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Estimates on arc-lengths of trajectory-fronts for surface magnetic fields

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Abstract. A trajectory-front for a surface magnetic field is formed by terminuses of trajectory-segments of given arc-radius which are emanating from a given point. In order to show how trajectories are spreaded we give estimates of their arc-lengths of trajectory-fronts.

Keywords: surface magnetic fields, trajectory-fronts, magnetic Jacobi field, comparison theorems

MSC 2010 classification: Primary 53C22, Secondary 53C20

Introduction

A closed 2-form on a Riemannian manifold is said to be a magnetic field because it can be regarded as a generalization of a static magnetic field on a Euclidean 3-space $\mathbb{R}^3$ (see [9, 12]). Typical examples of magnetic fields are constant multiples of the Kähler form on a Kähler manifold (see [1]), constant multiples of the canonical form on a real hypersurface in a Kähler manifold (see [7]), and 2-forms on an orientable Riemann surface. Motions of charged particle of unit mass and of unit speed under the influence of a magnetic field are said to be trajectories for this magnetic field. It is needless to say that properties of trajectories show the mixture of properties of a magnetic field and properties of the underlying Riemannian manifold.

In this paper we study trajectory-fronts for surface magnetic fields. A trajectory-front of arc-radius r consists of terminuses of trajectory-segments of arclength r which are emanating from a given point. It is also called a trajectory sphere. It shows how trajectories are spreaded. In their paper ([6]) Bai-Adachi gave estimates of areas of trajectory spheres for Kähler magnetic fields on a Kähler manifold. We note that Kähler magnetic fields are uniform magnetic fields. This means that the Lorentz force of a Kähler magnetic field does not depend on points. Hence trajectory spheres show essentially properties of underlying manifolds. On contrary, surface magnetic fields are not uniform, and are the simplest examples of non-uniform magnetic fields. We therefore study lengths of trajectory-fronts by investigating the influence of properties of magnetic fields.

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1 Trajectory-fronts

Let M be an orientable Riemann surface. It is naturally regarded as a 1-dimensional complex manifold with complex structure J. Given a smooth function h on M, we consider a 2-form $\mathbb{B}_h = h\, dvol_M$, where $dvol_M$ denotes the volume form on M. We call this a surface magnetic field on M. A smooth curve γ parameterized by its arc-length is said to be a trajectory for $\mathbb{B}_h$ if it satisfies the differential equation $\nabla_{\dot{\gamma}}\dot{\gamma} = h(\gamma)J\dot{\gamma}$, where $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇. For a unit tangent vector $u \in U_pM$ at a point $p \in M$, we denote by γ_u a trajectory for a surface magnetic field $\mathbb{B}_h$ with initial vector $\dot{\gamma_u}(0) = u$. We define a magnetic exponential map $\mathbb{B}_h\exp_p : T_pM \to M$ of the tangent space at p by

$$\mathbb{B}_h\exp_p(w) = \begin{cases} \gamma_{w/\|w\|}(\|w\|), & \text{if } w \neq 0_p, \\ p, & \text{if } w = 0_p. \end{cases}$$

For a trivial magnetic field $\mathbb{B}_0$ which is given as a surface magnetic field of null function, it is an ordinary exponential map $\exp_p : T_pM \to M$. By using magnetic exponential maps, we define a trajectory-front $F_r^h(p)$ of arc-radius r centered at p as

$$F_r^h(p) = \big\{\mathbb{B}_h\exp_p(ru) \bigm| u \in U_pM\big\}.$$

Since M of real dimension 2, we call it a trajectory-front. For a magnetic field on a Riemannian manifold of real dimension greater than 2, we can define such a set and call it a trajectory-sphere (cf. [5, 6]). We note that for a trivial magnetic field, its trajectory-sphere is nothing but a geodesic sphere. Also, we note that a trajectory-front $F_r^h(p)$ is contained in a geodesic ball $B_r(p) = \exp_p\big(\{tu \mid 0 \leq t \leq r,\ u \in U_pM\}\big)$ of radius r centered at p.

We here recall trajectory-fronts on a real space form for a surface magnetic field $\mathbb{B}_\kappa$ ($\kappa \in \mathbb{R}$) of constant Lorentz force (see [5]). Here, a real space form $\mathbb{R}M^2(c)$ of constant sectional curvature c is either one of a standard sphere $S^2(c)$, a Euclidean plane $\mathbb{R}^2$ or a real hyperbolic space $H^2(c)$ depending on c is positive, zero or negative.

Examples 1.1. On a Euclidean plane, the distance between two points $\gamma(0)$ and $\gamma(r)$ of a trajectory γ for $\mathbb{B}_\kappa$ is given as $d\big(\gamma(0),\gamma(r)\big) = \big(2/|\kappa|\big)\sin\big(|\kappa|r/2\big)$ when r satisfies $0 < r < 2\pi/|\kappa|$. Therefore, a trajectory-front $F_r^\kappa(p)$ coincides with a geodesic sphere $S_\rho(p)$ of radius $\rho = \big(2/|\kappa|\big)\sin\big(|\kappa|r/2\big)$.

Examples 1.2. On a standard sphere $S^2(c)$ of curvature c, when $0 < r < 2\pi/\sqrt{\kappa^2+c}$, the distance ρ between two points $\gamma(0)$ and $\gamma(r)$ of a trajectory γ for $\mathbb{B}_\kappa$ satisfies

$$\sqrt{\kappa^2+c}\sin\big(\sqrt{c}\,\rho/2\big) = \sqrt{c}\,\sin\big(\sqrt{\kappa^2+c}\,r/2\big).$$

Therefore, a trajectory-front $F_r^\kappa(p)$ coincides with a geodesic sphere $S_\rho(p)$.

Examples 1.3. On a real hyperbolic space $H^2(c)$ of curvature c, the distance ρ between two points $\gamma(0)$ and $\gamma(r)$ of a trajectory γ for $\mathbb{B}_\kappa$ satisfies

$$\begin{cases} \sqrt{|c|-\kappa^2}\,\sinh\big(\sqrt{|c|}\,\rho/2\big) = \sqrt{|c|}\,\sinh\big(\sqrt{|c|-\kappa^2}\,r/2\big), & \text{when } |\kappa| < \sqrt{|c|}, \\ 2\sinh\big(\sqrt{|c|}\,\rho/2\big) = \sqrt{|c|}\,r, & \text{when } \kappa = \pm\sqrt{|c|}, \\ \sqrt{\kappa^2+c}\,\sinh\big(\sqrt{|c|}\,\rho/2\big) = \sqrt{|c|}\,\sin\big(\sqrt{\kappa^2+c}\,r/2\big), & \text{when } |\kappa| > \sqrt{|c|}, \end{cases}$$

where $0 < r < 2\pi/\sqrt{\kappa^2+c}$ when $\kappa^2+c>0$.

2 Magnetic Jacobi fields

In order to study trajectory-fronts, we need to investigate magnetic exponential maps and so, variations of trajectories. A vector field Y along a trajectory γ for $\mathbb{B}_h$ is said to be a *magnetic Jacobi field* if it satisfies

$$\begin{cases} \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y + R(Y,\dot\gamma)\dot\gamma - (Yh)J\dot\gamma - h(\gamma)J\nabla_{\dot\gamma}Y = 0, \\ \langle \nabla_{\dot\gamma}Y, \dot\gamma\rangle \equiv 0. \end{cases} \tag{1}$$

We call a smooth map $\alpha : I\times(-\epsilon,\epsilon)\to M$ a variation of trajectory for $\mathbb{B}_h$ if for each $s\in(-\epsilon,\epsilon)$ the map $\alpha_s = \alpha(\ ,s) : I\to M$ is a trajectory for $\mathbb{B}_h$. Since α_s is parameterized by its arc-length, we have $\big\langle \nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}\big\rangle = 0$. By differentiating both sides of the equation $\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t} = h(\alpha)J\frac{\partial\alpha}{\partial t}$ by s, we find that a vector field $\frac{\partial\alpha}{\partial s}(\cdot,s)$ along a trajectory α_s satisfies the equations (1). Thus, a variation of trajectories gives a magnetic Jacobi field. One can easily check that the converse holds (see [2, 11]). Since M of real dimension 2, we decompose a vector field Y along γ into two components and denotes as $Y = f_Y\dot\gamma + g_Y J\dot\gamma$ with smooth functions f_Y, g_Y along γ. Then (1) turns to

$$\begin{cases} g_Y'' + g_Y\{K(\gamma)+h(\gamma)^2 - (J\dot\gamma)h\} = 0, \\ f_Y' = h(\gamma)g_Y, \end{cases} \tag{2}$$

where $K(\gamma(t))$ denotes the sectional curvature of the tangent plane at $\gamma(t)$. A point $\gamma(t_0)$ with $t_0 \neq 0$ is said to be a spherical magnetic conjugate point of $\gamma(0)$ along γ if there is a non-trivial magnetic Jacobi field Y for $\mathbb{B}_h$ along γ with $Y(0) = 0$ and $Y(t_0) = 0$. In this case we call t_0 a spherical magnetic conjugate value of $\gamma(0)$ along γ. Putting $p = \gamma(0)$ we denote by $t_s^h(p;\gamma)$ the minimal positive spherical magnetic conjugate value of p along γ. When there are no spherical magnetic conjugate points of p on the trajectory half-line $\gamma(0,\infty)$, we set $t_s^h(p;\gamma) = \infty$. By definition the differential of the map

$$\mathbb{B}_h\exp_p\big|_{rU_pM}\colon rU_pM = \{ru \mid u \in U_pM\} \to M$$

is singular at ru if and only if r is a spherical magnetic conjugate value along a trajectory γ_u.

Similarly, we say a point $\gamma(t_c)$ with $t_c \neq 0$ to be a magnetic conjugate point of $p = \gamma(0)$ along γ if there is a non-trivial magnetic Jacobi field $Y = f_Y\dot\gamma + g_Y J\dot\gamma$ for $\mathbb{B}_h$ along γ with $Y(0) = 0$ and $g_Y(t_c) = 0$. In this case we call t_0 a spherical magnetic conjugate value of $\gamma(0)$ along γ. We denote by $t_c^h(p;\gamma)$ the minimal positive magnetic conjugate value of p along γ. When there are no magnetic conjugate points of p on the trajectory half-line $\gamma(0,\infty)$, we set $t_c^h(p;\gamma) = \infty$. Clearly we have $0 < t_c^h(p;\gamma) \leq t_s^h(p;\gamma)$. We set $t_c^h(p) = \min\big\{t_c^h(p;\gamma_u) \bigm| u \in U_pM\big\}$.

We here make mention of magnetic Jacobi fields for uniform magnetic fields on a real space form $\mathbb{R}M^2(c)$. For constants κ and c, we define functions $\mathfrak{s}_\kappa(t;c)$, $\mathfrak{u}_\kappa(t;c) : \big[0, 2\pi/\sqrt{\kappa^2+c}\,\big] \to \mathbb{R}$ by

$$\mathfrak{s}_\kappa(t;c) = \begin{cases} \big(2/\sqrt{\kappa^2+c}\,\big)\ \sin\!\big(\sqrt{\kappa^2+c}\,t/2\big), & \text{if } \kappa^2+c>0,\\ t, & \text{if } \kappa^2+c=0,\\ \big(2/\sqrt{|c|-\kappa^2}\,\big)\ \sinh\!\big(\sqrt{|c|-\kappa^2}\,t/2\big), & \text{if } \kappa^2+c<0, \end{cases}$$

$$\mathfrak{u}_\kappa(t;c) = \begin{cases} \big(4/(\kappa^2+c)\big)\,\big\{1-\cos\!\big(\sqrt{\kappa^2+c}\,t/2\big)\big\}, & \text{if } \kappa^2+c>0,\\ t^2/2, & \text{if } \kappa^2+c=0,\\ \big(4/(|c|-\kappa^2)\big)\,\big\{\cosh\!\big(\sqrt{|c|-\kappa^2}\,t/2\big)-1\big\}, & \text{if } \kappa^2+c<0. \end{cases}$$

Here, we regard $2\pi/\sqrt{\kappa^2+c}$ as infinity when $\kappa^2 + c \leq 0$ (see [3]). We use such a convention throughout of this paper. We note that if $\kappa_1^2 + c_1 < \kappa_2^2 + c_2$ we see $\mathfrak{s}_{\kappa_1}(t;c_1) > \mathfrak{s}_{\kappa_2}(t;c_2)$. For a trajectory γ on a real space form $\mathbb{R}M^2(c)$, by solving the equations (2) under the condition that $Y(0) = 0$ (i.e. $f_Y(0) = g_Y(0) = 0$), we obtain that a magnetic Jacobi field Y along γ with $Y(0) = 0$ is given as

$$Y(t) = g_Y'(0)\big\{\kappa\mathfrak{u}_\kappa(t;c)\dot\gamma(t) + \mathfrak{s}(t;\kappa,c)J\dot\gamma(t)\big\}.$$

In particular, for an arbitrary trajectory γ for $\mathbb{B}_\kappa$ have $t_s^h\big(\gamma(0);\gamma\big) = 2\pi/\sqrt{\kappa^2+c}$. Since a trajectory-front is given as $F_r^\kappa(p) = \mathbb{B}_\kappa\mathrm{exp}_p(rU_pM)$, its arclength is given as $2\pi\Theta_\kappa(r;c)$, where

$$\Theta_\kappa(r;c) = \sqrt{\kappa^2\mathfrak{u}_\kappa(r;c)^2+\mathfrak{s}_\kappa(r;c)^2}$$
$$= \begin{cases} \dfrac{2}{\kappa^2+c}\sin\sqrt{\kappa^2+c}r/2\sqrt{\kappa^2+c\cos^2\frac{1}{2}\sqrt{\kappa^2+c}r}, & \text{if } \kappa^2+c>0,\\ r\sqrt{\kappa^2r^2+4}/2, & \text{if } \kappa^2+c=0,\\ \dfrac{2}{|c|-\kappa^2}\sinh\sqrt{|c|-\kappa^2}r/2\sqrt{|c|\cosh^2\sqrt{|c|-\kappa^2}r/2-\kappa^2}, & \text{if } \kappa^2+c<0. \end{cases}$$

when $M = \mathbb{R}M^2(c)$ and $r \le 2\pi/\sqrt{\kappa^2+c}$.

We now give estimate of arclengths of trajectory-fronts for surface magnetic fields on a general Riemann surface. For positive constants a, b we define a function $\Theta(r;a,b,c)$ by

$$\Theta(r;a,b,c)$$
$$= \sqrt{a^2\mathfrak{u}_b(r;c)^2+\mathfrak{s}_b(r;c)^2}$$
$$= \begin{cases} \dfrac{1}{b+c}\sqrt{a^2(1-\cos\sqrt{b+c}\,r)^2+(b+c)\sin^2\sqrt{b+c}\,r}, & \text{if } b+c>0,\\ r\sqrt{br^2+4}/2, & \text{if } b+c=0,\\ \dfrac{1}{|c|-b}\sqrt{a^2(\cosh\sqrt{|c|-b}\,r-1)^2+(|c|-b)\sinh^2\sqrt{|c|-b}\,r}, & \text{if } b+c<0. \end{cases}$$

Clearly it satisfies $\Theta(t;|\kappa|,\kappa^2,c) = \Theta_\kappa(t;c)$. Our results are the following.

Theorem 2.1. Let $\mathbb{B}_h$ be a surface magnetic field on a complete orientable Riemannian surface M whose sectional curvatures satisfy $\mathrm{Riem}^M \le c$ with some constant c. For an arbitrary point $p \in M$, we take a positive r with $r \le t_c^h(p)$. If we set $a = \max_{x\in B_r(p)}|h(x)|$ and $b := \min_{x\in B_r(p)}\big(h(x)^2 - \|\nabla h(x)\|\big)$, then the arclength of the curve of the trajectory-front $F_r^h(p)$ is estimated from above as $\mathrm{length}\big(F_r^h(p)\big) \le 2\pi\Theta(r;a,0,b+c)$.

Theorem 2.2. Let $\mathbb{B}_h$ be a surface magnetic field on a complete orientable Riemannian surface M whose sectional curvatures satisfy $\mathrm{Riem}^M \ge c$ with some constant c. For an arbitrary point $p \in M$, we take a positive r with $r \le t_c^h(p)$. Suppose $\hat{a} := \min_{x\in B_r(p)}|h(x)| > 0$. If we set $\hat{b} = \max_{x\in B_r(p)}\big(h(x)^2+\|\nabla h(x)\|\big)$, then the arclength of the curve of the trajectory-front $F_r^h(p)$ is estimated from below as $\mathrm{length}\big(F_r^h(p)\big) \ge 2\pi\Theta(r;\hat{a},\hat{b},c)$.

3 Proofs of Theorems

Let dS denote the ordinary area element of a standard circle $S^1 = U_pM \subset \mathbb{R}^2$. When $r < t_0^h(p) = \min\{t_0^h(p;\gamma_v) \mid v \in U_pM\}$, we have

$$\mathrm{length}\big(F_r^h(p)\big) = \int_{U_pM} \big\| \big(d\mathbb{B}_h \exp_p\big)_{rv} \big\| \, dS(v).$$

Therefore, in order to show our theorems it is enough to give estimates of norms of magnetic Jacobi fields. In the preceding paper ([11]), we give comparison theorems on magnetic Jacobi fields for surface magnetic fields (see [2, 4]). We here partially extend one of them by comparing magnetic Jacobi fields with ordinary Jacobi fields along geodesics.

Proposition 3.1. Let M a complete orientable Riemann surfaces whose sectional curvatures satisfy $\mathrm{Riem}^M \geq c$ with some constant c. We take a non-trivial magnetic Jacobi field Y along a trajectory γ for a surface magnetic field $\mathbb{B}_h$ which satisfies $g_Y(0) = 0$. For a positive T with $T \leq t_0^h\big(\gamma(0);\gamma\big)$, we set $b_\gamma^T = \min_{0\leq t\leq T}\big\{h(\gamma(t))^2 - \|(\nabla h)(\gamma(t))\|\big\}$. For $0 \leq t \leq T$, We then have

(1) $|g_Y(t)|/\mathfrak{s}_0(t; c + b_\gamma^T)$ is monotone decreasing.

(2) $g'(t)/g(t) \leq \mathfrak{s}_0'(t; c + b_\gamma^T)/\mathfrak{s}_0(t; c + b_\gamma^T)$.

(3) $|g_Y(t)| \leq |g_Y'(0)|\, \mathfrak{s}_0(t; c + b_\gamma^T)$.

In particular, we have $T \leq \pi/\sqrt{c + b_\gamma^T}$.

We study magnetic Jacobi fields along the same lines as for ordinary Jacobi fields (see [8, 10]). To show Proposition 1 we introduce a function of the set of vector fields along a trajectory which are orthogonal to the velocity vectors. Let γ be a trajectory for $\mathbb{B}_h$ on M and S be a positive number. For a vector field $X = g_X J\dot\gamma$ we set

$$Ind_0^S(X) = \int_0^S \Big\{ g_X'(t)^2 + g_X(t)^2 \big\{ \big((J\dot\gamma)h\big)(t) - K\big(\gamma(t)\big) - h\big(\gamma(t)\big)^2 \big\} \Big\}\, dt.$$

When $Y = f_Y\dot\gamma + g_Y J\dot\gamma$ is a magnetic Jacobi field along γ, its component $Y^\perp = g_Y J\dot\gamma$ orthogonal to $\dot\gamma$ satisfies

$$\begin{aligned} &g_Y'(S)g_Y(S) - g_Y'(0)g_Y(0) \\ &\qquad = \int_0^S \big\{g_Y'(t)g_Y(t)\big\}'\, dt = \int_0^S \big\{g_Y''(t)g_Y(t) + g_Y'(t)^2\big\}'\, dt = Ind_0^S(Y^\perp) \end{aligned}$$

by (2). Moreover we have the following:

Lemma 3.2 ([11])**.** Let Y be a magnetic Jacobi field along γ satisfying $Y(0)=0$. If $0<S\le t_c^h\big(\gamma(0);\gamma\big)$ and a vector field $X=g_X J\dot\gamma$ satisfies $X(0)=0$ and $X(S)=Y^{\perp}(S)$, then we have $Ind_0^S(X)\ge Ind_0^S(Y^{\perp})$.

Proof. Since $g_Y(t)\neq 0$ for $0<t\le t_c^h\big(\gamma(0);\gamma\big)$, we can choose a smooth function φ along γ so that $g_X(t)=\varphi(t)g_Y(t)$ holds on the interval $\big[0,t_c^h\big(\gamma(0);\gamma\big)\big]$. As $\varphi(S)=1$ and $g_Y(0)=0$, by direct calculation we obtain

$$\begin{aligned}
Ind_0^S(X) &= \int_0^S \Big\{(\varphi' g_Y+\varphi g_Y')^2+\varphi^2 g_Y^2\big\{(J\dot\gamma)h-K(\gamma)-h(\gamma)^2\big\}\Big\}\,dt\\
&= \int_0^S \varphi^2\Big\{{g_Y'}^2+g_Y^2\big\{(J\dot\gamma)h-K(\gamma)-h(\gamma)^2\big\}\Big\}\,dt\\
&\qquad +\int_0^S \big\{g_Y' g_Y(\varphi^2)'+{\varphi'}^2 g_Y^2\big\}\,dt\\
&= \int_0^S \varphi^2\Big\{{g_Y'}^2+g_Y^2\big\{(J\dot\gamma)h-K(\gamma)-h(\gamma)^2\big\}\Big\}\,dt\\
&\qquad + g_Y'(S)g_Y(S)-\int_0^S \varphi^2\{{g_Y'}^2+g_Y'' g_Y\}\,dt+\int_0^S {\varphi'}^2 {g_Y}^2\,dt\\
&= Ind_0^S(Y^{\perp})+\int_0^S {\varphi'}^2 {g_Y}^2\,dt \ge Ind_0^S(Y^{\perp}),
\end{aligned}$$

by making use of (2). QED

Proof of Proposition 3.1. We take a geodesic $\hat\gamma$ on a real space form $\widehat{M}=\mathbb{R}M^2(c+b_\gamma^T)$ and take a Jacobi field $\hat g J\dot{\hat\gamma}$ along this geodesic satisfying $\hat g(0)=0$ and $\hat g'(0)=|g_Y'(0)|$. That is, we put $\hat g(t)=|g_Y'(0)|\,\mathfrak{s}_0(t;c+b_\gamma^T)$. We here study the function $F(t)=\hat g(t)^2/g(t)^2$. By de l'Hôspital's rule, we have

$$\lim_{t\downarrow 0}F(t)=\lim_{t\downarrow 0}\frac{\hat g'(t)\hat g(t)}{g'(t)g(t)}=\lim_{t\downarrow 0}\frac{\hat g''(t)\hat g(t)+\hat g'(t)^2}{g''(t)g(t)+g'(t)^2}=1.$$

As we have

$$F'(t)=\frac{\hat g(t)^2}{g(t)^2}\Big(\frac{\hat g'(t)}{\hat g(t)}-\frac{g'(t)}{g(t)}\Big),$$

in order to show our assertion, it is enough to show that $\hat g'(t)/\hat g(t)\ge g'(t)/g(t)$ holds for an arbitrary t with $0<t\le T$.

For an arbitrary positive S with $S\le\min\big\{T,\,\pi/\sqrt{c+b_\gamma^T}\big\}$, we set a function $\widehat G_S(t):=g(t)/g(S)$. Since $\widehat G_S(t)J\dot{\hat\gamma}(t)$ is a Jacobi field along $\hat\gamma$ and $X=\widehat G_S J\dot\gamma$

is a vector field along γ, we have

$$\begin{aligned}\frac{\hat g'(S)}{\hat g(S)} &= \widehat G'_S(S)\widehat G_S(S)\\ &= \int_0^S \{\widehat G'_S(t)\widehat G_S(t)\}'\,dt = \int_0^S \big\{\widehat G'_S(t)^2 - (c+b^T_\gamma)\widehat G_S(t)^2\big\}\,dt\\ &\geq \int_0^S \Big\{\widehat G'_S(t)^2 - \big\{\mathrm{Riem}\big(\gamma(t)\big) + h(\gamma(t))^2 - \|(\nabla h)(\gamma(t))\|\big\}\widehat G_S(t)^2\Big\}\,dt\\ &= Ind_0^S(X).\end{aligned}$$

If we set a vector field Y_S by $Y_S(t) = Y(t)/g_Y(S)$, we have $g_{Y_S}(S) = 1 = g_X(S)$, hence obtain

$$Ind_0^S(X) \geq Ind_0^S(Y_S^\perp) = g'_{Y_S}(S)g_{Y_S}(S) = \frac{g'_Y(S)}{g_Y(S)}$$

by using Lemma 3.2. Thus we get the conclusion. QED

Remark 3.3. Since $\mathfrak{s}_0(t;C_1) > \mathfrak{s}_0(t;C_2)$ if $C_1 < C_2$, when $b^T_\gamma > b$ we have $|g_Y(t)| \leq |g'_Y(0)|\mathfrak{s}_0(t;c+b)$ for $0 \leq t \leq T$ in Proposition 3.1. In particular, if we set $b = \inf_{p\in M}\{h(p)^2 - \|(\nabla h)(p)\|\}$, this estimate holds for $0 \leq t \leq t^h_c\big(\gamma(0);\gamma\big)$.

We are now in the position to prove Theorem 2.1. For $v \in U_pM$, we denote by Y_v the magnetic Jacobi field along γ_v satisfying $Y_v(0) = 0,\ \|(\nabla_{\dot\gamma_v}Y_v)(0)\| = 1$. Since $\big\|\big(d\mathbb{B}_h\exp_p\big)_{rv}\big\|^2 = f_{Y_v}(r)^2 + g_{Y_v}(r)^2$, we apply Proposition 3.1 by noticing $|g'_{Y_v}(0)| = 1$. As $r \leq t^h_c(p;\gamma_v)$, we have

$$|f_{Y_v}(r)| \leq \int_0^r \big|h\big(\gamma(t)\big)\big|\,|g_{Y_v}(t)|\,dt \leq a\int_0^r \mathfrak{s}_0(t;c+b)\,dt = a\mathfrak{u}_0(r;c+b).$$

We hence get the conclusion of Theorem 2.1.

Next we obtain Theorem 2.2. Corresponding to Proposition 3.1 we have the following.

Proposition 3.4. Let M a complete orientable Riemann surfaces whose sectional curvatures satisfy $\mathrm{Riem}^M \leq c$ with some constant c. We take a magnetic Jacobi field Y along a trajectory γ for a surface magnetic field $\mathbb{B}_h$ which satisfy $g_Y(0) = 0$. For a positive T with $T \leq t^h_0\big(\gamma(0);\gamma\big)$, we set $\hat b^T_\gamma := \max_{0\leq t\leq T}\big\{h(\gamma(t))^2 + \|(\nabla h)(\gamma(t))\|\big\}$. We then have

$$|g_Y(t)| \geq |g'_Y(0)|\,\mathfrak{s}_{\sqrt{\hat b^T_\gamma}}(t;c) \qquad \text{for } 0 \leq t \leq T.$$

Proof of Theorem 2.2. We use the same notations as in the above proof of Theorem 2.1. We apply Proposition 3.4. Since $\hat{a} > 0$, we see that $h\big(\gamma(t)\big)$ does not vanish. Therefore

$$|f_{Y_v}(r)| = \int_0^r |h\big(\gamma(t)\big)|\,|g_{Y_v}(t)|\,dt \geq \hat{a}\int_0^r \mathfrak{s}_{\sqrt{\hat{b}_\gamma^T}}(t;c)\,dt = \hat{a}\mathfrak{u}_{\sqrt{\hat{b}_\gamma^T}}(r;c).$$

Thus we get the conclusion. *QED*

Remark 3.5. In Theorem 2.2, if we drop the assumption that $\hat{a} > 0$, we can only estimate the arclength as $\mathrm{length}\big(F_r^h(p)\big) \geq 2\pi\mathfrak{s}_{\sqrt{\hat{b}_\gamma^T}}(r;c)$.

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