Note di Matematica
Note Mat. 1(2008), suppl. n. 1, 147-168
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1supplp147
http://siba2.unile.it/notemat

Survey on homogeneous geodesics

Zdeněk Dušek[i]
Department of Algebra and Geometry, Palacky University,
Tomkova 40, 77900 *Olomouc, Czech Rebublic,*
dusek@prfnw.upol.cz

Received: 30/10/07; accepted: 6/2/08.

Abstract. A survey on homogeneous geodesics on homogeneous Riemannian and pseudo-Riemannian manifolds, Riemannian and pseudo-Riemannian g.o. spaces and pseudo-Riemannian almost g.o. spaces is presented.

Keywords: pseudo-Riemannian homogeneous space, homogeneous geodesic, g.o. space, almost g.o. space, geodesic graph

MSC 2000 classification: primary 53C30, secondary 53C35, 53C50

Dedicated to Professor Oldřich Kowalski on the occasion of his 70th birthday.

1 Introduction

The study of homogeneous geodesics (see Definition 1) on homogeneous Riemannian manifolds started a long time ago, in the works of E. Vinberg [39], B. Kostant [23], O. Kowalski and L. Vanhecke [29]. The big development of this study was done by O. Kowalski and his collaborators. In [28], O. Kowalski and J. Szenthe proved that any homogeneous Riemannian manifold admits at least one homogeneous geodesic. (For a Lie group this result was proved earlier by J. Kajzer in [22].) They also proved that a homogeneous Riemannian manifold $M = G/H$ with semisimple group G admits $n = \dim(M)$ mutually orthogonal homogeneous geodesics. Further, O. Kowalski, S. Nikčević and Z. Vlášek in [27] and R. Marinosci in [33] studied 3-dimensional Lie groups with Riemannian metrics and they illustrated the various possibilities for the number of homogeneous geodesics through the origin. In particular, for some of these examples, there is just one homogeneous geodesic through the origin. In [30], O. Kowalski and Z. Vlášek found such an example in arbitrary dimension $n \geq 4$, which proved that the first result from [28] cannot be improved in general.

A homogeneous Riemannian manifold whose all geodesics are homogeneous

[i]The author was supported by the grant GAČR 201/05/2707 and by the research project MSM 6198959214 financed by MŠMT.

is called g.o. manifold. It is well known that all Riemannian symmetric spaces and more generally all naturally reductive spaces are g.o. manifolds. Some years ago, it was generally believed that every g.o. manifold is naturally reductive. The first counter-example was given by A. Kaplan in [21]. The extensive study of g.o. manifolds started just with this paper. O. Kowalski and L. Vanhecke proved in [29], that up to dimension 5, every g.o. manifold is naturally reductive and they also classified all 6-dimensional g.o. manifolds which are not naturally reductive. In dimension 7, the examples of g.o. spaces which are not naturally reductive were given by C. Gordon in [20] and by the present author, O. Kowalski and S. Nikčević in [18]. One of the methods for describing g.o. spaces is based on the concept of geodesic graphs and the degree of a g.o. manifold. For naturally reductive manifolds, the degree is equal to zero. For the 6-dimensional and 7-dimensional examples mentioned above, the degree is equal to 2. In [13], the present author and O. Kowalski studied the 13-dimensional generalized Heisenberg group, which is a g.o. manifold, and they proved that its degree is equal to 3. In [11], the present author proved that the degree of the flag g.o. manifold SO(7)/U(3), is equal to 4.

The study of homogeneous geodesics on pseudo-Riemannian homogeneous manifolds started with the papers [19] and [35], written by physicists J. Figueroa-O'Farrill, P. Meessen, and S. Philip. They stated the Geodesic Lemma in the generalized pseudo-Riemannian version, but they did not give the proof. The refined, mathematical, formulation of the Geodesic Lemma with the proof was given by the present author and O. Kowalski in [14]. In the pseudo-Riemannian situation, for null (light-like) homogeneous geodesics, we observe a new phenomenon which is not present in the Riemannian situation. Some of these null homogeneous geodesics require a reparametrization and some do not (see Lemma 3 and Example 29 later). In [7] and [8], G. Calvaruso and R. Marinosci studied homogeneous geodesics on 3-dimensional Lie groups with Lorentzian metrics. These groups provide examples which admit various number of homogeneous geodesics. The most interesting examples will be presented in Section 5.

In [15], [16] and [10], the present author and O. Kowalski generalized the notion of a (Riemannian) g.o. space to the pseudo-Riemannian situation. Some suggestions in this direction were made by P. Meessen in the private communications. In fact, there are two different ways of this generalization. In the first one, the isotropy group remains compact, the corresponding pseudo-Riemannian homogeneous space is a g.o. space and geodesic graph has similar properties as in the Riemannian case. The examples in dimensions 6 and 7 were described in [15]. The second generalization is more interesting. Here the isotropy group is noncompact and the corresponding homogeneous space is no more a g.o. space. There is a nonempty set (of measure zero) in the tangent plane, such that geodesics with these initial vectors are not homogeneous. Hence we call these spaces almost g.o. spaces. The examples in dimensions 6 and 7 were given in [16] and [10]. In [16] it was shown that the behaviour of the geodesic graph is different than in the Riemannian case. In [10], various conjectures which compare pseudo-Riemannian g.o. spaces and almost g.o. spaces were formulated.

In this paper, the review of these results, important theorems, examples and conjectures and the relevant references according to the author's knowledge are presented. The order of sections is organized historically, with the only exception: In Section 4, pseudo-Riemannian g.o. spaces are included because they are similar to Riemannian g.o. spaces. Historically, they should be between Sections 5 and 6.

2 Preliminaries

In this section, we give the basic concepts and definitions. In Sections 3 and 4, special Riemannian versions are applicable, however, we give already here the general pseudo-Riemannian versions which will be necessary in Sections 5 and 6.

Let M be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a *homogeneous pseudo-Riemannian manifold*. Let $p \in M$ be a fixed point. If we denote by H the isotropy group at p, then M can be identified with the *homogeneous space* G/H. In general, there may exist more than one such transitive isometry group $G \subset I_0(M)$ and more presentations of M as a homogeneous space. For any fixed choice $M = G/H$, G acts effectively on G/H from the left. The pseudo-Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is then called a *pseudo-Riemannian homogeneous space*.

If the metric g is positive definite, then $(G/H, g)$ is always a *reductive* homogeneous space in the following sense: we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively, and we consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. We remark already here that if $\mathfrak{m}$ itself is a Lie algebra, we will denote it by $\mathfrak{n}$. If the metric g is indefinite, the reductive decomposition may not exist (see for instance [19] for the example of nonreductive pseudo-Riemannian homogeneous space). For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi\colon G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM, we obtain a scalar product $\langle\, , \rangle$ on $\mathfrak{m}$. This scalar product is obviously

$\mathrm{Ad}(H)$-invariant.

The definition of a homogeneous geodesic is well-known in the Riemannian case (see, e.g., [29]). In the pseudo-Riemannian case, the necessary generalized version was given in [14]:

1 Definition. Let $M = G/H$ be a pseudo-Riemannian reductive homogeneous space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and p the basic point of G/H. The geodesic $\gamma(s)$ through the point p defined in an open interval J (where s is an affine parameter) is said to be homogeneous if there exists
1) a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J;
2) a vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in (-\infty, +\infty)$.
The vector X is then called a geodesic vector.

2 Remark. In the Riemannian situation, the diffeomorphism from the condition 1 is always the identity map on the real line and hence the definition can be formulated more simply. An example (pseudo-Riemannian) with the nontrivial diffeomorphism φ will be shown in Section 5.

The basic formula characterizing geodesic vectors in the pseudo-Riemannian case appeared in [19] and [35], but without a proof. The correct mathematical formulation with the proof was given in [14]:

3 Lemma (Geodesic Lemma)**.** *Let $M = G/H$ be a reductive homogeneous pseudo-Riemannian space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and p the basic point of G/H. Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ (the orbit of a one-parameter group of isometries) is a geodesic curve with respect to some parameter s if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle \tag{1}$$

for all $Z \in \mathfrak{m}$, where $k \in \mathbb{R}$ is some constant.
Further, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a (properly) pseudo-Riemannian space.

4 Remark. The reparametrization $s = e^{-kt}$ is exactly the diffeomorphism φ in Definition 1.

It is clear from Definition 1 and Lemma 3, that for a *homogeneous space* G/H, we consider the isometry group G and homogeneous geodesics are considered with respect to this group. For a *homogeneous manifold* $M = G/H$, there may exist another expression $M = G'/H'$, where $G \subsetneq G'$. To investigate the properties of the *homogeneous manifold* M, it is necessary to consider the expression $M = G'/H'$, where G' is the identity component of the *maximal isometry group* ($G' = I_0(M)$).

3 Homogeneous geodesics on homogeneous Riemannian manifolds

In [22], V. Kajzer proved that every pseudo-Riemannian group space (G, g) admits at least one homogeneous geodesics through the identity element $e \in G$. In [28], O. Kowalski and J. Szenthe proved the following generalizations:

5 Theorem. *A homogeneous Riemannian manifold (M, g) admits at least one homogeneous geodesic through the origin $o \in M$.*

6 Theorem. *A homogeneous Riemannian manifold $(M = G/H, g)$ of dimension n with semi-simple group G admits n mutually orthogonal homogeneous geodesics through the origin $o \in M$.*

Let us now study the situation with a Lie group in dimension 3, as in [27] and [33]. First, we recall the results by J. Milnor:

7 Proposition ([34])**.** *Let (G, g) be a 3-dimensional non-unimodular Lie group equipped with a left-invariant Riemanian metric and let $\mathfrak{g}$ be the corresponding Lie algebra. Then there is an orthonormal basis $\{X, Y, Z\}$ of $\mathfrak{g}$ such that the multiplication table has the form*

$$[X, Y] = \alpha Y + \beta Z, \qquad [X, Z] = \gamma Y + \delta Z, \qquad [Y, Z] = 0, \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers such that $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$.

The basis $\{X, Y, Z\}$ also diagonalizes the Ricci form and the original paper contains the expressions for the principal Ricci curvatures. See [34] or [27] for details. In general, the principal Ricci curvatures are distinct.

8 Proposition ([34])**.** *Let (G, g) be a 3-dimensional unimodular Lie group equipped with a left-invariant Riemanian metric and let $\mathfrak{g}$ be the corresponding Lie algebra. Then there is an orthonormal basis $\{X, Y, Z\}$ of $\mathfrak{g}$ such that the multiplication table has the form*

$$[X, Y] = \lambda_1 Z, \qquad [Y, Z] = \lambda_2 X, \qquad [Z, X] = \lambda_3 Y. \tag{3}$$

The basis $\{X, Y, Z\}$ also diagonalizes the Ricci form and the original paper contains the expressions for the principal Ricci curvatures. See [34] or [33] for details. Again, the principal Ricci curvatures are distinct, in general.

For non-unimodular Lie groups, O. Kowalski, S. Nikčević and Z. Vlášek proved the following:

9 Theorem ([27])**.** *Let (G, g) be a 3-dimensional non-unimodular Lie group equipped with a left-invariant Riemanian metric and let $\mathfrak{g}$ be the corresponding Lie algebra with the multiplication table given by the formulas (2). Let $\alpha, \beta, \gamma, \delta$ be such that all principal Ricci curvatures are distinct. Denote $D = (\beta + \gamma)^2 -$*

$4\alpha\delta$. *Then, up to a reparametrization, the space* (G, g) *admits*
a) *just one homogeneous geodesic through a point if* $D < 0$*;*
b) *just two homogeneous geodesics through a point if* $D = 0$*; they are mutually orthogonal;*
c) *just three homogeneous geodesics through a point if* $D > 0$*; they are linearly independent but never mutually orthogonal.*

The case with only two distinct principal Ricci curvatures was investigated by R. Marinosci:

10 Theorem ([33]). *Let* (G, g) *be a* 3*-dimensional non-unimodular Lie group equipped with a left-invariant Riemanian metric and with two distinct principal Ricci curvatures.*
If both principal curvatures are nonzero, then there exist always three linearly independent homogeneous geodesics through each point, they are never mutually orthogonal and there are no other homogeneous geodesics.
If one principal curvature is equal to zero, then the geodesic vectors form a 2*-dimensional subspace of the Lie algebra* $\mathfrak{g}$*, i.e., there are infinitely many homogeneous geodesics through each point but every three of them are linearly dependent.*

For the unimodular Lie group, O. Kowalski, S. Nikčević and Z. Vlášek investigated in [27] only one special situation, the general situation was described by R. Marinosci:

11 Theorem ([33]). *In a* 3*-dimensional, connected and unimodular Lie group* G *endowed with a left invariant metric* g*, there always exist three mutually orthogonal homogeneous geodesics through each point. Moreover, if all* λ_i *in formulas* (3) *are distinct, then there are no other homogeneous geodesics.*

Further, in [27], the authors study homogeneous geodesics on the 4-dimensional homogeneous Riemannian manifold $(M = G/H, g)$, where G is the 5-dimensional group of equiaffine transformations of an Euclidean space and H is the subgroup of all rotations of the plane around the origin. The manifold itself is diffeomorphic to $\mathbb{R}^4$ and at the Lie algebra level it can be described with the orthonormal basis $\{X_1, Y_1, X_2, Y_2\}$ of $\mathfrak{m}$, the generator B of $\mathfrak{h}$ and the Lie bracket on $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ given by the formulas

$$\begin{array}{lll} [X_1, Y_1] = 0, & [X_1, X_2] = -X_1, & [X_1, Y_2] = X_1, \\ [Y_1, X_2] = Y_1, & [Y_1, Y_2] = X_1, & [X_2, Y_2] = -2B, \\ [B, X_1] = -Y_1, & [B, Y_1] = X_1, & \\ [B, X_2] = 2Y_2, & [B, Y_2] = -2X_2 & \end{array} \tag{4}$$

(see [24], [25] for details about this manifold). The result is the following:

12 Theorem ([27]). $(\mathbb{R}^4, g)$ *admits a continuum of quadruplets of linearly* *independent homogeneous geodesics through the origin o but never an orthogonal quadruplet.*

In [6], G. Calvaruso, O. Kowalski and R. Marinosci study a class of unimodular solvable Lie groups (G_n, g) with left-invariant metric in arbitrary odd dimension $2n+1$, introduced by M. Božek in [3]. Their result is connected with the well-known conjecture on Hadamard matrices:

13 Conjecture. *A Hadamard matrix of degree $k > 2$ exists if and only if k is divisible by four.*

14 Theorem ([6])**.** *Suppose that the conjecture about Hadamard matrices is true. Then the "Božek space" (G_n, g) of dimension $2n+1$ has the following maximal number of mutually orthogonal geodesics through a point:*
(i) *exactly $n+1$ geodesics if $n+1$ is odd;*
(ii) *exactly $n+2$ geodesics if $n+1$ is even but not divisible by 4;*
(iii) $2n+1 = \dim G$ *geodesics if $n+1$ is divisible by 4.*

In [30], O. Kowalski and Z. Vlášek constructed the following example: Consider for each $n \geq 3$ a Lie algebra $\mathfrak{g}_n$ of dimension $n+1$ given with respect to the basis $\{X_1, \dots, X_{n+1}\}$ by the multiplication table

$$\begin{aligned} [X_i, X_j] &= 0 \quad \text{for} \quad i, j = 1, \dots, n, \\ [X_{n+1}, X_i] &= a_i X_i + X_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1, \\ [X_{n+1}, X_n] &= a_n X_n, \end{aligned} \tag{5}$$

where $a_1, \dots, a_{n+1}$ are arbitrary parameters. Define a scalar product on $\mathfrak{g}_n$ for which the above basis is orthonormal. The family of the algebras $\mathfrak{g}_n$ give rise to an n-parameter family of solvable Lie groups G_n with an invariant Riemannian metrics g. We can assume that G_n is always diffeomorphic to $\mathbb{R}^{(n+1)}$. In [30], it is shown that we can choose the parameters $a_1, \dots, a_{n+1}$ such that all principal Ricci curvatures of (G_n, g) are distinct and hence the group G_n acting on itself by the left translations is the identity component of the full isometry group. Hence, the group G_n admits only the expression $G_n = G_n/\{e\}$ and all geodesic vectors belong to $\mathfrak{g}_n$. Then, by applying Geodesic Lemma, we obtain:

15 Theorem ([30])**.** *For the space (G_n, g) as above, the parameters $a_1, \dots, a_{n+1}$ can be chosen in a way that all geodesic vectors are multiples of the vector X_{n+1}. Consequently, there is (up to a reparametrization) only one homogeneous geodesic through each point.*

4 Riemannian g.o. manifolds

In this section, we present results about Riemannian g.o. spaces and g.o. manifolds. It appears that pseudo-Riemannian g.o. spaces (with compact isotropy group) have similar properties, hence we adapt the notation also to this situation and we include pseudo-Riemannian examples here.

16 Definition. A pseudo-Riemannian homogeneous space $(G/H, g)$ is called a g.o. space if every geodesic of $(G/H, g)$ is homogeneous. A homogeneous pseudo-Riemannian manifold (M, g) is a g.o. manifold if $M = G/H$ is a g.o. space for $G = I_0(M)$. Here "g.o." means "geodesics are orbits".

17 Remark. If a homogeneous manifold M admits several transitive groups of isometries, and hence several expressions $M = G/H$ and $M = G'/H'$ as a homogeneous space, it may happen that G/H is not a g.o. space and G'/H' (where $G \subsetneq G'$) is a g.o. space (see [18] for an example).

One of the techniques used for the characterization of g.o. spaces is based on the concept of "geodesic graph". The original idea (not using any explicit name) comes from J. Szenthe [38]:

18 Definition. Let $(G/H, g)$ be a pseudo-Riemannian g.o. space with compact isotropy group H and let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. A geodesic graph is an $\mathrm{Ad}(H)$-equivariant map $\eta\colon \mathfrak{m} \to \mathfrak{h}$ which is rational on an open dense subset U of $\mathfrak{m}$ and such that $X + \eta(X)$ is a geodesic vector for each $X \in \mathfrak{m}$.

According to Lemma 10 in [38], for every reductive g.o. space $(G/H, g)$ as above, there exists at least one geodesic graph. The construction of canonical and general geodesic graphs is based on Geodesic Lemma and is described in details in [26] or [13]. We will show the construction of the canonical geodesic graph on the example later in this section. On the open dense subset U of $\mathfrak{m}$, with respect to a basis $\{E_1, \dots, E_n\}$ of $\mathfrak{m}$ and a basis $\{F_1, \dots, F_h\}$ of $\mathfrak{h}$, the components of a geodesic graph η are rational functions of the coordinates on $\mathfrak{m}$. They are of the form $\eta_k = P_k/P$, where P_k and P are homogeneous polynomials and $\deg(P_k) = \deg(P) + 1$.

19 Definition. Let $(G/H, g)$ be a Riemannian g.o. space and let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. Let $E_1, \dots E_n$ and $F_1, \dots F_h$ be the bases of $\mathfrak{m}$ and $\mathfrak{h}$, respectively. Let $\eta\colon \mathfrak{m} \to \mathfrak{h}$ be a geodesic graph with the components $\eta_k = P_k/P$, where P_k and P have no nontrivial common factor. The degree of the geodesic graph η is $\deg(\eta) = \deg(P)$. The degree of the g.o. space G/H is

$$\deg(G/H) = \min\{\deg(\eta)\colon \eta \text{ is a geodesic graph on } G/H\}.$$

Let (M, g) be a homogeneous Riemannian manifold. The degree of M is

$$\deg(M) = \min\{\deg(G/H)\colon M = G/H\}.$$

20 Remark. We recall that a homogeneous space $(G/H, g)$ is *naturally reductive* if there exist a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ such that

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle [X, Z]_{\mathfrak{m}}, Y\rangle = 0 \tag{6}$$

for all $X, Y, Z \in \mathfrak{m}$. A homogeneous Riemannian manifold (M, g) is *naturally reductive* if $M = G/H$ is naturally reductive for $G = I_0(M)$. According to Proposition 2.10 in [29], the homogeneous space G/H is naturally reductive if there exists a linear geodesic graph (and, according to Definition 19 above, $\deg(G/H) = 0$).

In [29], O. Kowalski and L. Vanhecke obtained the following results:

21 Theorem ([29])**.** *Every simply connected Riemannian g.o. space* $(G/H,g)$ *of dimension* $n \leq 5$ *is a naturally reductive Riemannian manifold.*

22 Remark. In dimension 5, there are examples such that G/H is a g.o. space of degree 2, but it becomes naturally reductive if we extend the group G (see [29] or [26]).

23 Theorem ([29])**.** *The following* 6*-dimensional simply connected Riemannian g.o. spaces* (*and only those*) *are never naturally reductive:*
(i) (M, g) *is a two-step nilpotent Lie group with* 2*-dimensional center, provided with a left-invariant Riemannian metric such that the maximal connected isotropy group is isomorphic to* $\mathrm{SU}(2)$ *or* $\mathrm{U}(2)$*. All these Riemannian g.o. spaces depend on three real parameters;*
(ii) $(\widetilde{M}, g)$ *is the universal covering space of a homogeneous Riemannian manifold of the form* $M = \mathrm{SO}(5)/\mathrm{U}(2)$ *or* $M = \mathrm{SO}(1, 4)/\mathrm{U}(2)$*, where* $\mathrm{SO}(5)$*, or* $\mathrm{SO}(1, 4)$*, respectively, is the identity component of the full isometry group. In each case, all admissible Riemannian metrics depend on two real parameters.*

We will show here the 6-dimensional nilpotent example in the generalized pseudo-Riemannian form, as it was presented in [15]. For $\varepsilon_1 = \varepsilon_2 = 1$, we obtain the Riemannian examples considered in [29], [26] and in the first part of Theorem 23. For $\varepsilon_1 = \varepsilon_2 = 1$, $\alpha = \gamma = 1$ and $\beta = 0$, we obtain the first example of a g.o. space which is not naturally reductive, given by A. Kaplan in [21].

24 Example. Let us consider the 6-dimensional vector space $\mathfrak{n}$ with the pseudo-orthonormal basis $\{E_1, \dots, E_4, Z_1, Z_2\}$ with the signature $(1,1,1,1,\varepsilon_1,\varepsilon_2)$, where $\varepsilon_1, \varepsilon_2 = \pm 1$. We denote A_{ij} (for $1 \leq i < j \leq 4$) the elements of $\mathfrak{so}(\mathfrak{n})$, with the corresponding action given by the formulas

$$A_{ij}(E_k) = \delta_{ik}E_j - \delta_{jk}E_i \qquad \text{for } k = 1, \dots, 4. \tag{7}$$

Further, we denote

$$A = A_{34} - A_{12}, \quad B = A_{13} + A_{24}, \quad C = A_{14} - A_{23}. \tag{8}$$

We notice the Lie bracket relations

$$[A,B]=2C, \quad [B,C]=2A, \quad [C,A]=2B \tag{9}$$

and we consider the algebra $\mathfrak{h}=\mathrm{span}(A,B,C)\simeq\mathfrak{su}(2)$ of the operators on $\mathfrak{n}$. We put $\mathfrak{g}=\mathfrak{n}+\mathfrak{h}$ and we define the Lie bracket on $\mathfrak{g}$ by the additional relations

$$\begin{aligned} [E_1,E_2]&=0, & [E_2,E_3]&=\beta Z_1+\gamma Z_2,\\ [E_1,E_3]&=\alpha Z_1, & [E_2,E_4]&=-\alpha Z_1,\\ [E_1,E_4]&=\beta Z_1+\gamma Z_2, & [E_3,E_4]&=0,\end{aligned}$$

$$[Z_1,E_i]=[Z_2,E_i]=[Z_1,Z_2]=0 \qquad \text{for } i=1,\dots,4 \tag{10}$$

for arbitrary parameters α,β,γ ($\alpha\neq 0\neq\gamma$). The scalar product on $\mathfrak{n}$ is $\mathrm{ad}(H)$-invariant for all possibilities of $\varepsilon_1,\varepsilon_2=\pm1$. If we consider N as the unique connected and simply connected group whose Lie algebra is $\mathfrak{n}$, $H=\mathrm{SU}(2)$ and $G=N\rtimes\mathrm{SU}(2)$, we obtain a 3-parameter family of invariant pseudo-Riemannian metrics on the homogeneous space G/H.

With respect to the above bases of $\mathfrak{n}$ and $\mathfrak{h}$, we can write each vector $X\in\mathfrak{n}$ in the form $X=x_1E_1+\cdots+x_4E_4+z_1Z_1+z_2Z_2$ and each vector $\xi(X)\in\mathfrak{h}$ in the form $\xi(X)=\xi_1A+\xi_2B+\xi_3C$. In this way, we identify these vectors with the arithmetic vectors $X=(x_1,\dots,x_4,z_1,z_2)$ and $\xi(X)=(\xi_1,\dots,\xi_3)$. The components of the canonical geodesic graph, which is denoted by ξ instead of η, are

$$\begin{aligned}\xi_1 &= \frac{-2\alpha\varepsilon_1z_1(x_1x_4+x_2x_3)+2(\beta\varepsilon_1z_1+\gamma\varepsilon_2z_2)(x_1x_3-x_2x_4)}{\|x\|^2},\\ \xi_2 &= \frac{\alpha\varepsilon_1z_1(x_1^2-x_2^2+x_3^2-x_4^2)+2(\beta\varepsilon_1z_1+\gamma\varepsilon_2z_2)(x_1x_2+x_3x_4)}{\|x\|^2},\\ \xi_3 &= \frac{2\alpha\varepsilon_1z_1(x_3x_4-x_1x_2)+(\beta\varepsilon_1z_1+\gamma\varepsilon_2z_2)(x_1^2-x_2^2-x_3^2+x_4^2)}{\|x\|^2},\end{aligned} \tag{11}$$

where we put $\|x\|^2=x_1^2+x_2^2+x_3^2+x_4^2$. (See [26] or [15] for details.) The formulas (11) describe the canonical geodesic graph of degree 2 on the subset $U=\{X\in\mathfrak{n};\ \|x\|^2\neq 0\}$. For $\|x\|^2=0$ we can put $\xi(X)=0$. It was proved in [15] that the geodesic graph ξ is continuous at the origin. On the subset $V=\{(0,0,0,0,z_1,z_2)\in\mathfrak{n};\ z_1^2+z_2^2\neq 0\}$ it cannot be continuous: For example, for $z_1\neq 0$ we can consider the component ξ_2 and the curves $\gamma_1(t)=(t,0,0,0,z_1,z_2)$ and $\gamma_2(t)=(0,t,0,0,z_1,z_2)$, which lie in U for $t\neq 0$ and go through $(0,0,0,0,z_1,z_2)\in V$ for $t=0$. If we calculate limits of the component ξ_2 along γ_1 and along γ_2, we obtain

$$\lim_{t\to 0}\xi_2(\gamma_1(t)) \;=\; \lim_{t\to 0}\frac{\alpha\varepsilon z_1t^2+0}{t^2}=\alpha\varepsilon z_1,$$

$$\lim_{t\to 0} \xi_2(\gamma_2(t)) \quad = \quad \lim_{t\to 0} \frac{-\alpha\varepsilon z_1 t^2 + 0}{t^2} = -\alpha\varepsilon z_1 \tag{12}$$

and we see that the limits are different. Other points in V can be treated similarly.

Because ξ is the unique geodesic graph on G/H, we obtain $\deg(G/H) = 2$. In the Riemannian case, if $\alpha^2 \neq \gamma^2$ or $\beta \neq 0$, then G is the identity component of the maximal isometry group. If $\alpha^2 = \gamma^2$ and $\beta = 0$, then G can be enlarged to $G' = N \rtimes \mathrm{U}(2)$. In the corresponding isotropy algebra $\mathfrak{h}' \simeq \mathfrak{u}(2)$, the additional operator is $D = 2\,B_{12} + A_{12} + A_{34}$ (where $B_{12}(Z_1) = Z_2$ and $B_{12}(Z_2) = -Z_1$). But, from Geodesic Lemma it follows that the component of any geodesic graph to this operator is zero. Hence also $\deg(G'/H') = 2$ and $\deg(M) = 2$ for any α, β, γ.

There are other examples of Riemannian g.o. manifolds of degree 2. Here we will only make a list of them:

– *H-type groups* (2-step nilpotent Lie groups with a special left-invariant Riemannian metric) of dimension $2 + 4n$ with 2-dimensional center and H-type groups of dimension $3 + 4n$ with 3-dimensional center. (For $n = 1$ we obtain the 6-dimensional example by A. Kaplan.) Geodesic graphs on these groups were described in [11].

– The 6-dimensional compact example $M = \mathrm{SO}(5)/\mathrm{U}(2)$ (and its dual $M = \mathrm{SO}(1,4)/\mathrm{U}(2)$) with a 2-parameter family of left-invariant Riemannian metrics from the second part of Theorem 23. Geodesic graphs were described in [26];

– The 7-dimensional nilpotent group $N = N \rtimes \mathrm{SU}(2)/\mathrm{SU}(2)$ with a 3-parameter family of left-invariant Riemannian metrics constructed by C. Gordon in [20]. Geodesic graph was described in [26];

– The 7-dimensional compact example $M = \big(\mathrm{SO}(5) \times \mathrm{SO}(2)\big)/\big(\mathrm{SU}(2) \times \mathrm{SO}(2)_\varphi\big)$ (and its dual $M = (\mathrm{SO}(1,4) \times \mathrm{SO}(2))/(\mathrm{SU}(2) \times \mathrm{SO}(2)_\varphi))$ with a 2-parameter family of left-invariant Riemannian metrics constructed by the present author, O. Kowalski and S. Nikčević in [18] and in special cases considered in [17].

25 Remark. The general Riemannian metrics in the first part of Theorem 23 correspond to the so called *modified H-type groups*. These manifolds were investigated by J. Lauret in [31]. He described which of modified H-type groups are g.o. manifolds. The result is a generalization of the result for H-type groups, which will be mentioned later in this section.

26 Remark. In [18], the authors stated a conjecture that in dimension 7, except the two kinds of examples in the above list, there are no other g.o. manifolds which are not naturally reductive.

In [15], the present author and O. Kowalski generalized the metrics on the above examples and obtained pseudo-Riemannian g.o. spaces (with compact isotropy group). The generalization of the metric was similar to the possibilities for $\varepsilon_1, \varepsilon_2$ in Example 24. Geodesic graphs on these g.o. spaces are similar to the geodesic graph described by the formulas (11). All geodesic graphs are continuous at the origin and for each example, there are points in $\mathfrak{m}$, such that geodesic graph is discontinuous there. These features will be described later in Conjecture 40.

In [13], the present author and O. Kowalski investigated the 13-dimensional generalized Heisenberg group (H-type group) with 5-dimensional center. H-type groups which are g.o. manifolds were classified by C. Riehm in [37]. There are 2 series and 5 additional examples. The 2 series with 2-dimensional or 3-dimensional center were mentioned earlier in the list of g.o. manifolds of degree 2. The dimensions of the 5 additional examples are 13, 14, 15, 23 and 31. (For modified H-type groups which are g.o. manifolds, the dimensions are the same, only the H-type metric is generalized to the family of modified H-type metrics in each case.) The 13-dimensional H-type group admits 2 transitive groups of isometries, $G = I_0(M)$ and $G' \subsetneq G$. The isotropy groups H and H' corresponding to G and G', respectively, are $H = \mathrm{SO}(5) \times \mathrm{SO}(2)$ and $H' = \mathrm{SO}(5)$. Hence, the group N admits two presentations $N = G/H$ and $N = G'/H'$ as a homogeneous space and both these spaces are g.o. spaces. In G'/H', geodesic graph is unique and $\deg(G'/H') = 6$. In G/H, there are more geodesic graphs. For the canonical geodesic graph ξ it holds $\deg(\xi) = 6$, but there is a general geodesic graph of degree 3. Hence, $\deg(M) = \deg(G/H) = 3$ (see [13] for details about general geodesic graphs). Unfortunately, the other examples of H-type groups cannot be described by this method. To solve the equations given by Geodesic Lemma, it is necessary to calculate big determinants. In dimension 13, this was at the limits of the computer possibilities.

In [1] and [2], D. Alekseevsky and A. Arvanitoyeorgos classified Riemannian flag manifolds which are g.o. manifolds. There are two series, namely $\mathrm{SO}(2n+1)/\mathrm{U}(n)$ and $\mathrm{Sp}(n)/\mathrm{U}(1) \cdot \mathrm{Sp}(n-1)$, for $n \geq 2$. For $n = 2$, both these manifolds coincide with the 6-dimensional compact example in the classification in Theorem 23. In [12], the present author described the second example in the first series, namely $\mathrm{SO}(7)/\mathrm{U}(3)$, explicitly in terms of Lie algebras and calculated the canonical geodesic graph. The group $\mathrm{SO}(7)$ is the maximal isometry group and the canonical geodesic graph ξ is unique in this example and hence it holds $\deg(M) = \deg(\mathrm{SO}(7)/\mathrm{U}(3)) = \deg(\xi) = 4$. For the example $\mathrm{Sp}(3)/\mathrm{U}(1) \cdot \mathrm{Sp}(2)$, there is not unique geodesic graph and the computation of the canonical geodesic graph is again too complicated for the computer.

An important observation shows that the denominator P in the components of a geodesic graph is always an algebraic invariant with respect to the action of H on $\mathfrak{m}$. In Example 24 and formulas (11) it is obvious. For the 13-dimensional

H-type group, in the canonical geodesic graph, there is an invariant of degree 6 and for the general geodesic graph, there is an invariant of degree 3. For the flag manifold SO(7)/U(3), there is an invariant of degree 4. All these invariants are more complicated than in the formula (11), but they can be expressed using the Hilbert basis of invariants. See [13] and [12] for details.

27 Conjecture ([13]). *For the components η_k of a geodesic graph written in the form $\eta_k = P_k/P$, where P_k and P have no nontrivial common factor, the denominator P is always an algebraic invariant.*

5 Homogeneous geodesics on homogeneous pseudo-Riemannian manifolds

As it was mentioned in Introduction, the first result about homogeneous geodesics on homogeneous pseudo-Riemannian manifolds was the generalization of Geodesic Lemma. The formula (1) first appeared in [19] and [35], the proof was given in [14]. The first examples were given by G. Calvaruso and R. Marinosci in [4], [7] and [8]. In [4], G. Calvaruso proved that any 3-dimensional complete, connected and simply connected homogeneous Lorentzian manifold is either symmetric or it is a Lie group with the left-invariant metric. Using the classification of 3-dimensional unimodular Lie groups given by S. Rahmani in [36] and the results on non-unimodular Lie algebras by L. Cordero and P. Parker in [9], he obtained the following theorem:

28 Theorem ([4]). *Let (M, g) be a 3-dimensional connected, simply connected, complete homogeneous Lorentzian manifold. If (M, g) is not symmetric, then $M = G$ is a 3-dimensional Lie group, g is the left-invariant metric and one of the following cases occurs:*
1) *If G is unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra $\mathfrak{g}$ of G is one of the following:*

$$\begin{aligned}\mathfrak{g}_1 \;:\; & [e_1, e_2] = \alpha e_1 - \beta e_3,\\ & [e_1, e_3] = -\alpha e_1 - \beta e_2,\\ & [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \qquad (\alpha \neq 0);\end{aligned}$$

$$\begin{aligned}\mathfrak{g}_2 \;:\; & [e_1, e_2] = \gamma e_2 - \beta e_3,\\ & [e_1, e_3] = -\beta e_2 + \gamma e_3,\\ & [e_2, e_3] = \alpha e_1, \qquad (\gamma \neq 0);\end{aligned}$$

$$\mathfrak{g}_3 \;:\; [e_1, e_2] = -\gamma e_3,$$

$$
\begin{aligned}
&[e_1, e_3] = -\beta e_2,\\
&[e_2, e_3] = \alpha e_1;
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{g}_4 \quad : \quad &[e_1, e_2] = -e_2 + (2\varepsilon - \beta)e_3,\\
&[e_1, e_3] = -\beta e_2 + e_3,\\
&[e_2, e_3] = \alpha e_1, \qquad (\varepsilon = \pm 1);
\end{aligned}
$$

2) *If G is non-unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra $\mathfrak{g}$ of G is one of the following:*

$$
\begin{aligned}
\mathfrak{g}_5 \quad : \quad &[e_1, e_2] = 0,\\
&[e_1, e_3] = \alpha e_1 + \beta e_2,\\
&[e_2, e_3] = \gamma e_1 + \delta e_2, \qquad \alpha + \delta \neq 0, \qquad \alpha\gamma + \beta\delta = 0;
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{g}_6 \quad : \quad &[e_1, e_2] = \alpha e_2 + \beta e_3,\\
&[e_1, e_3] = \gamma e_2 + \delta e_3,\\
&[e_2, e_3] = 0, \qquad \alpha + \delta \neq 0, \qquad \alpha\gamma - \beta\delta = 0;
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{g}_7 \quad : \quad &[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3,\\
&[e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3,\\
&[e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \qquad \alpha + \delta \neq 0, \qquad \alpha\gamma = 0.
\end{aligned}
$$

In [7] and [8], these cases were studied separately and for every algebra, the sets of homogeneous geodesics were described. The results are very variable, we will show here just two interesting cases which were treated separately in [14] and in [5]. In Example 29, we will show the Lie group with the Lorentzian metric which admits just two homogeneous geodesics through the origin, both these geodesics are light-like and for one of them we have $k = 0$ and for the other we have $k \neq 0$. Hence, for the second geodesic, the parameter of the orbit is not the affine parameter of the geodesic and it must be reparametrized. In Example 30, we will show the Lie group N which admits the presentation $N = G/H$ where initial vectors of homogeneous geodesics form just the light-cone and a hyperplane in the tangent space. In particular, $N = G/H$ is not a g.o. space, but all light-like geodesics are homogeneous (with nonzero k).

29 Example ([14])**.** Let us consider the 3-dimensional unimodular Lie group $G = E(1, 1)$ with a left-invariant Lorentzian metric whose Lie algebra $\mathfrak{g}$ admits the pseudo-orthonormal basis $\{E_1, E_2, E_3\}$ with E_3 timelike and the Lie bracket given by the formulas

$$
[E_1, E_2] = E_1, \qquad [E_1, E_3] = -E_1, \qquad [E_2, E_3] = E_2 + E_3. \tag{13}
$$

(This is a special case of the algebra $\mathfrak{g}_1$ above.) We consider G acting on itself as group of isometries by left translations. We see easily (from Geodesic Lemma) that geodesic vectors are just the vectors $X = \frac{E_2+E_3}{2}$ and $Z = \frac{E_2-E_3}{2}$, if we put, in the formula (1), $k = 0$ or $k = -1$, respectively. Both these vectors are light-like. We are going to investigate the orbits of the 1-parameter groups $\exp(tX)$ and $\exp(tZ)$.

The group G can be identified with the matrices of the form

$$\begin{pmatrix} e^{-x^3} & 0 & x^1 \\ 0 & e^{x^3} & x^2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

where (x^1, x^2, x^3) form a global coordinate system. The 1-parameter subgroups of G generated by the vectors X and Z are

$$\exp(tX) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \exp(tZ) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

The unit matrix corresponds to the origin $p = (0,0,0)$ of the coordinate system (x^1, x^2, x^3). The orbits of the subgroups (15) starting at the origin are

$$\gamma_1(t) = \exp(tX)(p) = (t,0,0), \qquad \gamma_2(t) = \exp(tZ)(p) = (0,0,t). \tag{16}$$

Because the nonzero components of the Levi-Civita connection are

$$\Gamma^1_{22} = -2e^{-3x^3}, \qquad \Gamma^2_{23} = \Gamma^2_{32} = -\Gamma^3_{33} = -1, \tag{17}$$

we obtain for the tangent vectors $\gamma_1'(t) = \frac{\partial}{\partial x^1}$ and $\gamma_2'(t) = \frac{\partial}{\partial x^3}$ the relations

$$\nabla_{\gamma_1'(t)}\gamma_1'(t) = \Gamma^k_{11}\frac{\partial}{\partial x^k} = 0, \qquad \nabla_{\gamma_2'(t)}\gamma_2'(t) = \Gamma^k_{33}\frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^3} = \gamma_2'(t). \tag{18}$$

We see that the curve γ_1 is a geodesic with the affine parameter t. The direct computation shows that the curve

$$\gamma_3(t) = (0,0,\log(t)) \tag{19}$$

is the reparametrization of the curve γ_2 by the affine parameter.

30 Example ([5]). Let us consider the 3-dimensional Lie algebra $\mathfrak{n}$ with the pseudo-orthonormal basis $\{E_1, E_2, E_3\}$ with E_3 timelike and the multiplication defined by the relations

$$[E_1, E_2] = 0, \quad [E_1, E_3] = \alpha E_1 + \beta E_2, \quad [E_2, E_3] = -\beta E_1 + \alpha E_2, \tag{20}$$

for $\alpha \neq 0$ (the special case of the algebra $\mathfrak{g}_5$ above). We denote by N the unique connected and simply connected Lie group corresponding to $\mathfrak{n}$. Now we introduce a linear operator A on $\mathfrak{n}$, which acts by the relations

$$A(E_1) = E_2, \quad A(E_2) = -E_1, \quad A(E_3) = 0. \tag{21}$$

This is the unique operator on $\mathfrak{n}$ which preserves the scalar product and the Lie algebra structure on $\mathfrak{n}$. We put $\mathfrak{h} = \mathrm{span}(A) \simeq \mathfrak{so}(2)$. Then, $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ is a reductive decomposition and the scalar product on $\mathfrak{n}$ induces a left-invariant Lorentzian metric g on $N = G/H = (N \rtimes \mathrm{SO}(2))/\mathrm{SO}(2)$.

Again, we write each vector $X \in \mathfrak{n}$ and each vector $\eta(X) \in \mathfrak{h}$ in the form $X = \sum x_i E_i$ and $\eta(X) = \eta_1 A$ and we consider the coordinates $(x_1, \ldots, x_3)$ on $\mathfrak{n}$ and (η_1) on $\mathfrak{h}$. Now, a "geodesic graph" on $\mathfrak{n}$ can be defined only on the hyperplane $x_1 = x_2 = 0$, for example as a zero map, and on the light-cone, by the formula

$$k = -\alpha x_3, \qquad \eta_1 = \beta x_3. \tag{22}$$

On the other hand, (N, g) is a space of constant sectional curvature $\alpha^2 > 0$. Having supposed N is connected and simply connected, it is a Lorentzian sphere of constant curvature and in particular, it is a symmetric space. It is well-known that its isometry group is $\mathrm{O}(1,3)$ and so, N can be realized as $N = G'/H'$, where $G' = I_0(M)$ is a 6-dimensional Lie group (the identity component of $\mathrm{O}(1,3)$) and H' is the identity component of the group $\mathrm{O}(1,2)$. For a symmetric space G'/H', all geodesics are homogeneous (with respect to the group G'). However, $\mathfrak{g}'$ can not be obtained by adding some derivations to $\mathfrak{g}$, hence the Lie group structure on G' is not compatible with the Lie group structure on G (and on N).

6 Pseudo-Riemannian almost g.o. spaces

Now we present the modification of Example 24 and we obtain the homogeneous space with the noncompact isotropy group. This example and other examples in dimensions 6 and 7 lead us to new definitions.

31 Example ([16])**.** Let us consider the 6-dimensional vector space $\mathfrak{n}$ with the pseudo-orthonormal basis $\{E_1, \ldots, E_4, Z_1, Z_2\}$ and with the signature $(-1, -1, 1, 1, \varepsilon, 1)$, where $\varepsilon = \pm 1$. Let us define the Lie bracket on $\mathfrak{n}$ by the relations

$$\begin{array}{lll} [E_1, E_2] = 0 \ , & [E_1, E_3] = Z_1 \ , & [E_1, E_4] = Z_2, \\ [E_2, E_3] = Z_2, & [E_2, E_4] = -Z_1, & [E_3, E_4] = 0, \end{array}$$

$$[Z_1, E_i] = [Z_2, E_i] = [Z_1, Z_2] = 0 \quad \text{for } i = 1, \ldots, 4. \tag{23}$$

We denote by N the unique connected and simply connected Lie group whose Lie algebra is $\mathfrak{n}$. Further, we denote by A_{ij} (for $1 \leq i < j \leq 4$) the endomorphisms of $\mathfrak{n}$ as before and we denote by $\bar{A}_{ij}$ (for $1 \leq i < j \leq 4$) the endomorphisms of $\mathfrak{n}$, with the corresponding action given by the formulas

$$\bar{A}_{ij}(E_k) = \delta_{ik}E_j + \delta_{jk}E_i \quad \text{for } k = 1, \dots, 4.$$

We consider the operators

$$A = A_{34} - A_{12}, \quad B = \bar{A}_{13} + \bar{A}_{24}, \quad C = \bar{A}_{14} - \bar{A}_{23} \tag{24}$$

whose Lie bracket satisfy the relations

$$[A, B] = 2C, \quad [B, C] = -2A, \quad [C, A] = 2B. \tag{25}$$

We obtain the isomorphism $\mathfrak{h} = \mathrm{span}(A, B, C) \simeq \mathfrak{so}(1, 2)$ and we choose $H = \mathrm{SO}(1, 2)$. It is easy to verify that the algebra $\mathfrak{h}$ acts on $\mathfrak{n}$ by derivations and the scalar product on $\mathfrak{n}$ is invariant with respect to this action (for both choices $\varepsilon = \pm 1$). Hence, the manifold N can be expressed as a homogeneous space G/H, where $G = N \rtimes H$.

In this case, the components of the canonical geodesic graph are

$$\begin{aligned}
\xi_1 &= \frac{2\varepsilon z_1(x_1x_4 + x_2x_3) - 2z_2(x_1x_3 - x_2x_4)}{-\|x\|^2}, \\
\xi_2 &= \frac{\varepsilon z_1(x_1^2 - x_2^2 - x_3^2 + x_4^2) + 2z_2(x_1x_2 - x_3x_4)}{-\|x\|^2}, \\
\xi_3 &= \frac{-2\varepsilon z_1(x_1x_2 + x_3x_4) + z_2(x_1^2 - x_2^2 + x_3^2 - x_4^2)}{-\|x\|^2}.
\end{aligned} \tag{26}$$

Here we put $\|x\|^2 = x_3^2 + x_4^2 - x_1^2 - x_2^2$. The above formulas describe the geodesic graph on $U = \{X \in \mathfrak{n};\ \|x\|^2 \neq 0\}$.

32 Remark. In accordance with Conjecture 27, $\|x\|^2$ is an invariant with respect to the transformation group $\mathrm{Ad}(H)$.

Further, we put $V = \mathfrak{n} \setminus U = \{X \in \mathfrak{n};\ \|x\|^2 = 0\}$. For a g.o. space, it must be possible to define a geodesic graph also on V. However, the next theorem shows that it is not possible in this case.

33 Theorem ([16]). *Let $(G/H, g)$ be the homogeneous space from Example 31. The subset $V = \mathfrak{n} \setminus U$ of $\mathfrak{n}$ can be decomposed as $V = V_0 + V_1$, where V_0 is an open dense subset of V. The geodesic graph can be defined on V_1 and it cannot be defined on V_0. In particular, because V_0 is nonempty, G/H is not a g.o. space (for each $\varepsilon = \pm 1$).*

34 Remark. If the denominator in the components (26) of geodesic graph ξ is zero, then geodesic graph can be defined only if all the numerators are zero as well (the set V_1). If the denominator is zero and at least one of the numerators is nonzero, then the system of equations given by the Geodesic Lemma and the formula (1) is unsolvable (the set V_0).

This observation lead us to the following definition:

35 Definition. A pseudo-Riemannian homogeneous space $(G/H, g)$ with the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is called an almost g.o. space if geodesic graph can be defined on the open dense subset $U \subset \mathfrak{m}$, but not on all $\mathfrak{m}$.

The next feature of the above example is the following:

36 Theorem ([16])**.** *Let $(G/H, g)$ be the homogeneous space from Example* 31*. Let us denote by N the null-cone in $\mathfrak{n}$. For $\varepsilon = 1$, it holds $N \cap V = N \cap V_1$. In particular, all null geodesics are homogeneous. For $\varepsilon = -1$, the set $N \cap V_0$ is nonempty and there are null geodesics which are not homogeneous.*

37 Definition ([32], reformulated)**.** A pseudo-Riemannian homogeneous space $(G/H, g)$ with the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is called a n.g.o. space if geodesic graph can be defined on the null cone $N \subset \mathfrak{m}$, but not on all $\mathfrak{m}$. Here "n.g.o." means "null geodesics are orbits".

38 Remark. It is worth mentioning that the homogeneous space G/H from Example 31 (for $\varepsilon = 1$) is a n.g.o. space and an almost g.o. space. The homogeneous space from Example 30 is a n.g.o. space and not an almost g.o. space.

The last feature of the above example are infinite limits of geodesic graph:

39 Theorem ([16])**.** *Let $(G/H, g)$ be the homogeneous space from Example* 31*. For any vector $\mathring{X} \in V$, there is a curve $\gamma(t)$ with the values in $\mathfrak{n}$ and defined on an interval $\langle 0, \delta)$ such that $\gamma(0) = \mathring{X}$, $\gamma(t) \in U$ for $t \in (0, \delta)$ and the limit of $\xi_1(\gamma(t))$ is infinite for $t \to 0_+$.*

For example, let $\mathring{X} = (x_1, x_2, x_3, x_4, z_1, z_2) \in V_0$ and let us consider the curve

$$\gamma = (x_1 + t^2, x_2, x_3, x_4, z_1 + t, z_2), \tag{27}$$

which lies in U for $t \neq 0$ and which goes through $\mathring{X} \in V$ for $t = 0$. Now we calculate

$$\begin{aligned}
\xi_1(\gamma_1(t)) &= \frac{2\varepsilon(z_1 + t)((x_1 + t^2)x_4 + x_2x_3) + 2z_2(-(x_1 + t^2)x_3 + x_2x_4)}{(x_1 + t^2)^2 + x_2^2 - x_3^2 - x_4^2} = \\
&= \frac{2\varepsilon(z_1 + t)(x_4t^2 + x_1x_4 + x_2x_3) + 2z_2(-x_3t^2 - x_1x_3 + x_2x_4)}{t^4 + 2x_1t^2 + x_1^2 + x_2^2 - x_3^2 - x_4^2} = \\
&= \frac{2\varepsilon t^3x_4 + 2t^2(\varepsilon z_1x_4 - z_2x_3) + 2\varepsilon t(x_1x_4 + x_2x_3) + P_1}{t^4 + 2x_1t^2},
\end{aligned} \tag{28}$$

where P_1 is exactly the numerator in the component of ξ_1 in formulas (26). We see that the limit of $\xi_1(\gamma(t))$ for $t \to 0$ is infinite for $P_1 \neq 0$. Other points in V can be treated similarly, see [16] for details.

In [10], the present author modified other 6-dimensional and 7-dimensional g.o. spaces and obtained other homogeneous spaces with the noncompact isotropy group. There are 6-dimensional examples
- $\mathrm{SO}(2,3)/\mathrm{U}(1,1)$ with the signature $(2,4)$;
- $\mathrm{SO}(2,3)/(\mathrm{SU}(1,1) \times \mathbb{R})$ with the signature $(3,3)$;

and 7-dimensional examples
- $(\mathrm{SO}(2,3) \times \mathrm{SO}(2))/(\mathrm{SU}(1,1) \times \mathrm{SO}(2)_\varphi)$, with signatures $(2,5)$ or $(3,4)$;
- $(\mathrm{SO}(2,3) \times \mathbb{R})/(\mathrm{SU}(1,1) \times \mathbb{R}_\varphi)$, with the signature $(3,4)$.

All these homogeneous spaces are almost g.o. spaces. The 6-dimensional example with the signature $(2,4)$ and the 7-dimensional example with the signature $(2,5)$ are n.g.o. spaces, other examples are not n.g.o. spaces. For all these examples, the components of geodesic graphs are very similar to the components of geodesic graph (26). It is not hard to find curves analogous to curves (27) with similar properties. If we compare g.o. spaces and geodesic graphs on them with these almost g.o. spaces, we are led to the following conjecture:

40 Conjecture ([10]). *Let $(G/H, g)$ be a pseudo-Riemannian homogeneous space, let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be a fixed reductive decomposition. Let ξ be the geodesic graph which is nonlinear and unique on an open dense subset $U \subset \mathfrak{m}$.*
1) *If the isotropy group H is compact, then geodesic graph can be defined also on $V = \mathfrak{m} \setminus U$ and $(G/H, g)$ is a g.o. space. Further, geodesic graph is continuous at the origin and if geodesic graph ξ is discontinuous at some point $\mathring{X} \in V$, then the limits of $\xi(X)$ for $X \to \mathring{X}$ are finite.*
2) *If the isotropy group H is noncompact, then there is a set $V_0 \subset V = \mathfrak{m} \setminus U$, where geodesic graph cannot be defined and hence $(G/H, g)$ is not a g.o. space, but only an almost g.o. space. Further, for any point $\mathring{X} \in V$, there is a curve $\gamma(t)$ with the values in $\mathfrak{m}$ and defined on an interval $\langle 0, \delta)$ such that $\gamma(0) = \mathring{X}, \gamma(t) \in U$ for $t \in (0, \delta)$ and the limit of some component $\xi_k(\gamma(t))$ of geodesic graph ξ is infinite for $t \to 0_+$.*

For all examples of pseudo-Riemannian homogeneous spaces with noncompact isotropy group, there is a family of metrics. For some of these metrics, we observed that a geodesic graph can be defined linearly on all $\mathfrak{m}$ (and the existence of a linear geodesic graph implies natural reductivity).

41 Conjecture ([10]). *If a pseudo-Riemannian reductive homogeneous space G/H with noncompact isotropy group H is a g.o. space, then it is naturally reductive.*

And finally, for all our examples of almost g.o. spaces, for all null homogeneous geodesics it holds $k = 0$. There are examples of homogeneous spaces where null homogeneous geodesics with nonzero k exist (see Example 30), but these examples are not g.o. spaces or almost g.o. spaces.

42 Conjecture ([10]). *Let G/H be a pseudo-Riemannian g.o. space or almost g.o. space. For all null homogeneous geodesics it holds $k = 0$ in Lemma 3.*

References

[1] Alekseevsky, D. and Arvanitoyeorgos, A.: *Metrics with homogeneous geodesics on flag manifolds*, Comment. Math. Univ. Carolinae, **43**, 2 (2002), 189–199.

[2] Alekseevsky, D. and Arvanitoyeorgos, A.: *Riemannian flag manifolds with homogeneous geodesics*, Trans. Amer. Math. Soc., **359** (2007), 3769–3789.

[3] Božek, M.: *Existence of generalized symmetric Riemannian spaces with solvable isometry groups*, Casopis Pest. Mat. **105** (1980), 368–384.

[4] Calvaruso, G.: *Homogeneous structures on three-dimensional Lorentzian manifolds*, J. Geom. Phys., **57** (2007), 1279–1291.

[5] Calvaruso, G. and Dušek, Z.: *A null geodesic orbit space whose null orbits require a reparametrization*, Geometry, Integrability and Quantization IX, I. Mladenov (Ed.), (2008), 167–174.

[6] Calvaruso, G., Kowalski, O. and Marinosci, R.A.: *Homogeneous geodesics in solvable Lie groups*, Acta Math. Hungarica **101** (4) (2003), 313–322.

[7] Calvaruso, G. and Marinosci, R.A.: *Homogeneous geodesics of three-dimensional unimodular Lorentzian Lie groups*, Mediterranean J. Math. (3-4) **3** (2006), 467–481.

[8] Calvaruso, G. and Marinosci, R.A.: *Homogeneous geodesics of three-dimensional non-unimodular Lorentzian Lie groups and naturally reductive Lorentzian spaces in dimension three*, Adv. Geom., **8** (2008), 473–489.

[9] Cordero, L.A. and Parker, P.E.: *Left-invariant Lorentzian metrics on* 3*-dimensional Lie groups*, Rend. Mat. Appl. (7) **17** (1997), 129–155.

[10] Dušek, Z.: *Almost g.o. spaces in dimensions* 6 *and* 7, Adv. Geom., **9** (2009), 99–110.

[11] Dušek, Z.: *Explicit geodesic graphs on some H-type groups*, Rend. Circ. Mat. Palermo, Serie II, Suppl. **69** (2002), 77–88.

[12] Dušek, Z.: *Structure of geodesics in the flag manifold* SO(7)/U(3), Differential Geometry and its Applications, O. Kowalski, D. Krupka, O. Krupkova and J. Slovak (Eds.), World Scientific (2008), 89–98.

[13] Dušek, Z. and Kowalski, O.: *Geodesic graphs on the* 13*-dimensional group of Heisenberg type*, Math. Nachr., **254-255** (2003), 87–96.

[14] Dušek, Z. and Kowalski, O.: *Light-like homogeneous geodesics and the geodesic lemma for any signature*, Publ. Math. Debrecen., **71/1–2** (2007), 245–252.

[15] Dušek, Z. and Kowalski, O.: *Examples of pseudo-Riemannian g.o. manifolds*, Geometry, Integrability and Quantization VIII, I. Mladenov and M. de León (Eds.), (2007), 144–155.

[16] Dušek, Z. and Kowalski, O.: *On six-dimensional pseudo-Riemannian almost g.o. spaces*, J. Geom. Phys., **57** (2007), 2014–2023.

[17] Dušek, Z. and Kowalski, O.: *Geodesic graphs on special* 7*-dimensional Riemannian g.o. manifolds*, Archivum Mathematicum, **42** (2006), Supplement, 213–227.

[18] Dušek, Z., Kowalski, O. and Nikčević, S.: *New examples of Riemannian g.o. manifolds in dimension* 7, Differ. Geom. Appl. **21** (2004), 65–78.

[19] Figueroa-O'Farrill, J., Meessen, P. and Philip, S.: *Homogeneity and plane-wave limits*, J. High Energy Physics **05** (2005), *050*.

[20] Gordon, C.: *Homogeneous Riemannian Manifolds Whose Geodesics are Orbits*, Prog. Nonlinear Differ. Eq. Appl. **20** (1996), 155–174.

[21] Kaplan, A.: *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.

[22] Kajzer, V. V.: *Conjugate points of left-invariant metrics on Lie groups*, Soviet Math. **34** (1990), 32–44; transl. from Izv. Vyssh Uchebn. Zaved. Mat. **342** (1990), 27–37.

[23] Kostant, B.: *Holonomy and Lie algebra of motions in Riemannian manifolds*, Trans. Amer. Math. Soc. **80** (1995), 520–542.

[24] Kowalski, O.: *Classification of generalized symmetric Riemannian spaces of dimension* $n \leq 5$, (Lecture Note, 61 pages), Rozpravy ČSAV, Řada MPV **85** (1975), no. 8.

[25] Kowalski, O.: *Generalized Symmetric Spaces*, Lecture Notes in Math., vol. 805, Springer-Verlag, Berlin - Heidelberg - New York, 1980.

[26] Kowalski, O. and Nikčević, S.: *On geodesic graphs of Riemannian g.o. spaces*, Arch. Math. **73** (1999), 223–234; Appendix: Arch. Math. **79** (2002), 158–160.

[27] Kowalski, O., Nikčević, S. and Vlášek, Z.: *Homogeneous geodesics in homogeneous Riemannian manifolds - Examples*, Preprint Reihe Mathematik, TU Berlin, No. 665/2000.

[28] Kowalski, O. and Szenthe, J.: *On the existence of homogeneous geodesics in homogeneous Riemannian manifolds*, Geom. Dedicata **81** (2000), 209–214, Erratum: Geom. Dedicata **84** (2001), 331–332.

[29] Kowalski, O. and Vanhecke, L.: *Riemannian manifolds with homogeneous geodesics*, Boll. Un. Math. Ital. B(7) **5** (1991), 189–246.

[30] Kowalski, O. and Vlášek, Z.: *Homogeneous Riemannian manifolds with only one homogeneous geodesic*, Publ. Math. Debrecen **62/3–4** (2003), 437–446.

[31] Lauret, J.: *Modified H-type groups and symmetric-like Riemannian spaces*, Diff. Geom. Appl. **10** (1999), 121–143.

[32] Meessen P.: *Homogeneous Lorentzian spaces whose null-geodesics are canonically homogeneous*, Letters in Math. Phys. **75** (2006), 209–212.

[33] Marinosci, R.A.: *Homogeneous geodesics in a three-dimensional Lie group*, Comment. Math. Univ. Carolinae **43**,2 (2002), 261–270.

[34] Milnor, J.: *Curvatures of left-invariant metrics on Lie groups*, Adv. Math. **21** (1976), 293–329.

[35] Philip, S.: *Penrose limits of homogeneous spaces*, J. Geom. Phys. **56** (2006), 1516–1533.

[36] Rahmani, S.: *Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension trois*, J. Geom. Phys. **9** (1992), 295–302.

[37] Riehm, C.: *Explicit spin representations and Lie algebras of Heisenberg type*, J. London Math. Soc. (2) **32** (1985), 265–271.

[38] Szenthe, J.: *Sur la connection naturelle à torsion nulle*, Acta Sci. Math. (Szeged) **38** (1976), 383–398.

[39] Vinberg, E.B.: *Invariant linear connections in a homogeneous manifold*, Trudy MMO **9** (1960), 191–210.