -\tfrac{2}{\sqrt{1+\lambda}}M_3, \qquad [K_1, M_3] = \tfrac{2}{\sqrt{1+\lambda}}M_2,$$
$$[K_2, M_1] = \tfrac{2}{\sqrt{1+\lambda}}M_3, \qquad [K_2, M_2] = 0, \qquad [K_2, M_3] = -\tfrac{2}{\sqrt{1+\lambda}}M_1,$$
$$[K_3, M_1] = -\tfrac{2}{\sqrt{1+\lambda}}M_2, \qquad [K_3, M_2] = \tfrac{2}{\sqrt{1+\lambda}}M_1, \qquad [K_3, M_3] = 0.$$

It is proved in [13] the following result.

16 Theorem. *The 3-dimensional real projective space* $I\!RP^3(\frac{\lambda+1}{\lambda})$ *can be isometrically embedded as a totally geodesic submanifold in* (V_3, g_λ) *and any geodesic in this submanifold admits Jacobi fields on* (V_3, g_λ) *vanishing at two points which are not isotropic.*

Aloff and Wallach [1] have introduced (V_3, g_λ) from a different point of view.

They studied for positive integers k and l the groups

$$T_{kl} := \left\{ \begin{pmatrix} z^k & 0 & 0 \\ 0 & z^l & 0 \\ 0 & 0 & \bar{z}^{k+l} \end{pmatrix} \mid z \in S^1 \subset \mathbb{C} \right\} \subset U(2) \subset SU(3),$$

and the one-parameter family of left-invariant, $Ad(U(2))$-invariant metrics on $SU(3)$, given by

$$g_t(u+x, v+y) := -(1+t)B_{\mathfrak{su}(3)}(u,v) - B_{\mathfrak{su}(3)}(x,y),$$

where $t \in]-1, \infty[$, $u, v \in \mathfrak{u}(2)$, $x, y \in \mathfrak{u}(2)^{\perp}$ and $B_{\mathfrak{su}}(3)$ denotes the Killing form of $SU(3)$. In [1], Aloff and Wallach have shown that the space $(M^7_{k,l}, g_t) := (SU(3), g_t)/T_{k,l}$ has positive sectional curvature if and only if $t \in]-1, 0[$.

17 Proposition. [37] *(V_3, g_λ) is isometric to $(M^7_{1,1}, g_t)$, for $t = -\frac{3}{2\lambda+3}$.*

The following results can be found in [24].

18 Lemma. *For $\lambda = 2/3$ at $\gamma(0)$, we have:*

$$5R_0^{3)} = -2\|\gamma'\|^2 R_0^{1)} = -2R_0^{1)}, \quad 5R_0^{4)} = -2\|\gamma'\|^2 R_0^{2)} = -2R_0^{2)}.$$

19 Proposition. *For $\lambda = 2/3$ at $\gamma(0)$, we have:*

$$5R_0^{2n)} = (-1)^{n-1} 2R_0^{2)}, \quad 5R_0^{2n+1)} = (-1)^n 2R_0^{1)}.$$

20 Corollary. *V_3 has constant osculating rank 2 if and only if $\lambda = 2/3$.*

21 Corollary. *For $\lambda = 2/3$ along the geodesic γ the Jacobi operator can be written as*

$$R_t = R_0 - \frac{5}{2}R_0^{2)} + \sqrt{\frac{5}{2}}R_0^{1)} \sin\sqrt{\frac{2}{5}}t + \frac{5}{2}R_0^{2)} \cos\sqrt{\frac{2}{5}}t.$$

Now, proceeding as in 4.1, we are able to determine for the manifold V_3 the explicit form of the Jacobi operator along an arbitrary geodesic γ with initial vector v at the origin o.

Conjecture. *All examples of normal homogeneous spaces coming from the Berger's classification are of constant osculating rank zero or two.*

4.3 The Berger's manifold $V_2 = SU(5)/Sp(2) \times S^1$

M. Berger in [6] has shown that the Lie algebra $\mathfrak{h}$ of the isotropy subgroup in V_2 is given up to isomorphism by $\mathfrak{h} = \mathfrak{s}p(2) \oplus R$, where R is the centralizer of $\mathfrak{s}u(4)$ in $\mathfrak{s}u(5)$. Following [17], an orthonormal basis of $\mathfrak{s}u(5)$ with respect to the bi-invariant inner product $< X, Y >= -\frac{1}{4}\,\mathrm{trace}\, XY$ and adapted to the reductive decomposition $\mathfrak{su}(5) = \mathfrak{h} \oplus \mathfrak{m}$ is $\{H_1, \ldots, H_{11}; M_1, \ldots, M_{13}\}$ where $\{H_1, \ldots, H_{11}\}$ is basis of $\mathfrak{h}$ given by

$$\begin{array}{lll} H_1 = P_1 + P_2 - P_3 - P_4, & H_2 = Q_{13} + Q_{24}, & H_3 = R_{13} + R_{24}, \\ H_4 = P_1 - P_2 - P_3 + P_4, & H_5 = Q_{13} - Q_{24}, & H_6 = R_{13} - R_{24}, \\ H_7 = R_{12} - R_{34}, & H_8 = Q_{14} + Q_{23}, & H_9 = R_{14} + R_{23}, \\ H_{10} = 1/\sqrt{5}(P_1 + P_2 + P_3 + P_4), & H_{11} = Q_{12} + Q_{34}. & \end{array}$$

Here, H_{10} is the generator of R and

$$\begin{array}{cccc} M_1 = \sqrt{2}Q_{15}, & M_2 = \sqrt{2}Q_{25}, & M_3 = \sqrt{2}Q_{35}, & M_4 = \sqrt{2}Q_{45}, \\ M_5 = \sqrt{2}R_{15}, & M_6 = \sqrt{2}R_{25}, & M_7 = \sqrt{2}R_{35}, & M_8 = \sqrt{2}R_{45}, \\ M_9 = Q_{12} - Q_{34}, & M_{10} = Q_{14} - Q_{23}, & M_{11} = R_{12} + R_{34}, & \end{array}$$

$$M_{12} = R_{14} - R_{23}, \qquad M_{13} = P_1 - P_2 + P_3 - P_4$$

constitute an orthonormal basis for $\mathfrak{m} = \mathfrak{h}^{\perp}$.

22 Lemma. [13] *The subspace ν of $\mathfrak{m}$ generated by $\{M_9, \ldots, M_{13}\}$ is a symmetric Lie triple system and the corresponding Lie subalgebra $\mathfrak{s}$ is isomorphic to $\mathfrak{s}u(4)$.*

Hence, one can show that $SU(4)/Sp(2)$ becomes into a totally geodesic submanifold of V_2, which is isometric to the 5-dimensional sphere S^5 with constant sectional curvature 4, or equivalently, with radius 1/2. Then it proves the following

23 Theorem. [13] *The 5-dimensional sphere S^5 of radius $\frac{1}{2}$ can be isometrically embedded as a totally geodesic submanifold in V_2 and any geodesic in this submanifold admits Jacobi fields on V_2 vanishing at two points which are not isotropic.*

4.4 The flag manifold $F^6 = U(3)/(U(1) \times U(1) \times U(1))$

Here we study the constant osculating rank of the Jacobi operator on the flag manifold F^6 with a bi-invariant metric. We have the corresponding Lie algebras $\mathfrak{g} = \mathfrak{su}(3)$, $\mathfrak{h} = \mathfrak{su}(1) \oplus \mathfrak{su}(1) \oplus \mathfrak{su}(1)$ and $\mathfrak{m} = \mathfrak{h}^{\perp}$. It is well known that this manifold is a six-dimensional 3-symmetric space [15]. On $\mathfrak{m}$ we consider the bi-invariant metric $< A, B >= -Re(\mathrm{trace}\, AB)$. An adapted orthonormal basis to $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is given by $\{Q_i\}$, $i = 1, \dots, 9$, where

$$Q_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad Q_3 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$Q_4 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_5 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ ci & 0 & 0 \end{pmatrix}, \qquad Q_6 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

span $\mathfrak{m}$ and $\mathfrak{h}$ is generated by

$$Q_7 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

The multiplication table associated to the Lie algebra $\mathfrak{g}$ is the following:

$$\begin{aligned}
&[Q_1,Q_2] = -\tfrac{\sqrt{2}}{2}Q_3, && [Q_1,Q_3] = \tfrac{\sqrt{2}}{2}Q_2, && [Q_1,Q_4] = Q_7 - Q_8,\\
&[Q_1,Q_5] = \tfrac{\sqrt{2}}{2}Q_6, && [Q_1,Q_6] = -\tfrac{\sqrt{2}}{2}Q_5, && [Q_1,Q_7] = -Q_4,\\
&[Q_1,Q_8] = Q_4, && [Q_1,Q_9] = 0, && [Q_2,Q_3] = -\tfrac{\sqrt{2}}{2}Q_1,\\
&[Q_2,Q_4] = -\tfrac{\sqrt{2}}{2}Q_6, && [Q_2,Q_5] = -Q_7 + Q_9, && [Q_2,Q_6] = \tfrac{\sqrt{2}}{2}Q_4,\\
&[Q_2,Q_7] = Q_5, && [Q_2,Q_8] = 0, && [Q_2,Q_9] = -Q_5,\\
&[Q_3,Q_4] = \tfrac{\sqrt{2}}{2}Q_5, && [Q_3,Q_5] = -\tfrac{\sqrt{2}}{2}Q_4, && [Q_3,Q_6] = Q_8 - Q_9,\\
&[Q_3,Q_7] = 0, && [Q_3,Q_8] = -Q_6, && [Q_3,Q_9] = Q_6,\\
&[Q_4,Q_5] = \tfrac{\sqrt{2}}{2}Q_3, && [Q_4,Q_6] = -\tfrac{\sqrt{2}}{2}Q_2, && [Q_4,Q_7] = Q_1,\\
&[Q_4,Q_8] = -Q_1, && [Q_4,Q_9] = 0, && [Q_5,Q_6] = \tfrac{\sqrt{2}}{2}Q_1,\\
&[Q_5,Q_7] = -Q_2, && [Q_5,Q_8] = 0, && [Q_5,Q_9] = Q_2,\\
&[Q_6,Q_7] = 0, && [Q_6,Q_8] = Q_3, && [Q_6,Q_9] = -Q_3,\\
&[Q_7,Q_8] = 0, && [Q_7,Q_9] = 0, && [Q_6,Q_9] = 0.
\end{aligned}$$

Now proceeding as in [26], in [3] the authors had proved

24 Proposition. *On the manifold F^6 the Jacobi operator has constant osculating rank of order 4 and the relations among $R_0^{i)}$ are*

$$\frac{1}{16}\|\gamma'\|^4 R_0^{1)} + \frac{5}{8}\|\gamma'\|^2 R_0^{3)} + R_0^{5)} = 0, \quad \frac{1}{16}\|\gamma'\|^4 R_0^{2)} + \frac{5}{8}\|\gamma'\|^2 R_0^{4)} + R_0^{6)} = 0$$

and so on.

Moreover, F^6 is naturally reductive and compact. For naturally reductive Riemannian 3-symmetric spaces in [12] it is proved the following theorem, which gives in this case a positive answer for the Ziller's conjecture already mentioned:

25 Theorem. *A naturally reductive compact* 3*-symmetric space with the property that all Jacobi fields vanishing at two points are isotropic is a* (*Hermitian*) *symmetric space.*

26 Theorem. *On the manifold* F_6 *the Jacobi operator along the geodesic* γ *has constant osculating rank* 4 *and it can be written as*

$$\begin{aligned} R_t &= R_0 + 10R_0^{2)} + \tfrac{16}{3}R_0^{4)} \\ &\quad -\tfrac{\sqrt{2}}{3}(R_0^{1)} + 8R_0^{3)})\sin\tfrac{t}{\sqrt{2}} + \tfrac{2}{3}(8R_0^{4)} + R_0^{2)})\sin\tfrac{t}{\sqrt{2}} \\ &\quad +8\tfrac{\sqrt{2}}{3}(R_0^{1)} + 2R_0^{3)})\sin\tfrac{t}{2\sqrt{2}} - \tfrac{32}{3}(R_0^{2)} + 2R_0^{4)})\cos\tfrac{t}{2\sqrt{2}}. \end{aligned}$$

4.5 The osculating rank of the Jacobi operator for the Kaplan's example

The property of natural reductibility is still a special case of a more general property:

" Each geodesic of $(M,<,>) = G/H$*, is an orbit of a one-parameter group of isometries* $\exp tZ$, $Z \in \mathfrak{g}$*"* (g. o.)

Riemannian homogeneous spaces $(M,<,>) = G/H$ with this property (g.o.) will be called (Riemannian) g.o. spaces. The extensive study of g.o. spaces start just with the Kaplan's papers [19], [20], because he gives the first counterexample of a Riemannian g.o. manifold which is not naturally reductive. This is a six-dimensional Riemannian nilmanifold with a two-dimensional center, one of the so-called *generalized Heisenberg groups*. Later this class of manifolds has provided a large number of counter-examples . The first known 7-dimensional compact examples of Riemannian g.o. manifolds which are not naturally reductive have been found in [11].

The basic definitions and results about generalized Heisenberg groups can be obtained in [19] and [20]. According to [26], the following was proved in [2]:

27 Theorem. *Let* $(M,<,>)$ *be a Riemannian g.o. space. Then there always exists a finite real number* r *such that the curvature operator has constant osculating rank* r.

28 Corollary. *Let* $(M,<,>)$ *be a Riemannian g.o. space. Then the curvature operator and the Jacobi operator have the same constant osculating rank* r.

Now we consider the Kaplan's example N^6 [19], [20]. Let $\mathfrak{n}$ be a six-dimensional vector space equipped with a scalar product $<\,,\,>$ and let

$$\{E_1, E_2, E_3, E_4, E_5, E_6\}$$

be an orthonormal basis. The elements E_5 and E_6 span the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{n}$, whose structure is given by the following relations.

$$\begin{array}{rclrclrcl} [E_1, E_2] &=& 0, & [E_1, E_3] &=& E_5, & [E_2, E_3] &=& E_6, \\ [E_1, E_4] &=& E_6, & [E_2, E_4] &=& -E_5, & [E_3, E_4] &=& 0, \end{array}$$

$$[E_k, E_5] = [E_k, E_6] = 0, \quad k = 1, \ldots, 4, \quad [E_5, E_6] = 0.$$

The following results can be found in [2].

29 Corollary. *The osculating rank of the Jacobi operator R_t is constant in the Kaplan's example.*

Working with the covariant derivatives of the Jacobi operator as in [26], we have:

30 Lemma. *The relations satisfied among the n-covariant derivatives for $n = 1, \ldots, 6$ of the $(0,4)$-Jacobi operator along the arbitrary geodesic γ of the Kaplan's example are*

$$\frac{1}{4}\|\gamma'\|^4 R_0^{1)} + \frac{5}{4}\|\gamma'\|^2 R_0^{3)} + R_0^{5)} = 0, \quad \frac{1}{4}\|\gamma'\|^4 R_0^{2)} + \frac{5}{4}\|\gamma'\|^2 R_0^{4)} + R_0^{6)} = 0$$

and so on.

31 Theorem. *The Jacobi operator along the geodesic γ in the Kaplan's example has constant osculating rank 4 and it can be written as*

$$\begin{array}{rcl} R_t &=& (R_0 + 5R_0^{2)} + 4R_0^{4)}) \\ && -\frac{1}{3}(4R_0^{3)} + R_0^{1)}) \sin\frac{t}{2} - \frac{16}{3}(R_0^{4)} + R_0^{2)} \cos\frac{t}{2} \\ && -\frac{1}{3}(R_0^{1)} + 4R_0^{3)}) \sin t + \frac{1}{3}(R_0^{2)} + 4R_0^{4)}) \cos t. \end{array}$$

32 Proposition. *The Jacobi tensor field A_t along the geodesic γ in the Kaplan's example with initial values $A_0 = 0$, $A_0^{1)} = I$ is given by*

$$A_t = \sum_{k=0}^{\infty} \frac{1}{k!}\beta_k(0)t^k,$$

where $\alpha_0(t) = \alpha_1(t) = \beta_0(t) = 0$, $\beta_1(t) = I$ and

$$\beta_k(t) = \alpha_{k-1}(t) + \beta'_{k-1}(t), \quad \alpha_k(t) = \alpha'_{k-1}(t) - R_t\beta_{k-1}(t),$$

for $k \geq 2$*. Moreover, the coefficients* $\beta_k(0)$ *are only functions from* R_0, $R_0^{1)}$, $R_0^{2)}$, $R_0^{3)}$ *and* $R_0^{4)}$.

33 Remark. For this example, in [7] the authors obtained the expression for the Jacobi vector fields by a different method.

Open question. *What is the geometrical meaning of the constant rank of the Jacobi operator on a Riemannian homogeneous space?*

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