Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **37** (2017) suppl. 1, 187–200. 10.1112/blms/23.4.372

Determining the number of Killing tensors by (linear) algebra

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Received: 28-07-2016; accepted: 26-08-2016.

Abstract. It is a natural problem to determine the number of independent Killing tensors of a certain rank that a given metric admits. In this paper, it is discussed how this problem can be addressed in a computer-algebraic way. Pseudo-Riemannian metrics expressed in local coordinates are considered, with and without parameter dependence. No assumption is made about analyticity of the Killing tensors.

Keywords: Killing tensor; integral of motion; prolongation-projection method

MSC 2010 classification: 53-02; 37J30

1 Introduction

Let (M,g) be a pseudo-Riemannian manifold of dimension D.

Definition 1. A Killing tensor is a symmetric (0,d)-covariant tensor field $K : M \to T^*M^{\otimes d}$ such that for any vector field X on T^*M,

$$\nabla_X K(X, X, \dots, X) = 0. \tag{1}$$

Equivalently, the symmetrisation of the covariant derivative vanishes. The integer number d is called the *rank* (or valence) of K.

Remark 1. Killing tensors are equivalent to orbital *invariants* of the geodesic flow. Let $H : M \to T^*M$ be the Hamiltonian defined by $H(x,p) = \sum_i g^{ij} p_i p_j$, where (x,p) are the coordinates on T^*M (the fiber coordinates p are referred to as momenta). An orbital invariant (also called *first integral*) is a function $I : M \to T^*M$ such that $\{H, I\} = 0$, where $\{\cdot,\cdot\}$ is the standard Poisson bracket on T^*M. For a Killing tensor K, the function

$$F(x,p) = \sum K^{i_1,\dots,i_d} p_{i_1} \cdots p_{i_d}$$

[i]Andreas Vollmer is a research fellow of the Istituto Nazionale di Alta Matematica and acknowledges financial support by GRK 1523 (Deutsche Forschungsgemeinschaft) and by the project FIR-2013 Geometria delle equazioni differenziali.

is an orbital invariant.

Corresponding to the Poisson bracket, there exists the so-called *Schouten-Nijnhuis bracket* $[\cdot,\cdot] : T^*M^{q_1} \times T^*M^{q_2} \to T^*M^{q_1+q_2-1}$, such that

$$\{f_1, f_2\} = \sum [K_1, K_2]^{i_1,\dots,i_{q_1+q_2-1}}\, p_{i_1} \cdots p_{i_{q_1+q_2-1}}$$

where $f_j = \sum K_j^{i_1,\dots,i_{q_j}}\, p_{i_1} \cdots p_{i_{q_j}}$, for $j = 1, 2$. Particularly, we have that Equation (1) is equivalent to $[g, K] = 0$.

Definition 2. We say that two or more Killing tensors are *involutive* (or in involution) if their mutual Schouten-Nijnhuis brackets vanish.

In the present paper we address the following problems:

Problem 1.1. Given a metric g in local coordinates, how many independent irreducible Killing tensors of a given rank d do exist?

An *irreducible* Killing tensor is understood as a Killing tensor that cannot be decomposed into a linear combination of products of Killing vector fields[1] and the metric.

Actually, our examples in the following are specific to involutive Killing tensors. This additional assumption is not essential for our approach, but we keep it for sake of simplicity. The assumption is also justified if one's interest lies in the Liouville integrability of the metric g.

Remark 2. A metric g is Liouville integrable[2] if it admits D functionally independent orbital invariants in involution. The geodesic equations are then solvable by quadrature.

Liouville integrability plays a central role in many fields, e.g. the classical Kepler problem admits enough orbital invariants for Liouville integrabillity (making it solvable by a direct computation).

Remark 3. We note that the approach presented in this paper can also be applied to invariants of the Hamiltonian flow that are (non-homogeneous) polynomials in the momenta for a Hamiltonian that is a non-homogeneous polynomial in momenta. However, the system quickly has very many equations and unknowns such that the computations get computationally time-consuming.

Problem 1.2. Given a family of metrics g_α that depend on a real-valued parameter $\alpha \in \mathbb{R}$, determine the values of α (if any) for which there is a Killing tensor of rank d in addition to a collection of known ones.

The discussion is going to focus on metrics such that the Hamiltonian $\sum g^{ij} p_i p_j$ has a rational dependence on the parameter α (constant global factors

[1] More precisely, the (0,1)-tensor fields corresponding to Killing vectors

[2] By a slight misuse of terminology we refer to the metric as Liouville integrable. More precisely, it is the geodesic flow determined by g that is Liouville integrable.

can be ignored). Such a dependence occurs for many metrics, for instance the Kerr metric, the Nariai metric or the C-metric (the examples are taken from astrophysics). In the present paper we give a similar application within the theory of stationary and axially symmetric metrics where the metric does not depend rationally on the parameter.

Problems 1.1 and 1.2 have, of course, been asked and studied for a long time. Classically, the existence of an extra Killing tensor (or, more generally, orbital invariants that are polynomials in the momenta) has for instance been used to integrate the geodesic flow on the ellipsoid [12] or for the Kepler problem. Killing tensors also play a crucial role in projective geometry (projectively equivalent metrics can be related to one another by a Killing tensor) [4]. There are many other contexts where Killing tensors appear, e.g. c.f. the Maupertuis principle, or the problem of existence of separable coordinates [17], or superintegrability [16]. Note also that if the Hamiltonian is a homogeneous polynomial in the momenta and F is an orbital invariant that is polynomial in the momenta, then each homogeneous component of F is an orbital invariant itself [10, 11].

In astrophysics, the probably most famous example of a space-time that admits an extra Killing tensor of rank 2 is the family of Kerr metrics [13, 14]. Indeed, Brandon Carter has classified the stationary and axially symmetric space-times that permit to solve the Hamilton-Jacobi and the Schrödinger equation by separation of variables [15] (which is related to the existence of an extra quadratic Killing tensor, cf. [13]). Many papers have since been published on similar problems for higher rank Killing tensors.

Generally speaking, there exists only a limited list of methods that are able to answer questions as posed in Problem 1.1 or Problem 1.2. Classically, the relevant system of partial differential equations (PDE) has been solved using the method of characteristics, e.g. [11]. For quadratic Killing tensors it is possible to use separation of variables as in [15]. Other methods have been discussed in [19] or [33]. For analytic Killing tensors, methods from Differential Galois Theory can be applied [27]. There are also numerical approaches, for instance via surfaces of section, e.g. [23, 26].

The method used in this paper is taken from [3]. Similarly to the approach via Differential Galois Theory, it can be implemented on a computer. However, we do not need the assumption of analyticity here (of course at the cost of assumptions on the rank of the Killing tensor). For other uses of the method see [20, 29]. Similar ideas have been used, e.g., in [5].

The paper is organized as follows. First, we consider Problem 1.1. We describe the general idea of the method and discuss the Darmois metric as an example (this part is based on the references [3, 28, 29]). Second, we consider Problem 1.2. The problem is addressed specifically for metrics whose Hamiltonians depend rationally on the parameter. For such metrics the admissible parameter values are roots of one polynomial equation. We conclude the paper with a discussion of the Zipoy-Voorhees metric (a generalisation of the Darmois metric, see [28] for details on the example). The Hamiltonian for this latter example is not rational in the parameter, but structurally very similar.

2 Parameter-free metrics

Consider Equation (1) in local coordinates,

$$\nabla_{(j}K_{i_1,\dots,i_d)} = 0. \tag{2}$$

This is a system of partial differential equations on the components of the Killing tensor K. It contains $\binom{D+d}{D-1}$ equations and $\binom{d+D-1}{D-1}$ unknown functions, i.e. the system is overdetermined. It is also well-known that it is of finite type, i.e. that after differentiating $d+1$ times with respect to the base variables, a closed algebraic system is obtained [2]. Actually, the system is linear algebraic, since the system of PDE is of first order. The unknowns of this linear system are $\partial_{j_1}\cdots\partial_{j_{d+1}}K_{i_1,\dots,i_d}$, of which there are $n=\binom{d+D-1}{D-1}\binom{d+4}{2}$ many. The number of equations of the linear system is $m=\binom{D+d}{D-1}\binom{d+3}{2}$, i.e. the linear system is overdetermined. The problem is completely described by the $m\times n$-matrix $A[K]$ of coefficients. Solutions of the PDE problem (1) or (2) form a vector space[3].

Observation 2.1. The PDE problem (1) can be translated into a linear algebraic problem. Solutions of the PDE problem are solutions of the linear algebraic problem. Therefore, the number of independent solutions of the linear problem is an upper bound to the number of independent Killing tensors.

The rank of the matrix $A[K]$ depends on the position on the manifold. However, if the Killing vector K can be written as a linear combination in a neighbourhood of a point p, then it can be written as a linear combination everywhere. Actually, the only thing that needs to be determined is the generic rank of the matrix $A[K]$, and this can be achieved by (generically) choosing a point $p\in M$.

Once this has been done, the linear structure of the space of solutions permits as well to add constant multiples of known Killing tensors (i.e. to do the replacement $K_p\to K_p+c\mathring{K}_p$ where $\mathring{K}$ is a known Killing tensor of rank d and c a suitably chosen constant).

Observation 2.2. Let $p\in M$ be chosen generically. Assume there are N known independent Killing tensors $\mathring{K}^{[\nu]}$, $\nu=1,\dots,N$ of rank d. Then there are

[3]The space of Killing tensors of fixed rank d is a vector space. The space of all Killing tensors, however, forms an algebra.

constants c_ν, $\nu = 1, \ldots, N$, such that the (generic) rank of $A[K]$ is exactly

$$\mathrm{rk}(A[K]) = N + \mathrm{rk}(\mathring{A})$$

where $\mathring{A} = A[K - \sum c_\nu \mathring{K}^{[\nu]}](p)$ is a (smaller) matrix with entries in $\mathbb{R}$.

In fact we have entries in $\mathbb{Q}$ in our examples such that we can rewrite the problem, and without loss of generality we can work with integer-valued matrices. Note that in the above equation, the rank on the left is a generic rank (the maximum over all positions), while on the right we have a usual matrix rank. Thus, the problem is translated into a linear problem with real-valued coefficients, namely to determine whether $\mathring{A}$ has full rank.

Determining the number of independent solutions of a linear problem is, of course, well-understood: The solutions are elements in the kernel of $\mathring{A}$, so the nullity of the matrices bounds from above the number of independent Killing tensors. Thus, if one can show equality of the nullity of the matrix with some lower bound to the number of independent Killing tensors, this proves nonexistence of additional ones. On the other hand, if one can show that the nullity is larger than expected, this can be an indication of additional Killing tensors.

Instead of solving Problem 1.1 we can therefore study the conceptually simpler question: What is the rank of the matrix A (A being either $A[K]$ or $\mathring{A}$)? Of course, it has since long been well-understood how to do this in principle: Determine the Gauß echelon form for A and count the non-zero rows. Indeed, Gauß elimination is not optimal [1], and there are now several algorithms with lower complexity available to complete the task. However, doing the computation in practical situations can still be challenging because the matrix A quickly has huge dimensions, typically several thousands of equations and unknowns.

On the other hand, actual problems often permit additional simplifications if one takes into account the specific symmetries of the metric g. We illustrate this by the following example from astrophysics.

Example 1 (Darmois metric)**.** The Darmois metric is a solution to the Einstein vacuum field equation and a special case of the Zipoy-Voorhees metric (10). Using local coordinates, it can be expressed as

$$g = \left(\frac{x+1}{x-1}\right)^2 \left(\left(\frac{x^2-1}{x^2-y^2}\right)^3 \left(dx^2 + \frac{x^2-1}{1-y^2}\, dy^2\right) + (x^2-1)(1-y^2)d\phi^2 \right) - \left(\frac{x-1}{x+1}\right)^2 dt^2. \quad (3)$$

This solution has first been described by Darmois [30]. It is a special case of the family of Zipoy-Voorhees metrics [31, 32], see Equation (10). The Darmois metric admits two involutive Killing vector fields and thus lacks only one involutive invariant for Liouville integrability. Let us look into the existence of an irreducible, involutive Killing tensor in addition to the metric and the two Killing vector fields.

The existence of the two Killing vector fields permits to rewrite the problem as a problem on a 2-dimensional surface M_{red} with metric g_{red} (and potential V), defined by the symplectic quotient by the two involutive Killing vectors. The resulting 2-dimensional problem has a non-homogeneous Hamiltonian that has a term quadratic in (p_x, p_y), and one of degree zero. These observations allow one to make the following statement when the orbital invariant defined by the Killing tensor is decomposed into components:

$$I_K(x,p) = K_x(p,\dots,p) = \sum_{k=0}^{d} \left(I_e^{(k)}(x,p) + I_o^{(k)}(x,p) \right) \tag{4}$$

where $I_e^{(k)}$ and $I_o^{(k)}$ are the components of I_K of k-th degree in (p_x, p_y) and of, respectively, even or odd degree in (p_φ, p_t).

Proposition 1. *This problem for rank d Killing tensors can be solved by answering the analogous problem for integrals $I_e^{(k)}$ that are non-homogeneous polynomials in the momenta of rank $k = d-2, d-1, d$ for the Hamiltonian $H = \sum g_{red}^{ij} p_i p_j + V$.*

Proof. Consider the polynomial invariant corresponding to a Killing tensor of rank d. It is a homogeneous polynomial[4] in p_x, p_y, p_φ, p_t. The Hamiltonian is also a homogeneous polynomial and has the following properties: It is of even degree in (p_x, p_y), i.e. the degree of any homogeneous component w.r.t these momenta is even (2 or 0). Moreover, the Hamiltonian is of even degree in p_φ and p_t, respectively (the coefficient of $p_\varphi p_t$ is zero). Thus, the PDE system separates up into four subsystems:
(i) Firstly, one for the component of the invariant that is even in (p_x, p_y) and in p_φ (and p_t). For later reference, let us call it the principal component.
(ii) Secondly, one that is even in (p_x, p_y) and odd in p_φ (and p_t). It turns out that the system of PDE governing it is identical to the principal component of an invariant of degree $d-2$.
(iii) There are two components with odd degree in (p_x, p_y), the first one is even in p_φ (and odd in p_t), the other vice versa. However, it turns out that the systems of PDE governing them are identical, and identical to the system of PDE for the principal component of an invariant of degree $d-1$.

[4] We shall not distinguish by notation between the momenta p_φ, p_t and the constant values they assume when restricting to shells of constant $p_\varphi = c_\varphi$ and $p_t = c_t$.

$\boxed{QED}$

So, we need only consider separately the problem whether there is an additional invariant of "principal type" in degrees $d-2$, $d-1$ and d, and this is much easier to do than solving the original problem.

The computation has been done in [29] and [28], where the detailed proof of Proposition 1 can be found. It permits to extend the result of [3] to higher rank Killing tensors. See [23] for the original motivation to study the problem (existence of an extra invariant is suggested in this reference, see however [26]), and [3, 26, 27] for related works.

Proposition 2. *The Darmois metric* (3) *admits no irreducible involutive Killing tensor of rank* ≤ 11*, other than the metric itself.*

Proof. The detailed proof can be found in [29]. As suggested by Proposition 1, the proof is completed in three steps:

Degree d-2=9: By a straightforward computation, one finds the matrix $A[K]$ as an explicit 5005×4620 matrix. We evaluate it at $(x, y) = (1/2, 2)$ and obtain the matrix $\mathring{A}$. Solving the linear system partially (see [28] for details), we obtain a 1058×726 matrix with integer entries. We verify that the kernel of this matrix is trivial, and this confirms that the initial matrix has full rank.

Degree d-1=10: The matrix $A[K]$ is a 7392×7098 matrix. We evaluate it at $(x, y) = (1/2, 2)$ and reduce it to a 1358×1043 matrix with integer entries. The kernel of the remaining matrix is trivial, and therefore the initial matrix also has trivial kernel.

Degree d=11: The matrix $A[K]$ is a 10920×10192 matrix, and in $(x, y) = (1/2, 2)$ we can reduce it to a 2162×1510 integer matrix problem. The kernel of this latter matrix is trivial and thus the kernel of the initial matrix is trivial.

$\boxed{QED}$

3 Parameter-dependent metrics

Let us now move over to the more involved problem of parameter-dependent metrics. The general problem is similar to Problem 1.1. Indeed, if the metric g_α admits an additional Killing tensor for arbitrary values of α, the situation is identical to Problem 1.1: We just have to replace matrix A by a parameter-dependent matrix $A(\alpha)$. Let us assume that $A(\alpha)$ depends polynomially on α. This can be assured if the Hamiltonian is rational in α. Note however that in Example 3 we do not have a metric that is rational in α, and also $A(\alpha)$ is not polynomial in α. Yet, we can proceed similarly.

For the parameter-dependent matrix problem, one replaces usual Gaußian elimination (or a similar algorithm of one's choice) by suitable algorithms working on the ring $\mathbb{R}[\alpha]$ of (univariate) polynomials in α. For instance, one can use an algorithm that brings $A(\alpha)$ into Smith normal form.

Let us use the notation $\mathring{A}(\alpha)$ analogously to the previous section, i.e. let $\mathring{A}(\alpha)_p$ be the matrix obtained after removing the maximum number of unknowns with the help of known Killing tensors. We have $\mathring{A}(\alpha) = \sum_{k=0}^{r} A_k \alpha^k$.

We assume that generically $\mathring{A}(\alpha)$ has trivial kernel. In this case, though the generic rank of $\mathring{A}(\alpha)$ is full, extra solutions can exist for specific values of α. Obviously, admissible values of α lie in the algebraic variety

$$\mathcal{V} = \{\, \alpha \mid \det(B) = 0 \text{ for any } m \times m \text{ submatrix } B \text{ of } \mathring{A}(\alpha)_p,\ p \in M \,\}$$

defined by $A(\alpha)$. Therefore the determinants of all quadratic $m \times m$ submatrices of the $n \times m$ matrix $A(\alpha)_p$ $(n > m, p \in M)$ have to vanish, i.e.

$$\begin{aligned} \det(B(\alpha)) &= \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} B(\alpha)_{i\sigma(i)} \\ &= \sum_{k=0}^{rm} \alpha^d \left(\sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) \sum_{i_1+\cdots+i_m=d} \prod_{j=1}^{m} (B_{i_j})_{j\sigma(j)} \right) = 0 \end{aligned} \tag{5}$$

$\forall\ m \times m$ submatrices B of $A(\alpha)_p$. Here, $B(\alpha) = \sum B_k \alpha^k$. By $\mathfrak{S}_m$ we denote the set of permutations of the integers $1, \dots, m$. Note that the determinants are polynomials in α.

Lemma 1. *Though defined by infinitely many equations* (5)*, the (univariate) ideal defining the variety $\mathcal{V}$ is generated by only one polynomial equation.*

Proof. By the Hilbert basis theorem it is generated by finitely many equations. Actually one equation suffices, because via the Euclidean algorithm one can compute their greatest common divisor, e.g. [34]. Hence, the admissible α are the roots of one polynomial function. $\boxed{QED}$

As one sees here, it is preferable to have $A(\alpha)$ given as a dense matrix with few zero entries, such that one easily can produce non-vanishing equations (5).

Although in general it might be difficult to find this polynomial function explicitly, we suggest that in practical applications it is often sufficient to consider just a few of the (non-trivial) defining equations (cf. Example 3 for instance).

Remark 4. If we had a polynomial dependence on several parameters, we could still write down analogous equations for the variety that contains the admissible parameter values. Though there are still only finitely many generating equations by the Hilbert basis theorem, however, in general more than one equation is needed.

Example 2. The simplest example is the case $r = 1$, i.e. a Hamiltonian that depends linearly on the parameter, $A(\alpha)_p = A_1\alpha + A_0$. Consider the linear mapping $A_1 : \mathbb{R}^m \to \mathbb{R}^n$. Let $(e_1, \ldots, e_\kappa)$ be an orthonormal basis of its null space and extend it to an orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^m$. Furthermore, let $(f_1, \ldots, f_\eta)$ be an orthonormal basis of the image of A_1, and extend it to an orthonormal basis $(f_1, \ldots, f_n)$ of $\mathbb{R}^n$. In this basis,

$$A(\alpha)_p = \begin{pmatrix} A_1^{(1)} & 0 \\ 0 & 0 \end{pmatrix} \alpha + \begin{pmatrix} A_0^{(1)} & A_0^{(2)} \\ A_0^{(3)} & A_0^{(4)} \end{pmatrix} \tag{6}$$

where $\operatorname{rk} A_1^{(1)} = \eta$ and $\operatorname{rk}(A_0^{(3)}\ A_0^{(4)}) = \kappa$. Recall the assumption that $A(\alpha)_p$ generically has full rank. By solving the equations that do not contain α, we can reduce (6) to a linear system of the form (X denotes the vector of unknowns)

$$\left(A_1^{(1)}\alpha + \tilde{A}_0\right)\ X = 0 \tag{7}$$

or, equivalently,

$$\left(\alpha\,\mathrm{Id} + (A_1^{(1)})^{-1}\,\tilde{A}_0\right)\ X = 0. \tag{8}$$

In order that this system of equations can admit nontrivial solutions, we need

$$\det\left(\alpha\,\mathrm{Id} + (A_1^{(1)})^{-1}\,\tilde{A}_0\right) = 0, \tag{9}$$

which is the characteristic polynomial of the matrix $(A_1^{(1)})^{-1}\,\tilde{A}_0$.

Observation 3.1. Parameter values that admit nontrivial Killing tensors are eigenvalues of the matrix $(A_1^{(1)})^{-1}\,\tilde{A}_0$.

Of course, typically not every eigenvalue needs to represent a case with nontrivial Killing tensors. To see this one might just recall that we only considered $A(\alpha)_p$ for one particular value of $p \in M$. However, we suggest that in practical applications it will typically be enough to compute the obstructions from a few choices for $p \in M$ only. In general this might be sufficient to reduce the possible values of α to a number of alternatives that can be checked manually.

We finish our discussion with an example of a parameter-dependent metric that does not depend polynomially on the parameter.

Example 3 (Zipoy-Voorhees metric)**.** The family of Zipoy-Voorhees metrics generalises the Darmois metric.

$$g = \left(\frac{x+1}{x-1}\right)^{\delta} \left(\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2-1} \left(dx^2 + \frac{x^2-1}{1-y^2}\, dy^2 \right) + (x^2-1)(1-y^2) d\phi^2 \right) \\ - \left(\frac{x-1}{x+1}\right)^{\delta} dt^2 \quad (10)$$

It depends on a parameter δ, where $\delta = 0$ represents the flat metric and $\delta = 1$ represents the Schwarzschild metric. Let us consider rank-2 Killing tensors.

In contrast to our previous assumption, the metric (10) is not a polynomial in the parameter δ. However, in spite of this, we can deduce algebraic obstructions on δ along the same lines as outlined for polynomial metrics. A detailed treatment of the example can be found in [28].

In particular, we can use the simplification laid out in Proposition 1. We restrict to the point $(x, y) = (1/2, 2)$ and to Killing tensors with the following form in the coordinates of (10):

$$K = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \quad (11)$$

(here $*$ stands for a non-zero entry).

We can perform usual Gauß elimination with entries that are nonzero for any value of δ, e.g. terms r^{δ} with $r \neq 0$, see [28] for details. In this way, a smaller, parameter-dependent, only 4×3 matrix is obtained (remove uninteresting rows and columns). We can compute the determinants of the 4 submatrices as described above for polynomial metrics.

For $(x, y) = (1/2, 2)$, the following equations are obtained (these are the determinants of the submatrices):

$$\delta(\delta^2-1)(96\delta^6 - 144\delta^5 + 160\delta^4 - 105\delta^3 - 23\delta^2 + 24\delta + 28) = 0$$
$$\delta(\delta^2-1)(96\delta^4 - 324\delta^3 + 420\delta^2 - 459\delta + 159) = 0$$
$$\delta(\delta^2-1)(192\delta^6 - 720\delta^5 + 1288\delta^4 - 1266\delta^3 + 835\delta^2 + 114\delta - 56) = 0$$
$$\delta(\delta^2-1)(3328\delta^8 - 10560\delta^7 + 21664\delta^6 \\ -28104\delta^5 + 28788\delta^4 - 16665\delta^3 + 11306\delta^2 - 2451\delta + 1064) = 0$$

As was mentioned previously, univariate polynomial ideals are generated by the greatest common divisor of the generating equations [34]. Here, this equation is $\delta(\delta^2 - 1) = 0$ and thus we have three candidates for additional nontrivial quadratic Killing tensors, $\delta = 0, \pm 1$. We also see that we do not need the full information from all submatrices. Any 2 of the 4 equations would have produced the same candidates.

Let us analyse them separately: $\delta = 0$ is the flat case, for which we know that any Killing tensor is reducible (we have the four Killing vectors ∂_x, ∂_y, ∂_φ and ∂_t). The cases $\delta = \pm 1$ are essentially only one case because $x \to -x$ transforms (10) into itself if we replace δ by $-\delta$. This is the Schwarzschild metric, for which we know that there is a set of four involutive Killing tensors (2 of rank 1, 1 of rank 2 plus the metric). However, the rank-2 Killing tensor in this case is not irreducible. In fact there are two additional (non-involutive) Killing vectors,

$$\begin{aligned} K_1 &= \sin\varphi\,\partial_\theta + \cot\theta\,\cos\varphi\,\partial_\varphi, \\ K_2 &= \cos\varphi\,\partial_\theta - \cot\theta\,\sin\varphi\,\partial_\varphi, \end{aligned}$$

and the rank-2 Killing tensor is given by a linear combination $L = K_1^\flat \otimes K_1^\flat + K_2^\flat \otimes K_2^\flat + d\varphi^2$.

Acknowledgements. The author wishes to thank Giovanni Calvaruso, Boris Kruglikov, Vladimir Matveev, Konrad Schöbel and Galliano Valent for discussions.

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