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g-natural metrics: new horizons in the geometry of tangent bundles of Riemannian manifolds

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Abstract. Traditionally, the Riemannian geometry of tangent and unit tangent bundles was related to the Sasaki metric. The study of the relationship between the geometry of a manifold (M, g) and that of its tangent bundle TM equipped with the Sasaki metric g^s had shown some kinds of rigidity. The concept of naturality allowed O.Kowalski and M.Sekizawa to introduce a wide class of metrics on TM naturally constructed from some classical and non-classical lifts of g. This class contains the Sasaki metric as well as the well known Cheeger-Gromoll metric and the metrics of Oproiu-type.

We review some of the most interesting results, obtained recently, concerning the geometry of the tangent and the unit tangent bundles equipped with an arbitrary Riemannian g-natural metric.

Keywords: Riemannian manifold, tangent bundle, unit tangent sphere bundle, g-natural metric, contact structure, curvatures

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Introduction and historical review

It is well-known, from different models of Management sciences and business, that a typical life cycle of any product project passes through four main stages: the introduction, the growth, the maturity (or saturation) and the decline. A new product is first developed and then introduced to the market. Once the introduction is successful, a growth period follows with wider awareness of the product and increasing sales. The product enters maturity when sales stop growing and demand stabilizes. Eventually, sales may decline by the repeated facts of the competition, the economical hazards and the new tendencies until the product is finally withdrawn from the market or redeveloped. The life cycle of a research project or a research production in some field doesn't escape from this rule. Indeed, the introduction step of a research project or a research activity in a scientific field is the step when the motivations of the subject or the topic are stated and the first works on it are published. The growth stage is then the stage when the scientific community acknowledges the interest of the project and several groups of researchers are interested in the topic and a real competition is engaged to solve its questions and problems. With the stabilization of speed of competition and the identification of the bounds of the research and problems related to the topic, a kind of saturation begins to occur and the volume of works dedicated to the topic attains a regular level and begins, in some sense, standard. Several research groups in the topic become then disinterested and leave to other topics or subjects, and the production becomes counter-optimal. Other tendencies or research ways tend then to replace the initial topic or subject or to modify substantially its physiognomy, opening the door to its decline.

The history of research in the topic of differential geometry of tangent bundles over Riemannian manifolds looks very appropriate to illustrate explicitly this case of figure: The introduction of the topic begun with S. Sasaki who constructed, in its original paper [67] of 1958, a Riemannian metric g^s on the tangent bundle TM of a Riemannian manifold (M, g), which depends closely on the base metric g. More precisely, the components of the metric g^s depend only on the components of the metric g and their first derivatives, i.e., using the terminology related to jets, g^s depends on the first jet of the metric g. Geometrically speaking, the Sasaki metric g^s is completely characterized by the following properties:

(1) The natural projection $p_M : (TM, g^s) \to (M, g)$ is a Riemannian submersion;

(2) The horizontal and vertical distributions are orthogonal;

(3) The induced metric on each fiber of TM is Euclidean.

The introduction of the Sasaki metric can be considered as the first stage of the whole topic of differential geometry of tangent bundles, and we can even say that the life cycle of the topic was considered by the specialists as the life cycle of research on the Sasaki metric since all the works published on the topic considered TM equipped with the Sasaki metric, although the introduction during the sixteen's of the 20-th century of other metrics on TM (cf. [80] and [81]), using especially the various kinds of classical lifts of tensor fields from M to TM. According to this concept of lift, the Sasaki metric is no other than the diagonal lift of the base metric, but it is distinguished by the fact that it is Riemannian, when the other constructed metrics are pseudo-Riemannian.

The decades 60-70 of the twentieth century had been the growth period of the topic, with a massive interest of eminent geometers in geometrical properties of the tangent bundle, equipped with the Sasaki metric. According to the approaches adopted for research, we can distinguish between two schools, the Japanese one led by Sasaki, Sato, Tanno who, influenced by physics, had chosen to treat questions by means of coordinates, and the European school, represented by Dombrowski, Kowalski, Nagy, Walczak and others, and who chose to work with coordinates-free formulas.

The middle of the 80's of the previous century was actually the starting period of the maturity stage in the life cycle of the topic, since it has been shown in many papers that the Sasaki metric presents a kind of rigidity. In [39], O. Kowalski proved that if the Sasaki metric g^s is locally symmetric, then the base metric g is flat and hence g^s is also flat. In [44], E. Musso and F. Tricerri have demonstrated an extreme rigidity of g^s in the following sense: if (TM, g^s) is of constant scalar curvature, then (M, g) is flat. This made geometers a bit reticent to the study of the geometry of (TM, g^s), but some research groups (Borisenko, Yampolsky, Vanhecke, Boeckx, Blair, Kowalski, Sekizawa and others) focused on the study of (unit) tangent sphere bundles endowed with the Sasaki metric or with some homothetic one which confers to it the structure of a contact manifold. Up to now, geometers remain interested on the geometry of the unit tangent sphere bundle endowed with the Sasaki metric, especially in the framework of harmonicity (G. Wiegmink, C.M. Wood, O.Gil Medrano and others. . .), but the geometers begun, during the 90's of the 20-th century, more and more convinced that this period was the beginning of the stage of the decline of the life cycle of research on the geometry of the Sasaki metric. With the evident historical relationship between the Sasaki metric and the whole topic of differential geometry of tangent bundles, this could be also the decline of the life cycle of the whole topic. Fortunately, there were a natural thinking to the introduction of (Riemannian) metrics on the tangent bundle other than the Sasaki metric, for which the rigidity of TM stops to be true. A first step towards this end was initiated by Musso and Tricerri [44] in 1986, who gave a process of construction of Riemannian manifolds on TM from basic symmetric tensor fields of type $(2, 0)$ on $OM \times \mathbb{R}^m$, where OM is the bundle of orthonormal frames. As an example, they constructed a new Riemannian metric on TM, i.e., the Cheeger-Gromoll metric g_{CG}. M. Sekizawa [68] has shown that the scalar curvature of (TM, g_{CG}) is never constant if the original metric on the base manifold has constant sectional curvature (see also [34]). Furthermore, the author and M. Sarih have proved that (TM, g_{CG}) is never a space of constant sectional curvature (cf. [11]).

More generally, O. Kowalski and M. Sekizawa [40] used the developed concept of naturality to give a full classification of metrics which are 'naturally constructed' from a metric g on the base M, supposing that M is oriented. Other presentations of the basic results from [40] (involving also the non-oriented case and something more) can be found in [38] or [42]. These metrics had been called in [12] *g-natural metrics on TM*. These metrics had been extensively studied during these last years, and it has been proved that some subclasses of g-natural metrics offer very interesting geometrical features and research horizons. This refreshed our way of thinking and our narrow classical perception of the whole topic of geometry of tangent bundles, which was reduced to the study of the geometry of the Sasaki metric. In this way, the topic, whose life cycle was considered by specialists at the beginning of its decline stage, gains another breath and becomes younger, and actually one can even say that the life cycle of the whole topic is only at its growth stage.

This significant advance in the topic of the geometry of tangent bundles is mainly due to Professor Oldrich Kowalski, whose contributions were essential at least in two steps: on the one hand, he was the first to prove the rigidity of the Sasaki metric on TM [39], and on the other hand, he succeeded, with M.Sekizawa, to give explicit expressions of a broad family of Riemannian metrics on TM [40], i.e. the so-called g-natural metrics. Its other joint paper [41] with M.Sekizawa contributed to a best understanding of the relationship between the geometries of tangent sphere bundles of various radii.

In this paper, I give brief presentation of some new results obtained on Riemannian g-natural metrics on both tangent and unit tangent bundles over Riemannian manifolds, the full statements and proofs of which being detailed in the corresponding papers given as references. Some other important results on Riemannian g-natural metrics in the framework of harmonicity had been obtained recently offering significant advances in our understanding of harmonic maps and sections in the context of unit tangent sphere bundles. A detailed presentation of these results can be found in the paper by G. Calvaruso published in this volume.

Finally, I would like to thank Professor O. Kowalski who influenced my career as a researcher in geometry, and had a central role in my fruitful collaboration with the group of geometry at the university of Lecce. My contribution in the topic of differential geometry of tangent bundles would not be possible without his valuable encouragement, advices and guidance.

1 Basic formulas and g-natural metrics on tangent and unit tangent bundles

Let (M, g) be an m-dimensional Riemannian manifold and ∇ its Levi-Civita connection. At any point $(x, u) \in TM$, the tangent space of TM splits into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

For any vector $X \in M_x$, there exists a unique vector $X^h \in H_{(x,u)}$ such that $p_* X^h = X$, where $p : TM \to M$ is the natural projection. We call X^h the *horizontal lift* of X to the point $(x,u) \in TM$. The *vertical lift* of a vector $X \in M_x$ to $(x,u) \in TM$ is a vector $X^v \in V_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e., $(df)(x,u) = uf$). The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between M_x and $V_{(x,u)}$. Each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\tilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors.

Horizontal and vertical lifts of vector fields on M can be defined in an obvious way. They are uniquely defined vector fields on TM. Each system of local coordinates $\{(U; x^i, i = 1, \ldots, m)\}$ in M induces on TM a system of local coordinates $\{(p^{-1}(U); x^i, u^i, i = 1, \ldots, m)\}$. Let $X = \sum_i X^i \left(\frac{\partial}{\partial x^i}\right)_x$ be the local expression in $\{(U; x^i, i = 1, \ldots, m)\}$ of a vector X in M_x, $x \in M$. Then, the horizontal lift X^h and the vertical lift X^v of X to $(x,u) \in TM$ are given, with respect to the induced coordinates, by:

$$X^h = \sum X^i \left(\frac{\partial}{\partial x^i}\right)_{(x,u)} - \sum \Gamma^i_{jk} u^j X^k \left(\frac{\partial}{\partial u^i}\right)_{(x,u)}, \tag{1}$$

$$X^v = \sum X^i \left(\frac{\partial}{\partial u^i}\right)_{(x,u)}, \tag{2}$$

where (Γ^i_{jk}) denote the Christoffel's symbols of g.

The *canonical vertical vector field* on TM is defined, in terms of local coordinates, by $\mathcal{U} = \sum_i u^i \partial/\partial u^i$. Here, $\mathcal{U}$ does not depend on the choice of local coordinates and it is defined globally on TM. For a vector $u = \sum_i u^i (\partial/\partial x^i)_x \in M_x$, we see that $u^v_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^v_{(x,u)} = \mathcal{U}_{(x,u)}$ and $u^h_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^h_{(x,u)}$ is the *geodesic flow* on TM.

The Riemannian curvature R of g is defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{3}$$

Now, if we write $p_M : TM \to M$ for the natural projection and F for the natural bundle with $FM = p_M^*(T^* \otimes T^*)M \to M$, then $Ff(X_x, g_x) = (Tf.X_x, (T^* \otimes T^*)f.g_x)$ for all manifolds M, local diffeomorphisms f of M, $X_x \in T_xM$ and $g_x \in (T^* \otimes T^*)_x M$. The sections of the canonical projection $FM \to M$ are called *F-metrics* in literature. So, if we denote by $\oplus$ the fibered product of fibered manifolds, then the F-metrics are mappings $TM \oplus TM \oplus TM \to \mathbb{R}$ which are linear in the second and the third argument.

For a given F-metric δ on M, there are three distinguished constructions of metrics on the tangent bundle TM [40]:

(a) If δ is symmetric, then the *Sasaki lift* δ^s of δ is defined by

$$\begin{cases} \delta^s_{(x,u)}(X^h, Y^h) = \delta(u; X, Y), & \delta^s_{(x,u)}(X^h, Y^v) = 0, \\ \delta^s_{(x,u)}(X^v, Y^h) = 0, & \delta^s_{(x,u)}(X^v, Y^v) = \delta(u; X, Y), \end{cases}$$

for all X, $Y \in M_x$. If δ is non degenerate and positive definite, then the same holds for δ^s.

(b) The *horizontal lift* δ^h of δ is a pseudo-Riemannian metric on TM, given by

$$\begin{cases} \delta^h_{(x,u)}(X^h, Y^h) = 0, & \delta^h_{(x,u)}(X^h, Y^v) = \delta(u; X, Y), \\ \delta^h_{(x,u)}(X^v, Y^h) = \delta(u; X, Y), & \delta^h_{(x,u)}(X^v, Y^v) = 0, \end{cases}$$

for all X, $Y \in M_x$. If δ is positive definite, then δ^h is of signature (m, m).

(c) The *vertical lift* δ^v of δ is a degenerate metric on TM, given by

$$\begin{cases} \delta^v_{(x,u)}(X^h, Y^h) = \delta(u; X, Y), & \delta^v_{(x,u)}(X^h, Y^v) = 0, \\ \delta^v_{(x,u)}(X^v, Y^h) = 0, & \delta^v_{(x,u)}(X^v, Y^v) = 0, \end{cases}$$

for all X, $Y \in M_x$. The rank of δ^v is exactly that of δ.
If $\delta = g$ is a Riemannian metric on M, then these three lifts of δ coincide with the three well-known classical lifts of the metric g to TM.

1.1 g-natural metrics on tangent bundles

Now, we shall describe all first order natural operators $D : S^2_+T^* \rightsquigarrow (S^2T^*)T$ transforming Riemannian metrics on manifolds into metrics on their tangent bundles, where $S^2_+T^*$ and S^2T^* denote the bundle functors of all Riemannian metrics and all symmetric two-forms over m-manifolds respectively. For the concept of naturality and related notions, see [38] for more details.

Let us call every section $G : TM \to (S^2T^*)TM$ a (possibly degenerate) *metric*. Then, there is a bijective correspondence between the triples of first order natural F-metrics $(\zeta_1, \zeta_2, \zeta_3)$ and first order natural (possibly degenerate) metrics G on the tangent bundles given by (cf. [40]):

$$G = \zeta_1^s + \zeta_2^h + \zeta_3^v.$$

Therefore, to find all first order natural operators $S^2_+T^* \rightsquigarrow (S^2T^*)T$ transforming Riemannian metrics on manifolds into metrics on their tangent bundles, it suffices to describe all first order natural F-metrics, i.e. first order natural operators $S^2_+T^* \rightsquigarrow (T, F)$. In this sense, it is shown in [40] (see also [38] and

[1]) that all first order natural F-metrics ζ in dimension $m > 1$ form a family parametrized by two arbitrary smooth functions α_0, $\beta_0 : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers, in the following way: for every Riemannian manifold (M, g) and tangent vectors u, X, $Y \in M_x$

$$\zeta_{(M,g)}(u)(X,Y) = \alpha_0(g(u,u))g(X,Y) + \beta_0(g(u,u))g(u,X)g(u,Y). \tag{4}$$

If $m = 1$, then the same assertion holds, but we can always choose $\beta_0 = 0$. In particular, all first order natural F-metrics are symmetric.

1 Definition. Let (M, g) be a Riemannian manifold. We shall call a metric G on TM which comes from g by a first order natural operator $S^2_+T^* \rightsquigarrow (S^2T^*)T$ a *g-natural metric.*

Thus, all g-natural metrics on the tangent bundle of a Riemannian manifold (M, g) are completely determined as follows:

2 Proposition. *[12] Let (M, g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every u, X, $Y \in M_x$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2)g_x(X, Y) \\ \qquad\qquad\qquad\quad +(\beta_1 + \beta_3)(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(r^2)g_x(X, Y) + \beta_2(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2)g_x(X, Y) + \beta_1(r^2)g_x(X, u)g_x(Y, u), \end{cases} \tag{5}$$

where $r^2 = g_x(u, u)$.
For $m = 1$, the same holds with $\beta_i = 0$, $i = 1, 2, 3$.

3 Notations. In the sequel, we shall use the following notations:

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,
- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$.

Riemannian g-natural metrics are characterized as follows:

4 Proposition. *[12] The necessary and sufficient conditions for a g-natural metric G on the tangent bundle of a Riemannian manifold (M, g) to be Riemannian are that the functions of Proposition 2, defining G, satisfy the inequalities*

$$\begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases} \tag{6}$$

for all $t \in \mathbb{R}^+$.

For $m = 1$ the system reduces to $\alpha_1(t) > 0$ and $\alpha(t) > 0$, for all $t \in \mathbb{R}^+$.

1.2 g-natural metrics on (unit) tangent sphere bundles

On tangent sphere bundles, the restrictions of g-natural metrics possess a simpler form. Precisely, we have

5 Theorem. *[9] Let $r > 0$ and (M, g) be a Riemannian manifold. For every Riemannian metric $\tilde{G}$ on T_rM induced from a Riemannian g-natural G on TM, there exist four constants a, b, c and d, with $a > 0$, $a(a+c) - b^2 > 0$ and $a(a+c+dr^2) - b^2 > 0$, such that $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$, where*

- *k is the natural F-metric on M defined by*

$$k(u; X, Y) = g(u, X)g(u, Y), \quad \text{for all} \quad (u, X, Y) \in TM \oplus TM \oplus TM,$$

- *$\widetilde{g^s}$, $\widetilde{g^h}$, $\widetilde{g^v}$ and $\widetilde{k^s}$ are the metrics on T_rM induced by g^s, g^h, g^v and k^v, respectively.*

It is worth mentioning that such a metric $\tilde{G}$ on T_1M is necessarily induced by a metric G on TM of the form $G = a.g^s + b.g^h + c.g^v + \beta.k^v$ (i.e., $G_{(x,u)} = a.g^s_{(x,u)} + b.g^h_{(x,u)} + c.g^v_{(x,u)} + \beta(g_x(u,u)).k^v_{(x,u)}$, for all $(x, u) \in TM$), where a, b, c are constants and $\beta : [0, \infty) \to \mathbb{R}$ is a C^∞-function, such that

$$a > 0, \ \alpha := a(a+c) - b^2 > 0, \ \text{and} \ \phi(t) := a(a+c+t\beta(t)) - b^2 > 0,$$

for all $t \in [0, \infty)$ (see [9] for such a choice). The three preceding inequalities express the fact that G is Riemannian (cf. [9]).

By a simple calculation , using the Schmidt's orthonormalization process, it is easy to check that the vector field on TM defined by

$$N^G_{(x,u)} = \frac{1}{\sqrt{(a+c+d)\phi}}\,[-b.u^h + (a+c+d).u^v], \tag{7}$$

for all $(x, u) \in TM$, is normal to T_1M and unitary at any point of T_1M.

Now, we define the "tangential lift" X^{t_G} –with respect to G– of a vector $X \in M_x$ to $(x, u) \in T_1M$ as the tangential projection of the vertical lift of X to (x, u) –with respect to N^G–, that is,

$$X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})\, N^G_{(x,u)} = X^v - \sqrt{\frac{\phi}{a+c+d}} g_x(X, u)\, N^G_{(x,u)}. \tag{8}$$

If $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$.

The tangent space $(T_1M)_{(x,u)}$ of T_1M at (x, u) is spanned by vectors of the form X^h and Y^{t_G}, where X, $Y \in M_x$. Using this fact, the Riemannian metric

$\tilde{G}$ on T_1M, induced from G, is completely determined by the identities

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)g_x(X,Y) + dg_x(X,u)g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^{t_G}) &= bg_x(X,Y), \\ \tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= ag_x(X,Y) - \frac{\phi}{a+c+d}g_x(X,u)g_x(Y,u), \end{cases} \tag{9}$$

for all $(x,u) \in T_1M$ and X, $Y \in M_x$. It should be noted that, by (9), horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$ if and only if $b = 0$.

2 g-natural metrics by the scheme of Musso-Tricerri

Considering TM as a vector bundle associated with the bundle of orthonormal frames OM, E. Musso and F. Tricerri have constructed an interesting class of *Riemannian* natural metrics on TM [44]. This construction is not a classification *per se*, but it is a construction process of *Riemannian* metrics on TM from symmetric, positive semi-definite tensor fields Q of type $(2,0)$ and rank $2m$ on $OM \times \mathbb{R}^m$, which are basic for the natural submersion $\Phi : OM \times \mathbb{R}^m \to TM$, $\Phi(v,\varepsilon) = (x, \sum_i \varepsilon^i v_i)$, for $v = (x; v_1, \dots, v_m) \in OM$ and $\varepsilon = (\varepsilon^1, \dots, \varepsilon^m) \in \mathbb{R}^m$. Recall that Q is *basic* means that Q is $O(m)$-invariant and $Q(X,Y) = 0$, if X is tangent to a fiber of Φ. The construction can be presented as follows:

6 Proposition. *[44] Let Q be a symmetric, positive semi-definite tensor field of type $(2,0)$ and rank $2m$ on $OM \times \mathbb{R}^m$, which is basic for the natural submersion $\Phi : OM \times \mathbb{R}^m \to TM$. Then there is a unique Riemannian metric G^Q on TM such that $\Phi^*(G^Q) = Q$. It is given by*

$$G^Q_{(x,u)}(X,Y) = Q_{(v,\varepsilon)}(X', Y'), \tag{10}$$

where (v,ε) belongs to the fiber $\Phi^{-1}(x,u)$, X, Y are elements of $(TM)_{(x,u)}$, X', Y' are tangent vectors to $OM \times \mathbb{R}^m$ at (v,ε) with $\Phi_*(X') = X$ and $\Phi_*(Y') = Y$.

It was proved in [12] the following result:

7 Proposition. *Every g-natural metric on the tangent bundle TM of a Riemannian manifold (M,g) can be constructed by the Musso-Tricerri's generalized scheme, given by Proposition 6.*

On the other hand, Musso and Tricerri proposed a similar process for constructing Riemannian metrics on the unit tangent sphere bundle T_1M from symmetric, positive semi-definite tensor fields $\tilde{Q}$ of type $(2,0)$ and rank $2m-1$ on OM, which are basic for the natural submersion $\psi_m : OM \to T_1M$, $\psi_m(v) = (x, v_m)$, for $v = (x; v_1, \dots, v_m) \in OM$. Recall that $\tilde{Q}$ is *basic* means that $\tilde{Q}$ is $O(m-1)$-invariant and $\tilde{Q}(X,Y) = 0$, if X is tangent to a fiber of ψ_n. Note that ψ_m is a submersion whose fibers are diffeomorphic with $O(m-1)$, identified to the subgroup of $O(m)$ of the matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $A \in O(m-1)$. Then, T_1M can be regarded as the quotient space $OM/O(m-1)$, and ψ_m is the natural projection. The construction can be stated as follows:

8 Proposition. *[44] Let $\tilde{Q}$ be a symmetric, positive semi-definite tensor field of type $(2,0)$ and rank $2m-1$ on OM, which is basic for the natural submersion $\psi_m : OM \to T_1M$. Then, there is a unique Riemannian metric $\tilde{G}^{\tilde{Q}}$ on T_1M such that $\psi_m^*(\tilde{G}^{\tilde{Q}}) = \tilde{Q}$. It is given by*

$$\tilde{G}^{\tilde{Q}}_{(x,u)}(X,Y) = \tilde{Q}_{(v)}(X',Y'), \tag{11}$$

where v belongs to the fiber $\psi_m^{-1}(x,u)$, X, Y are elements of $(T_1M)_{(x,u)}$, X', Y' are tangent vectors to OM at v with $(\psi_m)_*(X') = X$ and $(\psi_m)_*(Y') = Y$.

Now, the Musso-Tricerri processes described by Propositions 6 and 8, respectively, are compatible in the following sense:

9 Proposition. *[2] If a Riemannian metric G on TM is induced from a quadratic form Q on $OM \times \mathbb{R}^m$, by the process of Musso-Tricerri described in Proposition 6, i.e., $\Phi^*(G) = Q$, then the induced metric $\tilde{G} := i^*(G)$ on T_1M, where $i : T_1M \to TM$ is the canonical injection, can be obtained from a quadratic form on OM, by the process of Musso-Tricerri described in Proposition 8.*

Combining Propositions 7 and 9, we obtain

10 Proposition. *[2] Every Riemannian g-natural metric on the unit tangent sphere bundle T_1M of a Riemannian manifold (M,g) can be constructed by the Musso-Tricerri's scheme, given by Proposition 8.*

3 Some geometrical properties of Riemannian g-natural metrics on tangent bundles

3.1 Hereditary properties of Riemannian g-natural metrics

It is well-known that if the tangent bundle TM of a Riemannian manifold (M,g) is endowed with the Sasaki metric g^s, then the flatness property on TM is inherited by the base manifold [39]. This motivates us to the general question if the flatness and also other simple geometrical properties remain "hereditary" if we replace g^s by the most general Riemannian "g-natural metric" on TM. In this direction, the following holds

11 Theorem. *[13] If (TM,G) is flat, or locally symmetric, or of constant sectional curvature, or of constant scalar curvature, or an Einstein manifold, respectively, then (M,g) possesses the same property, respectively.*

3.2 The flatness property of g-natural metrics

Theorem 11 deals only with the necessity conditions, the sufficiency part being very complicated and requiring a separated study for each case. Indeed, for the flatness property, very hard and tricky calculations give only partial results:

12 Theorem. *[14] Let (M, g) be a Riemannian manifold of dimension $m \geq 3$ and G be a Riemannian g-natural metric on TM, with respect to which horizontal and vertical distributions are orthogonal. Then, (TM, G) is flat if and only if the following conditions are satisfied:*

(i) G is strongly horizontally and vertically homothetic to g, i.e. of the form

$$G = a.g^s + c.g^v, \text{ where } a > 0 \text{ and } a + c > 0, \tag{12}$$

(ii) (M, g) is flat.

Note that we call G to be *strongly horizontally homothetic* to g if there is a constant $c \geq 0$ such that $G_{(x,u)}(X^h, Y^h) = c.g_x(X, Y)$, for all vectors X, $Y \in M_x$, where M_x denotes the tangent space of M at $x \in M$, and the horizontal lifts are taken at a point $(x, u) \in TM$. Analogously, we say that G is *strongly vertically homothetic* to g if there is a constant $c' \geq 0$ such that $G_{(x,u)}(X^v, Y^v) = c'.g_x(X, Y)$, for all vectors $X, Y \in M_x$, $x \in M$, where the vertical lifts are taken at a point $(x, u) \in TM$.

13 Remark. Note that the Sasaki metric g^s and the Cheeger-Gromoll metric g_{CG} on TM are examples of g-natural metrics with respect to which horizontal and vertical distributions are orthogonal. Yet, g_{CG} is not of the form (12) and consequently g_{CG} is never flat.

Now, if we consider the subclass of g-natural metrics on TM which are strongly horizontally and vertically homothetic to g (but with respect to which horizontal and vertical distributions are not necessarily orthogonal), then we have:

14 Theorem. *[14] Let (M, g) be a Riemannian manifold of dimension $m \geq 3$ and G be a Riemannian g-natural metric on TM, which is strongly horizontally and vertically homothetic to g. Then (TM, G) is flat if and only if the following conditions are satisfied:*

(i) G is of the form

$$G = a.g^s + b.g^h + c.g^v, \text{ where } a > 0 \text{ and } a(a + c) - b^2 > 0, \tag{13}$$

(ii) (M, g) is flat.

3.3 g-natural metrics with constant scalar curvature

Theorem 14 asserts that all Riemannian g-natural metrics of the form $G = a.g^s + b.g^h + c.g^v$, with a, b and c constants, are rigid in the sense that (TM, G) is flat if and only if (M, g) is also flat. All these metrics possess, actually, an extreme rigidity. Indeed, we have the following

15 Theorem. *[14] Let (M, g) be a Riemannian manifold and $G = a.g^s + b.g^h + c.g^v$, such that $a > 0$ and $a(a + c) - b^2 > 0$. Then (TM, G) is of constant scalar curvature if and only if (M, g) is flat.*

16 Remark. For $a = 1$ and $b = c = 0$ in Theorem 15, we obtain hence the result from [44]. Moreover, the proof in the present paper is quite different from that given in [44].

On the other hand, we can prove

17 Theorem. *[14] Let (M, g) be a space of negative scalar curvature and $G = a.g^s + b.g^h + c.g^v$ such that $a > 0$ and $a(a + c) - b^2 > 0$. Then (TM, G) is of negative scalar curvature.*

In [53], V.Oproiu considered an interesting family of Riemannian metrics on TM, which depends on two arbitrary functions of one variable. It was proved in [13] that the family of Riemannian metrics on TM considered by Oproiu is, exactly, the family of Riemannian g-natural metrics on TM characterized by:

* horizontal and vertical distributions are orthogonal,
* $\alpha = \phi = 1$, where α and ϕ are the functions defined by $\alpha = \alpha_1(\alpha_1 + \alpha_3) - \alpha_2^2$, $\phi = \phi_1(\phi_1 + \phi_3) - \phi_2^2$, and $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$, for all $t \in \mathbb{R}^+$ (cf. Notations 3).

Oproiu and its collaborators devoted a series of papers (cf. [51]-[57], [61]) to sort out, inside the previous family of metrics (not only on the tangent bundle but also on tubes in it and on the nonzero tangent bundle), those having a certain property: to be Einstein, or locally symmetric, with the additional condition of being Kähler with respect to a natural almost complex structure. They have used, for this, some quite long and hard computations made by means of the Package "RICCI".

As an application of their results, M. Sarih an I proved the following:

18 Theorem. *[13] Let (M, g) be an m-dimensional space of negative constant sectional curvature, where $m \geq 3$. Then there is a 1-parameter family $\mathcal{F}$ of Riemannian g-natural metrics on TM with nonconstant defining functions α_i and β_i such that, for every $G \in \mathcal{F}$, (TM, G) is a space of positive constant scalar curvature. Moreover, for each (M, g) as above, and each prescribed constant $S > 0$, there is a metric $G \in \mathcal{F}$ with the constant scalar curvature S.*

4 Some geometrical properties of Riemannian g-natural metrics on unit tangent bundles

4.1 g-natural metrics on T_1M with constant curvature

For the Sasaki metric g^S, it is well known that (T_1M, g^S) has constant sectional curvature if and only if the base manifold (M, g) is two-dimensional which either is flat or has constant Gaussian curvature equal to 1 [25]. When we replace g^S by the most general g-natural Riemannian metric $\tilde{G}$, we again find that (M, g) is necessarily two-dimensional and of constant Gaussian curvature $\bar{c}$, but we have much more freedom concerning the possible values of $\bar{c}$. Indeed, we have

19 Theorem. *[5] Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{G})$ has constant sectional curvature $\tilde{K}$ if and only if the base manifold is a Riemannian surface (M^2, g) of constant Gaussian curvature $\bar{c}$ and one of the following cases occurs:*

(i) $d = 0$ and $\bar{c} = 0$. In this case, $\tilde{K} = 0$.

(ii) $b = 0$, $d \neq 0$ and $\bar{c} = \frac{d}{a}$. In this case, $\tilde{K} = \frac{d}{a(a+c+d)}$.

(iii) $b = d = 0$ and $\bar{c} = \frac{a+c}{a} > 0$. In this case, $\tilde{K} = \frac{a+c}{4a(a+c+d)} > 0$.

¿From Theorem 19, we obtain at once the following classification of Riemannian g-natural metrics of constant sectional curvature in the unit tangent sphere bundle of a Riemannian surface (M^2, g).

20 Corollary. *[5] Let (M^2, g) be a Riemannian surface of constant sectional curvature $\bar{c}$. The following Riemannian g-natural metrics are those of constant sectional curvature on T_1M^2:*

- *if $\bar{c} = 0$, then Riemannian g-natural metrics of the form $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v}$, $a > 0$, $a(a+c) - b^2 > 0$, have constant sectional curvature $\tilde{K} = 0$.*
- *if $\bar{c} > 0$, then Riemannian g-natural metrics of the form either $\tilde{G} = a.\widetilde{g^s} + c.\widetilde{g^v} + (\bar{c}a).\widetilde{k^v}$, $a > 0$, $a + c > 0$, or $\tilde{G} = a.\widetilde{g^s} + a(\bar{c} - 1).\widetilde{g^v}$, $a > 0$, have constant sectional curvature $\tilde{K} > 0$.*
- *if $\bar{c} < 0$, then Riemannian g-natural metrics of the form $\tilde{G} = a.\widetilde{g^s} + c.\widetilde{g^v} + (\bar{c}a).\widetilde{k^v}$, $a > 0$, $c > -a(\bar{c} + 1)$, have constant sectional curvature $\tilde{K} < 0$.*

Now, by Theorem 19, only unit tangent sphere bundles of two-dimensional Riemannian manifold of constant Gaussian curvature can admit Riemannian g-natural metrics of constant sectional curvature. Moreover, by Corollary 20 it follows that only some Riemannian g-natural metrics, over a Riemannian surface (M^2, g) of constant Gaussian curvature $\bar{c}$, have constant sectional curvature. Therefore, it is natural to investigate some milder curvature conditions for a Riemannian g-natural metric $\tilde{G}$ on T_1M^2.

A Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *curvature homogeneous* if, for any points $x, y \in M$, there exists a linear isometry $f : T_xM \to T_yM$ such that $f_{*x}(R_x) = R_y$ (see the survey [22]). A locally homogeneous space is curvature homogeneous, but there are many well-known examples of curvature homogeneous Riemannian manifolds which are not locally homogeneous. If $\dim \bar{M} = 3$, then curvature homogeneity is equivalent to the constancy of the Ricci eigenvalues. In particular, a curvature homogeneous manifold $(\bar{M}, \bar{g})$ has constant *scalar curvature* $\bar{\tau}$. The constancy of the scalar curvature is itself a well-known curvature condition, which naturally appears in many fields of Riemannian Geometry.

As concerns Riemannian g-natural metrics on T_1M^2 we have the following

21 Theorem. *[5] Let (M^2, g) be a Riemannian surface. The following properties are equivalent:*

(i) $(M^2 g)$ has constant Gaussian curvature,

(ii) T_1M^2 admits a Riemannian g-natural metric of constant scalar curvature,

(iii) T_1M^2 admits a curvature homogeneous Riemannian g-natural metric.

Moreover, when one of the properties above is satisfied, then all *Riemannian g-natural metrics on T_1M^2 are curvature homogeneous.*

22 Remark. We note that Theorem 21 can be used to build many examples of three-dimensional curvature homogeneous Riemannian manifolds, as unit tangent sphere bundles over Riemannian surfaces of constant Gaussian curvature, equipped with a Riemannian g-natural metric.

4.2 Einstein g-natural metrics on T_1M

As concerns unit tangent sphere bundles of Riemannian manifolds, equipped with the induced Sasaki metric, E. Boeckx and L. Vanhecke proved that they can not be Einstein unless the base manifold is 2-dimensional [25]. When we consider some induced Riemannian g-natural metrics on T_1M, we obtain a strikingly opposite result to that in [25]. Precisely, we have:

23 Theorem. *[10] For each Riemannian manifold (M, g) of dimension $m > 2$ with positive constant sectional curvature, there exists a Riemannian g-natural metric G on the unit tangent sphere bundle T_1M over (M, g) such that (T_1M, G) is a locally homogeneous Einstein manifold.*

When (M, g) is the standard m-sphere S^m, with its standard Riemannian metric of constant curvature 1, then its unit tangent sphere bundle T_1S^m is diffeomorphic to the Stiefel manifold $V_2\mathbb{R}^{m+1} = SO(m+1)/SO(m-1)$ of orthonormal 2-frames in Euclidean $(m+1)$-space. Theorem 23, together with some classical results on invariant Einstein metrics on the Stiefel manifold $V_2\mathbb{R}^{m+1} = SO(m+1)/SO(m-1)$,, let us assert the following two corollaries [10]

24 Corollary. *For $m \geq 4$, there is a unique Riemannian g-natural metric on T_1S^m which is Einstein.*

25 Corollary. *Up to homotheties, there are exactly two Riemannian g-natural metrics on T_1S^3 which are Einstein, the first one is given by Theorem 23 and the second is with respect to which the horizontal and vertical distributions of T_1S^3 are not orthogonal.*

5 g-natural contact metrics on unit tangent sphere bundles

Now, let us recall some definitions related to the contact geometry. A $(2n+1)$-dimensional manifold $\bar{M}$ is called a *contact manifold* if it admits a global 1-form η (a *contact form*) such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $\bar{M}$. Given η, there exists a unique vector field ξ, called the *characteristic vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. Furthermore, a Riemannian metric g is said to be an *associated metric* if there exists a tensor φ, of type (1,1), such that

$$\eta = g(\xi, \cdot)\,, \quad d\eta = g(\cdot, \varphi\cdot)\,, \quad \varphi^2 = -I + \eta \otimes \xi\,. \tag{14}$$

(η, g, ξ, φ), or (η, g), is called a *contact metric structure* and $(\bar{M}, \eta, g, \xi, \varphi)$ a *contact metric manifold*.

As a contact metric manifold, T_1M has been traditionally equipped with the Riemannian metric $\bar{g}$ homothetic to $\widetilde{g^s}$ with the homothety factor 1/4, inducing the *standard contact metric structure* $(\eta, \bar{g})$ on T_1M. Note that, since $\bar{g}$ is homothetic to $\widetilde{g_S}$, these Riemannian metrics share essentially the same curvature properties.

Several curvature properties on T_1M, equipped with one of the metrics above, turn out to correspond to very rigid properties for the base manifold M. We can refer to [24] for a survey on the geometry of $(T_1M, \widetilde{g_S})$. For a survey on the contact metric geometry of $(T_1M, \eta, \bar{g})$, we can confer to [27].

In [3], the authors investigated under which conditions a Riemannian g-natural metric on T_1M may be seen as a Riemannian metric associated to a very "natural" contact form. An arbitrary Riemannian g-natural metric $\tilde{G}$ over

T_1M is induced by a Riemannian g-natural metric $G = a.g^s + b.g^h + c.g^v + \beta.k^v$ over TM. Hence, $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$, where $d = \beta(1)$. We already remarked that $N^G_{(x,u)} = \frac{1}{\sqrt{(a+c+d)\phi}}[-b.u^h + (a+c+d).u^v]$, for all $(x,u) \in TM$, is a vector field on TM, unit and normal on T_1M at any point of T_1M. The tangent space to T_1M at (x,u) is given by

$$(T_1M)_{(x,u)} = \mathrm{Span}(\tilde{\xi}) \oplus \{X^h | X \perp u\} \oplus \{X^{t_G} | X \perp u\},$$

where we put

$$\tilde{\xi}_{(x,u)} = ru^h, \tag{15}$$

r being a positive constant. We consider the triple $(\tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$, where $\tilde{\xi}$ is defined as in (15), $\tilde{\eta}$ is the 1-form dual to $\tilde{\xi}$ through $\tilde{G}$, and $\tilde{\varphi}$ is completely determined by $\tilde{G}(Z, \tilde{\varphi}W) = (d\tilde{\eta})(Z,W)$, for all Z, W vector fields on T_1M. Then, simply calculations show that

$$\begin{cases} \tilde{\eta}(X^h) = \dfrac{1}{r} g(X,u), \\ \tilde{\eta}(X^{t_G}) = brg(X,u) \end{cases} \tag{16}$$

and

$$\begin{cases} \tilde{\varphi}(X^h) = \frac{1}{2r\alpha}\left[-bX^h + (a+c)X^{t_G} + \frac{bd}{a+c+d} g(X,u)u^h\right], \\ \tilde{\varphi}(X^{t_G}) = \frac{1}{2r\alpha}\left[-aX^h + bX^{t_G} + \frac{\phi}{a+c+d} g(X,u)u^h\right], \end{cases} \tag{17}$$

for all $X \in M_x$. If

$$\frac{1}{r^2} = 4\alpha = a + c + d \tag{18}$$

holds, then $\tilde{\eta}$ is well-defined and it is a contact form on T_1M, homothetic –with homothety factor $1/r$– to the classical contact form on T_1M (see, for example, [17] for a definition). Indeed, it follows from (8) that

$$\begin{cases} X^{t_G} = X^v, \quad \text{for} \quad X \perp u, \\ u^{t_G} = \frac{b}{a+c+d} u^h. \end{cases} \tag{19}$$

The second identity of (19) , together with (18), guarantee the compatibility of the two identities of (16). This shows that $\tilde{\eta}$ is well-defined. On the other hand, (19) implies that (16) is equivalent to

$$\begin{cases} \tilde{\eta}(X^h) = \dfrac{1}{r} g(X,u), \\ \tilde{\eta}(X^v) = 0, \quad \text{for} \quad X \perp u. \end{cases} \tag{20}$$

We deduce that $\tilde{\eta}$ is homothetic –with homothety factor $1/r$– to the classical contact form on T_1M, and is, hence, a contact form on T_1M.

From the definition of α and (18) it follows $d = (a + c)(4a - 1) - 4b^2$. So, among Riemannian g-natural metrics on T_1M, the ones satisfying (18) are contact metrics associated to the contact structures described by (15)-(17). In this way, we have proved the following:

26 Theorem ([3])**.** *The set* $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$*, described by* (15)-(18)*, is a family of contact metric structures over* T_1M*, depending on three real parameters* a*,* b *and* c*.*

More details can be found in [3], where the authors also proved that the class of g-natural contact metric structures on T_1M is invariant under *D-homothetic deformations*.

The tensor $h = \frac{1}{2}\mathcal{L}_\xi\varphi$, where $\mathcal{L}$ denotes the Lie derivative, plays a very important role in describing the geometry of a contact metric manifold $(\bar{M}, \eta, g)$. h is symmetric and satisfies

$$\nabla\xi = -\varphi - \varphi h, \quad h\varphi = -\varphi h, \quad h\xi = 0. \tag{21}$$

At any point (x, u) of the contact metric manifold $(T_1M, \tilde{\eta}, \tilde{G})$, the tensor $\tilde{h} = \frac{1}{2}\mathcal{L}_{\tilde{\xi}}\tilde{\varphi}$ is described as follows:

$$\begin{cases} \tilde{h}(X^h) = \dfrac{1}{4\alpha}\left[-(a+c)(X - g(X,u)u)^h + a(R_uX)^h - 2b(R_uX)^{t_G}\right], \\ \tilde{h}(X^{t_G}) = \dfrac{1}{4\alpha}\left[-2bX^h + b\left(1 + \dfrac{d}{a+c+d}\right)g(X,u)u^h + \right. \\ \qquad\qquad\qquad\qquad\qquad\qquad \left. (a+c)X^{t_G} - a(R_uX)^{t_G}\right], \end{cases} \tag{22}$$

for all $X \in M_x$, where $R_uX = R(X, u)u$ denotes the *Jacobi operator* associated to u.

Throughout the rest of this section, we shall assume that (M, g) is a Riemannian manifold of dimension $m \geq 3$.

5.1 K-contact and Sasakian structures

Let $(\bar{M}, \eta, \bar{g})$ be a contact metric manifold. If the almost complex structure J on $\bar{M} \times I\!R$ defined by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, $\bar{M}$ is called *Sasakian*. A well-known characterization states that $(\bar{M}, \eta, \bar{g})$ is Sasakian if and only if the covariant derivative of its tensor φ satisfies

$$(\nabla_Z\varphi)W = \bar{g}(Z, W)\xi - \eta(W)Z, \tag{23}$$

for all Z, W vector fields on $\bar{M}$.

A *K-contact manifold* is a contact metric manifold $(\bar{M}, \eta, \bar{g})$ whose characteristic vector field ξ is a Killing vector field with respect to $\bar{g}$. This is equivalent to the condition $h = 0$. Any Sasakian manifold is K-contact and the converse also holds for three-dimensional spaces. We refer to [17] for more information about K-contact and Sasakian manifolds.

Y. Tashiro [76] proved that T_1M, equipped with its standard contact metric structure $(\eta, \bar{g})$, is K-contact if and only if the base manifold (M, g) has constant sectional curvature 1. Moreover, in this case, T_1M is also Sasakian. With respect to the contact metric structures $(\tilde{\eta}, \tilde{G})$ over T_1M, we have the following:

27 Theorem. *[3] Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ is K-contact if and only if $b = 0$ and the base manifold (M, g) has constant sectional curvature $\frac{a+c}{a} > 0$. In this case, T_1M is Sasakian.*

¿From Theorem 27 one obtains the following classification of the K-contact (or Sasakian) g-natural contact metric structures on T_1M, (M, g) being of positive constant sectional curvature:

28 Theorem. *[3] A Riemannian manifold (M, g) has constant sectional curvature $k > 0$ if and only if there exists a Riemannian g-natural metric $\tilde{G}$ on T_1M, such that $(T_1M, \tilde{\eta}, \tilde{G})$ is K-contact (*equivalently*, Sasakian). The K-contact (or equivalently, Sasakian) g-natural contact metric structures on T_1M, described by* (15)-(18)*, are exactly the ones determined by Riemannian g-natural metrics of the form*

$$\tilde{G} = a.\widetilde{g^s} + (k-1)a.\widetilde{g^v} + ka(4a-1).\widetilde{k^v}, \ \textit{with}\, a > 0.$$

5.2 Contact structures giving rise to a strongly pseudo-convex CR-structure

A *strongly pseudo-convex CR-structure* on a manifold $\bar{M}$ is a contact form η together with an integrable complex structure J on the contact subbundle $D := \ker \eta$ (i.e., a bundle map $J : D \to D$ such that $J^2 = -I$), such that the associated *Levi form* L_η, defined by

$$L_\eta(X, Y) = -d\eta(X, JY), \quad X, Y \in D,$$

is definite positive. In this case, $(\bar{M}, \eta, J)$ is called a *strongly pseudo-convex manifold*.

A strongly pseudo-convex manifold $(\bar{M}, \eta, J)$ carries a contact metric structure $(\varphi, \xi, \eta, \bar{g})$, where $\bar{g}$ is the so-called *Webster metric* on $\bar{M}$, obtained extending L_η on $T\bar{M} = \mathrm{Span}\{\xi\} \oplus D$ by putting $\bar{g}(\xi, \xi) = 1$ and $\bar{g}(\xi, X) = 0$ for all

$X \in D$, and φ is defined by $\varphi\xi = 0$ and $\varphi X = JX$ for all $X \in D$. We can refer to Section 6.4 of [17] for more details on CR-manifolds.

Let now $(\bar{M}, g, \eta, \varphi, \xi)$ be a contact metric manifold and $D = \ker \eta$ the contact subbundle. Then, φ determines a complex structure $J := \varphi|_D$. The following result of Tanno gives a necessary and sufficient condition for $(\bar{M}, \eta, J)$ to be a strongly pseudo-convex CR-manifold:

29 Theorem ([74]). *Let $(\bar{M}, g, \eta, \varphi, \xi)$ be a contact metric manifold and $D = \ker \eta$ the contact subbundle. Then $(\bar{M}, \eta, J := \varphi|_D)$ is a strongly pseudo-convex CR-manifold if and only if*

$$(\nabla_Z \varphi)W = g(Z + hZ, W)\xi - \eta(W)(Z + hZ), \tag{24}$$

for all Z, W vector fields on $\bar{M}$.

For the case of the unit tangent sphere bundle T_1M, equipped with its standard contact metric structure $(\eta, \bar{g})$, Tanno proved in [75] that it gives rise to a strongly pseudo-convex CR-manifold if and only if the base manifold (M, g) has constant sectional curvature. The following extension of Tanno's result holds:

30 Theorem. *[3] Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ gives rise to a strongly pseudo-convex CR-structure if and only if the base manifold (M, g) has constant sectional curvature.*

Comparing the results of Theorems 28 and 30, we obtain the following:

31 Corollary. *[3] A Riemannian manifold (M, g) has constant sectional curvature $k \leq 0$ if and only if there exists a non K-contact (*equivalently*, non Sasakian) g-natural contact metric structure on T_1M, which gives rise to a strongly pseudo-convex CR-structure.*

5.3 Contact structures satisfying a critical point condition

Let $(\bar{M}, \eta)$ be a contact manifold with characteristic vector field ξ, and denote by $\mathcal{A}$ the set of all Riemannian metrics associated to η. When $\bar{M}$ is compact, we can consider, for all $g \in \mathcal{A}$, the functional

$$L(g) = \int_{\bar{M}} \mathrm{Ric}(\xi) dV,$$

where $\mathrm{Ric}(\xi) = \varrho(\xi, \xi)$ and ϱ is the Ricci tensor of $\bar{M}$. Now, by definition of associated metrics, each element g of $\mathcal{A}$ is defined together with a tensor field φ of type $(1, 1)$ on $\bar{M}$ given by (14). D. Blair [15] proved that $g \in \mathcal{A}$ is a critical point for L if and only the tensor $h := \mathcal{L}_\xi \varphi$ associated to $(\bar{M}, \eta, g)$ satisfies

$$\nabla_\xi h = 2h\varphi. \tag{25}$$

Note that the condition (25) also makes sense when $\bar{M}$ is not compact. Blair proved in [16] that the standard contact metric $\bar{g}$ on the unit tangent sphere bundle T_1M of a compact manifold M is critical for L if and only if the base manifold (M,g) has constant sectional curvature $+1$ or -1. Replacing the standard contact metric structure on T_1M by a g-natural contact metric structure, we have the following

32 Theorem. *[3] Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ satisfies $\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = 2\tilde{h}\tilde{\varphi}$ if and only if $b = 0$ and (M,g) has constant sectional curvature $\frac{a+c}{a} > 0$ or $-\frac{a+c}{a} < 0$.*

¿From Theorem 32 we can derive the following classification of g-natural contact metric structures on T_1M satisfying (25):

33 Theorem. *[3] A Riemannian manifold (M,g) has constant sectional curvature $k \neq 0$ if and only if there exists a Riemannian g-natural metric $\tilde{G}$ on T_1M, such that $(T_1M, \tilde{\eta}, \tilde{G})$ satisfies $\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = 2\tilde{h}\tilde{\varphi}$. The g-natural contact metric structures on T_1M described by (15)-(18), for which $\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = 2\tilde{h}\tilde{\varphi}$, are exactly the ones determined by the Riemannian g-natural metrics $\tilde{G}$ of the form*

- $\tilde{G} = a.\widetilde{g^s} + (k-1)a.\widetilde{g^v} + ka(4a-1).\widetilde{k^v}$ *if* $k > 0$, *or*
- $\tilde{G} = a.\widetilde{g^s} - (k+1)a.\widetilde{g^v} - ka(4a-1).\widetilde{k^v}$ *if* $k < 0$,

where $a > 0$.

Comparing the results obtained in Theorems 27 and 33, we get the following

34 Corollary. *[3] A Riemannian manifold (M,g) has constant sectional curvature $k < 0$ if and only if there exists a Riemannian g-natural metric $\tilde{G}$ on T_1M, such that $(T_1M, \tilde{\eta}, \tilde{G})$ satisfies $\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = 2\tilde{h}\tilde{\varphi}$ but $\tilde{h} \neq 0$.*

Next, we consider contact metric $(2n+1)$-manifolds $(\bar{M}, \eta, g)$ satisfying

$$\nabla_\xi h = 0\,. \tag{26}$$

The condition (26) has many special features. It is equivalent to requiring that at a given point, all planes perpendicular to the contact subbundle Kerη have the same sectional curvature. D. Perrone [63] proved that, when M is compact, (26) is the critical point condition for the functional

$$I(g) = \int_{\bar{M}} (r(g) + r_1(g))dV,$$

where $r(g)$ and $r_1(g)$, defined in the set $\mathcal{A}$ of all metrics associated to η, denote, respectively, the scalar curvature of (M,g) and the quantity $r_1(g) =$

$r^*(g) + 2n\mathrm{Ric}(\xi)$, $r^*(g)$ being the **-scalar curvature* obtained by contracting the curvature tensor by φ instead of the metric g (see also [17]). Note that when M is 3-dimensional or Sasakian, then $r_1(g) = r(g)$, and, consequently, (26) is the critical point condition for the functional $I^*(g) = \int_{\bar{M}} r(g)dV$. Finally, (26) is also a necessary condition for a contact metric manifold to be locally symmetric [64].

D. Perrone [64] proved that $(T_1M, \eta, \bar{g})$ satisfies (26) if and only if the base manifold (M, g) has constant sectional curvature 0 or +1. The extension of this result to the case of a g-natural metric on T_1M, turns out to be related to the the fact that base manifold (M, g) is globally Osserman.

We recall that a Riemannian manifold (M, g) is called *globally Osserman* if the eigenvalues of the Jacobi operator R_u are independent of both the unit tangent vector $u \in M_x$ and the point $x \in M$. The well-known *Osserman conjecture* states that any globally Osserman manifold is locally isometric to a *two-point homogeneous space*, that is, either a flat space or a rank one symmetric space. We recall that the complete list of rank-one symmetric spaces is formed by $\mathbb{RP}^n$, S^n, $C\mathbb{P}^n$, $H\mathbb{P}^n$, $Cay\mathbb{P}^2$ ant their non-compact duals. Actually, thanks to the works of Chi [30] and Nikolayevsky [46], [47], the Osserman conjecture has been proved to be true for all manifolds of dimension $n \neq 16$. Moreover, also in dimension 16 there are some partial results. In particular, if (M, g) is a Riemannian manifold such that R_u admits at most two distinct eigenvalues (besides 0), then it is locally isometric to a two-point homogeneous space [48].

The idea of characterizing two-point homogeneous spaces through the properties of their unit tangent sphere bundles, was already investigated in [24], [23] and [28], equipping T_1M with its standard contact metric structure. We can now characterize such Riemannian manifolds, via the existence of g-natural contact metric structures on T_1M satisfying (26):

35 Theorem ([3])**.** *There exists a Riemannian g-natural metric $\tilde{G}$ on T_1M, such that $(T_1M, \tilde{\eta}, \tilde{G})$ satisfies $\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = 0$, if and only if either (M, g) has constant sectional curvature, or it is locally isometric to a compact rank one symmetric space.*

Taking into account the fact that any locally symmetric contact metric manifold satisfies (26), from Theorem 35 we get at once the following

36 Corollary ([3])**.** *Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural contact metric on T_1M. If $\tilde{G}$ is locally symmetric, then either (M, g) has constant sectional curvature, or it is locally isometric to a compact rank one symmetric space.*

5.4 g-natural contact structures of constant ξ-sectional curvature

Let $(\bar{M}, \eta, \bar{g})$ be a contact metric manifold. The sectional curvature of plane sections containing the characteristic vector field ξ, is called *ξ-sectional curvature* (see Section 11.1 of [17]). Clearly, if π is a plane section containing ξ, we can determine the sectional curvature of π at a point $x \in \bar{M}$ as $K(Z, \xi_x)$, where Z is a vector of π_x, orthogonal to ξ_x. As it was proved in [35] (see also Theorem 7.2 of [17]), a contact metric manifold is K-contact if and only if it has constant ξ-sectional curvature equal to 1. Now, for the g-natural contact metric structures, the following holds

37 Theorem ([4])**.** *Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature $\widetilde{K}$ if and only if the base manifold (M, g) has constant sectional curvature $\bar{c}$ either equal to $\frac{d}{a}$ or to $\frac{a+c}{a} > 0$.*

In [66], D. Perrone investigated three-dimensional contact metric manifolds $(M^3, \eta, \bar{g})$ of constant ξ-sectional curvature. In particular, he characterized such spaces as contact metric manifolds of constant *scalar torsion* $||\tau||$ satisfying $\nabla_\xi \tau = 0$ [66], where the *torsion* $\tau := \mathcal{L}_\xi \bar{g}$ is the Lie derivative of $\bar{g}$ in the direction of the characteristic vector field ξ. It is also interesting to remark that, among three-dimensional contact metric manifolds satisfying $\nabla_\xi \tau = 2\tau\varphi$, K-contact spaces are the only ones having constant ξ-sectional curvature ([66], Corollary 4.6).

On any contact metric manifold $(M, \eta, \bar{g})$, the torsion τ is related to the tensor h by the formula $\tau = 2\bar{g}(h\varphi\cdot, \cdot)$, from which it follows

$$\nabla_\xi \tau = 2\bar{g}((\nabla_\xi h)\varphi\cdot, \cdot),$$

and so, critical point conditions given in the subsection 5.3 can be expressed in terms of the tensor τ. Taking into account Theorems 33, 35 and 37 above, the following results follow easily [4]:

38 Proposition. *Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. If $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature, then $\tilde{\nabla}_{\tilde{\xi}} \tilde{h} = 0$.*

39 Corollary. *Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M, such that $\tilde{\nabla}_{\tilde{\xi}} \tilde{h} = 2\tilde{h}\tilde{\varphi}$. Then, $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature if and only if it is K-contact.*

5.5 g-natural contact structures of constant φ-sectional curvature

Let $(\bar{M}, \eta, \bar{g}, \xi, \varphi)$ be a contact metric manifold and $Z \in \ker \eta$. The φ-*sectional curvature* determined by Z is the sectional curvature $K(Z, \varphi Z)$ along the plane spanned by Z and φZ. The φ-sectional curvature of a Sasakian manifold determines the curvature completely. A *Sasakian space form* is a Sasakian manifold of constant φ-sectional curvature. We refer to Section 7.3 of [17] for further details and results.

As concerns the standard contact metric structure of the unit tangent sphere bundle, the following result holds:

40 Theorem ([36])**.** *If (M, g) has constant sectional curvature $\bar{c}$ and $\dim M \geq 3$, the standard contact metric structure of T_1M has constant φ-sectional curvature (equal to $(2 \pm \sqrt{5})^2$) if and only if $\bar{c} = 2 \pm \sqrt{5}$.*

For the g-natural contact metric structures of T_1M, we can prove the following:

41 Theorem ([4])**.** *Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. If $(T_1M, \tilde{\eta}, \tilde{G})$ has constant φ-sectional curvature, then the base manifold (M, g) is locally isometric to a two-point homogeneous space.*

The converse of Theorem 41 would provide an interesting characterization of two-point homogeneous spaces in terms of their unit tangent sphere bundles. However, the calculations involved are really hard. A partial result, which extends Theorem 40 to an arbitrary g-natural contact metric structure, is given by the following

42 Theorem. *[4] Let (M, g) be a Riemannian manifold of constant sectional curvature $\bar{c}$ and $dimM \geq 3$, and $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ has constant φ-sectional curvature $\widetilde{K}$ if and only one of the following cases occurs:*

(i) $\bar{c} = 0$, $b = \pm\sqrt{(a+c)(a - \frac{1}{8})}$ and $d = -\frac{a+c}{2}$. In this case, $\widetilde{K} = 5$.

(ii) $\bar{c} \neq 0$, $a = \frac{1}{4}$, $b = d = 0$ and $c = -\frac{1}{4} - \frac{2\pm\sqrt{5}}{4}\bar{c}$. In this case, $\widetilde{K} = (2 \pm \sqrt{5})^2$.

5.6 g-natural contact structures of T_1M whose tensor l annihilates the vertical distribution

The $(1,1)$-tensor field l on $\bar{M}$, defined by $l(X) = R(X, \xi, \xi)$ for all $X \in \mathfrak{X}(M)$, naturally appears in the study of the geometry of $(\bar{M}, \eta, g)$. For example, K-contact spaces are characterized by the equation $l = -\varphi^2$. If $l = 0$, then sectional curvatures of all planes containing ξ are equal to zero. We may refer to [65] for these and further results on l. Note that there are many contact metric manifolds satisfying $l = 0$ ([17], p. 153).

D. Blair [15] proved that T_1M, equipped with its standard contact metric structure $(\eta, \bar{g})$, satisfies $lU = 0$ for all vertical vector field U on T_1M if and only if the base manifold (M, g) is flat. Moreover, in this case ξ is a *nullity vector field*, that is, $R(Z, W)\xi = 0$ for all $Z, W \in \mathfrak{X}(T_1M)$. These results can be extended to any g-natural contact metric structure $(\tilde{\eta}, \tilde{G})$ over T_1M, proving the following

43 Theorem ([4]). *Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ satisfies $lU = \tilde{R}(U, \tilde{\xi})\tilde{\xi} = 0$ for all vertical vector fields U on T_1M if and only if $d = 0$ and the base manifold (M, g) is flat. Moreover, in this case $\tilde{R}(Z, W)\tilde{\xi} = 0$ for all vector fields Z, W on T_1M.*

5.7 g-natural contact structures for which T_1M is a (k, μ)-space

Generally speaking, a (k, μ)-*space* $(\bar{M}, \eta, \bar{g})$ is a contact metric manifold whose characteristic vector field ξ belongs to the so-called (k, μ)-*nullity distribution*, that is, satisfies

$$R(Z, W)\xi = k(\eta(W)Z - \eta(Z)W) + \mu(\eta(W)hZ - \eta(Z)hW), \tag{27}$$

for some real constants k and μ and for all vector fields Z, W on $\bar{M}$. We can refer to [17] for a survey on (k, μ)-spaces. Here we just recall that they generalize Sasakian manifolds, and that non-Sasakian (k, μ)-spaces have been completely classified [19]. Note that on any (k, μ)-space we have $k \leq 1$, and $k = 1$ if and only if $(\bar{M}, \eta, \bar{g})$ is Sasakian. Moreover, any (k, μ)-space is a strongly pseudo-convex CR-manifold [17].

It was proved in [18] that T_1M, equipped with its standard contact metric structure $(\eta, \bar{g})$, is a (k, μ)-space if and only if the base manifold (M, g) has constant curvature $\bar{c}$. In this case, $k = \bar{c}(2 - \bar{c})$ and $\mu = -2\bar{c}$. This result was extended to g-natural contact metric structures over T_1M by the following

44 Theorem ([4]). *Let $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ is a (k, μ)-space if and only if (M, g) has constant sectional curvature $\bar{c}$. In this case, if $(T_1M, \tilde{\eta}, \tilde{G})$ is not Sasakian, then*

$$\begin{cases} k = \dfrac{1}{16\alpha^2}\left[-a^2\bar{c}^2 + 2(\alpha - b^2)\bar{c} + d(2(a + c) + d)\right], \\ \\ \mu = \dfrac{1}{2\alpha}(d - a\bar{c}), \end{cases} \tag{28}$$

where $\alpha := a(a + c) - b^2$.

In [19], E. Boeckx showed that non-Sasakian (k,μ)-spaces are determined, up to isometries, by the value of the invariant

$$I_{(k,\mu)} = \frac{1-\mu/2}{\sqrt{1-k^2}}.$$

Using (28), we can determine $I_{(k,\mu)}$ for all g-natural contact metric structures corresponding to non-Sasakian (k,μ)-spaces. Taking into account Theorem 44, standard calculations lead to the following

45 Theorem. *Let (M,g) be a Riemannian manifold of constant sectional curvature $\bar{c}$, and (k,μ) any real pair with $k<1$. There exists a g-natural contact metric structure $(\tilde{\eta},\tilde{G})$ such that $(T_1M,\tilde{\eta},\tilde{G})$ is a (k,μ)-space if and only if*

(i) either $I_{(k,\mu)} > -1$ and $(I^2_{(k,\mu)}-1)\bar{c} > 0$, or

(ii) $I_{(k,\mu)} = 1$ and $\bar{c} = 0$.

In particular, all non-Sasakian (k,μ)-spaces such that $I_{(k,\mu)} > -1$, can be realized as g-natural contact metric structures on a Riemannian manifold (M,g) of (suitable) constant sectional curvature.

5.8 Locally symmetric g-natural contact structures

Locally symmetric spaces are one of the main topics in Riemannian geometry. In the framework of contact metric geometry, local symmetry has been extensively investigated, obtaining many rigidity results. As concerns the unit tangent sphere bundle, Blair proved the following

46 Theorem ([16])**.** *$(T_1M,\eta,\bar{g})$ is locally symmetric if and only if either (M,g) is flat or it is a surface of constant sectional curvature* 1*.*

Theorem 46 has been extended by replacing local symmetry by semi-symmetry ([20], [28]). Recently, Boeckx and Cho [21] showed definitively the rigidity of the hypothesis of local symmetry in contact Riemannian geometry, by proving the following

47 Theorem ([21])**.** *A locally symmetric contact metric space is either Sasakian and of constant curvature* 1*, or locally isometric to the unit tangent sphere bundle of a Euclidean space with its standard contact metric structure.*

Taking into account Theorem 47, we have the following

48 Theorem ([4])**.** *A g-natural contact metric structure $(\tilde{\eta},\tilde{G})$ on T_1M is locally symmetric if and only if $(\tilde{\eta},\tilde{G}) = (\bar{\eta},\bar{g})$ is the standard contact metric structure of T_1M and (M,g) is flat.*

D. Perrone [65] proved that a locally symmetric contact metric manifold $(\bar{M}, \eta, g, \xi, \varphi)$ satisfies $\nabla_\xi h = 0$, where $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. In [4], the authors proved the following

49 Corollary. *A g-natural contact metric structure $(\tilde{\eta}, \tilde{G})$ on T_1M satisfies $\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = 0$ and is not locally symmetric if and only if*

- *either (M, g) is flat and $d = 0$ but $\tilde{G} \neq \bar{g}$, or*
- *(M, g) has constant curvature $\bar{c} > 0$ and $\tilde{G} = a.\widetilde{g^s} + (\bar{c} - 1)a.\widetilde{g^v} + \bar{c}a(4a - 1).\widetilde{k^v}$, or*
- *(M, g) is locally isometric to a compact rank-one symmetric space (of non-constant sectional curvature and Jacobi eigenvalues $(p, 4p)$ with $p > 0$), and*

$$\text{either } \tilde{G} = a.\widetilde{g^s} + (p-1)a.\widetilde{g^v} 4pa.\widetilde{k^v} \quad \text{or} \quad \tilde{G} = a.\widetilde{g^s} + (4p-1)a.\widetilde{g^v} + pa.\widetilde{k^v}.$$

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