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Local controllability of trident snake robot based on sub-Riemannian extremals

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Abstract. To solve trident snake robot local controllability by differential geometry tools, we construct a privileged system of coordinates with respect to the distribution given by Pffaf system based on local nonholonomic conditions and, furthermore, we construct a nilpotent approximation of the transformed distribution with respect to the given filtration. We compute normal extremals of sub-Riemanian structure, where the Hamiltonian point of view was used. We demonstrated that the extremals of sub-Riemannian structure based on this distribution play the similar role as classical periodic imputs in control theory with respect of our mechanism.

Keywords: local controllability, nonholonomic mechanics, planar mechanisms, sub–Riemannian geometry, differential geometry.

MSC 2010 classification: primary 53A17, secondary 70H05, 70Q05

1 Introduction

Originally, the general trident snake robot has been introduced in [5]. It is a planar robot with a body in the shape of a triangle and with three legs consisting of ℓ links. Then, its simplest non–trivial version (see figure 1), corresponding to $\ell = 1$, has been mainly discussed, see e.g. [6],[4]. Local controllability of such robot is given by the appropriate Pfaff system of ODEs. The solution with respect to

$$\dot{q} = (\dot{x}, \dot{y}, \dot{\theta}, \dot{\Phi}_1, \dot{\Phi}_2, \dot{\Phi}_3)$$

gives a control system $\dot{q} = Gu$. In the case length one links the control matrix G is a 6×3 matrix spanned by vector fields g_1, g_2, g_3, where

$$\begin{aligned}
g_1 &= \cos\theta\partial_x - \sin\theta\partial_y + \sin\Phi_1\partial_{\Phi_1} + \sin(\Phi_2 + \tfrac{2\pi}{3})\partial_{\Phi_2} + \sin(\Phi_3 + \tfrac{4\pi}{3})\partial_{\Phi_3},\\
g_2 &= \sin\theta\partial_x + \cos\theta\partial_y - \cos\Phi_1\partial_{\Phi_1} - \cos(\Phi_2 + \tfrac{2\pi}{3})\partial_{\Phi_2} - \cos(\Phi_3 + \tfrac{4\pi}{3})\partial_{\Phi_3},\\
g_3 &= \partial_\theta - (1+\cos\Phi_1)\partial_{\Phi_1} - (1+\cos\Phi_2)\partial_{\Phi_2} - (1+\cos\Phi_3)\partial_{\Phi_3}
\end{aligned}$$

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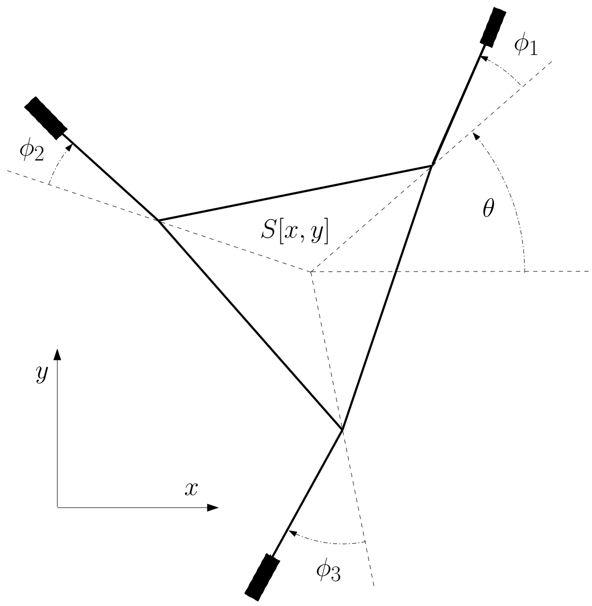

Figure 1. 1–link trident snake robot model

and $u : \mathbb{R} \to \mathbb{R}^3$ is the control of our system. It is easy to check that in regular points these vector fields define a (bracket generating) distribution H with growth vector $(3, 6)$. It means that in each regular point the vectors g_1, g_2, g_3 together with their Lie brackets span the whole tangent space. Consequently, the system is controllable by Chow–Rashevsky theorem.

The sub-Riemannian structure on manifold M is a generalization of Riemannian manifold. Given a distribution $H \subset TM$ equipped with metrics based on control u we can define so called Carnot-Carathéodory metric on M and sub-Remannian Hamiltonian.

2 Nilpotent aproximation

In order to simplify the trident snake robot control we construct a nilpotent approximation of the transformed distribution with respect to the given filtration. Note that all constructions are local in the neighborhood of 0. Following [3], we group together the monomial vector fields of the same weighted degree and thus we express $g_i, i = 1, 2, 3$ as a series

$$g_i = g_i^{(-1)} + g_i^{(0)} + g_i^{(1)} + \cdots ,$$

where $g_i^{(s)}$ is a homogeneous vector field of order s. Then the following proposition holds, [7]. Set $X_i = g_i^{(-1)}, i = 1, 2, 3$. The family of vector fields (X_1, X_2, X_3) is a first order approximation of (g_1, g_2, g_3) at 0 and generates a nilpotent Lie algebra of step $r = 2$, i.e. all brackets of length greater than 2 are zero. In our case [3], we obtain the following vector fields:

$$\begin{aligned}
X_1 &= \partial_x + \left(-\frac{y}{2}\right)\partial_a + \left(-\frac{y}{2} - d\right)\partial_b + \left(-\frac{x}{2}\right)\partial_c,\\
X_2 &= \partial_y + \left(\frac{x}{2}\right)\partial_a - \left(\frac{x}{2}\right)\partial_b + \left(\frac{y}{2} - d\right)\partial_c,\\
X_3 &= \partial_d.
\end{aligned}$$

with respect to a new coordinates (x, y, d, a, b, c). In particular, the family of vector fields (X_1, X_2, X_3) is the nilpotent approximation of (g_1, g_2, g_3) at 0 associated with the coordinates (x, y, d, a, b, c). The remaining three vector fields are generated by Lie brackets of (X_1, X_2, X_3). Note that due to linearity of the three latter coordinates of (X_1, X_2, X_3), the coordinates of (X_4, X_5, X_6) must be constant. We get

$$\begin{aligned}
X_4 &= \partial_a,\\
X_5 &= \partial_b,\\
X_6 &= \partial_c.
\end{aligned}$$

Note that the vector fields $(X_1, X_2, X_3, X_4, X_5, X_6)$ in $(x, y, \theta, \Phi_1, \Phi_2, \Phi_3)$ coordinates are of the form

$$\begin{aligned}
X_1 &= \partial_x + \theta\partial_{\Phi_1} - \left(-\tfrac{\sqrt{3}}{2} + \tfrac{\theta}{2}\right)\partial_{\Phi_2} - \left(-\tfrac{\sqrt{3}}{2} + \tfrac{\theta}{2}\right)\partial_{\Phi_3},\\
X_2 &= \partial_y - \partial_{\Phi_1} + \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}\theta}{2}\right)\partial_{\Phi_2} - \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}\theta}{2}\right)\partial_{\Phi_3},\\
X_3 &= \partial_\theta - 2\partial_{\Phi_1} - 2\partial_{\Phi_2} - 2\partial_{\Phi_3}\\
X_4 &= \partial_{\Phi_1} + \partial_{\Phi_2} + \partial_{\Phi_3},\\
X_5 &= -\tfrac{\sqrt{3}}{2}\partial_{\Phi_2} + \tfrac{\sqrt{3}}{2}\partial_{\Phi_3},\\
X_6 &= -\partial_\theta + \tfrac{1}{2}\partial_{\Phi_2} + \tfrac{1}{2}\partial_{\Phi_3}.
\end{aligned}$$

To show how nilpotent approximation affects on integral curves of the distributions and the resulting control we computed the Lie brackets of relevant vector fields. In Fig. 2, there is a comparison of the Lie bracket g_5 motions realized in $(x, y, \theta, \Phi_1, \Phi_2, \Phi_3)$ coordinates (dotted line) and in nilpotent approximation. Fig. 3 show the comparison of g_6 motions. The following figures show the trajectories of the root center point, vertices and wheels when a particular Lie bracket motion is realized.

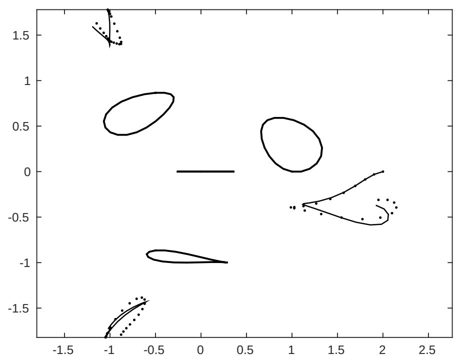

Figure 2. g_5 motion

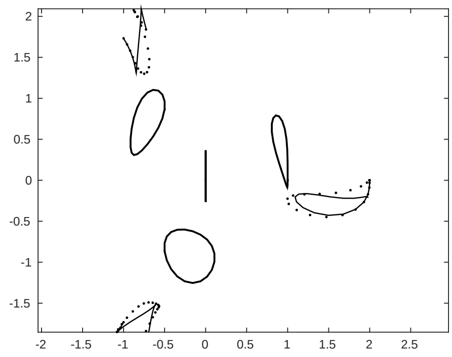

Figure 3. g_6 motion

3 sub-Riemannian Hamiltonian

Note that the functions in $C^\infty(M)$ are in one–to–one correspondence with functions in $C^\infty(T^*M)$ that are constant on fibers:

$$C^\infty(M) \cong C^\infty_{const}(T^*M) = \{\pi^*\alpha | \alpha \in C^\infty(M)\} \subset C^\infty(T^*M),$$

where $\pi : T^*M \to M$ denotes the canonical projection. In what follows, with no abuse of notation, we often identify the function $\pi^*\alpha \in C^\infty(T^*M)$ with the function $\alpha \in C^\infty(M)$. In a similar way, smooth vector fields on M are in a one–to–one correspondence with functions in $C^\infty(T^*M)$ that are linear on fibers via the map $Y \mapsto a_Y$, where $a_Y(\lambda) := \langle \lambda, Y(q)\rangle$ and $q = \pi(\lambda)$, i.e.

$$Vec(M) \cong C^\infty_{lin}(T^*M) = \{a_Y | Y \in Vec(M)\} \subset C^\infty(T^*M).$$

Our mechanism can be understood as a model of sub-Riemannian structure of constant rank, i.e. the triple $(M, D, \langle\cdot,\cdot\rangle)$, where D is a vector subbundle of TM locally generated by family of vector fields $\{f_1, \dots, f_m\}$ (in our case generated by X_1, X_2 and X_3) and $\langle\cdot,\cdot\rangle_q$ is a family of scalar products on D_q. Then the sub-Riemannian Hamiltonian is the function on T^*M defined as follows [1]

$$h : T^*M \to \mathbb{R}$$
$$h(\lambda) = \max_{u \in U_q} \left(\langle \lambda, f_u(q)\rangle - \frac{1}{2}|u|^2 \right), \;\; q = \pi(\lambda),$$

where $f_u = \sum_i u_i f_i$. For every generating family $\{f_1, \dots, f_m\}$ of the sub-Riemannian structure, the sub-Riemannian Hamiltonian H can be rewritten [1] as follows

$$h(\lambda) = \frac{1}{2}\sum_i \langle \lambda, f_i(q)\rangle^2, \;\; \lambda \in T^*_q M, \;\; q = \pi(\lambda).$$

To simplify our equations below we can define the Poisson bracket as an binary operation $\{,\}$ on functions in $C^\infty(T^*M)$. In fact, we can define

$$\{a_X, a_Y\} := a_{[X,Y]} \tag{1}$$

and there exists a unique bilinear and skew–symmetric map

$$\{\cdot,\cdot\} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$$

that extends Poisson bracket (1) on $C^\infty(T^*M)$, and that is a derivative (i.e. satisfies the Lipschitz rule) in each argument. Let (x, p) denote coordinates on T^*M, the formula for Poisson bracket of two functions $a, b \in C^\infty(T^*M)$ reads

$$\{a, b\} = \sum_{i=1}^{n} \frac{\partial a}{\partial p_i}\frac{\partial b}{\partial x_i} - \frac{\partial a}{\partial x_i}\frac{\partial b}{\partial p_i}$$

and the Hamiltonian vector field associated with the smooth function $a \in C^\infty(T^*M)$ is defines as the linear operation

$$\vec{a} : C^\infty(T^*M) \to C^\infty(T^*M), \quad \vec{a}(v) = \{a, b\}.$$

We can easily write the coordinate expression of $\vec{a}$ for an arbitrary function $a \in C^\infty(T^*M)$ as

$$\vec{a} = \{a, \cdot\} = \sum_{i=1}^{n} \frac{\partial a}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial a}{\partial x_i}\frac{\partial}{\partial p_i}.$$

Let $\gamma : [0, T] \to M$ be an admissible curve with respect to controlling distribution which is lenght-minimizer, parametrized by constant speed. Let $\bar{u}$ be the corresponding minimal control. Then there exists a Lipschitz curve $\lambda(t) \in T^*_{\gamma(t)}M$ such that

$$\dot{\lambda} = \sum_{i=1}^{m} \bar{u}_i(t)\vec{h}_i(\lambda(t)) = \vec{h}(t)(\lambda(t)) = \{h, \lambda\},$$

where $\vec{h} = \sum_{i=1}^{m} \bar{u}_i(t)\vec{h}_i$. On the controlling vector fields X_1, X_2 and X_3 and the general 1–form $\lambda \in T^*M$

$$\lambda = \lambda_x \mathrm{d}x + \lambda_y \mathrm{d}y + \lambda_d \mathrm{d}d + \lambda_a \mathrm{d}a + \lambda_b \mathrm{d}b + \lambda_c \mathrm{d}c$$

we define functions $h_i = \langle \lambda, X_i \rangle$

$$\begin{aligned}
h_1(\lambda) &:= \lambda_x - \frac{y}{2}\lambda_a - \left(\frac{y}{2} + d\right)\lambda_b - \frac{x}{2}\lambda_c,\\
h_2(\lambda) &:= \lambda_y + \frac{x}{2}\lambda_a - \frac{x}{2}\lambda_b + \left(\frac{y}{2} - d\right)\lambda_c,\\
h_3(\lambda) &:= \lambda_d.
\end{aligned}$$

In our case the sub-Riemannian Hamiltonian with respect to corresponding minimal control is then of the form

$$\begin{aligned} h(\lambda) &= u_1 h_1(\lambda) + u_2 h_2(\lambda) + u_3 h_3(\lambda) \\ &= u_1 \left(\lambda_x - \frac{y}{2}\lambda_a - \left(\frac{y}{2} + d\right)\lambda_b - \frac{x}{2}\lambda_c \right) \\ &+ u_2 \left(\lambda_y + \frac{x}{2}\lambda_a - \frac{x}{2}\lambda_b + \left(\frac{y}{2} - d\right)\lambda_c \right) + u_3 \lambda_d . \end{aligned}$$

In canonical coordinates (p, x), the Hamiltonian vector filed associated with h is expressed as follows

$$\vec{h} = \{h, \cdot\} = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial p_i}$$

and the Hamiltonian system $\dot{\lambda} = \vec{h}(\lambda)$ is rewritten as

$$\dot{x_i} = \frac{\partial h}{\partial p_i}, \;\; \dot{p_i} = -\frac{\partial h}{\partial x_i}$$

i.e. in our case

$$\begin{aligned} \dot{x} &= \frac{\partial h}{\partial \lambda_x} = \frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial \lambda_x} = u_1 \\ \dot{y} &= \frac{\partial h}{\partial \lambda_y} = \frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial \lambda_y} = u_2 \\ \dot{d} &= \frac{\partial h}{\partial \lambda_d} = \frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial \lambda_d} = u_3 \end{aligned}$$

$$\begin{aligned} \dot{\lambda}_x &= -\frac{\partial h}{\partial x} = -\frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial x} = \frac{1}{2}(-\lambda_c u_1 + (\lambda_a - \lambda_b)u_2) \\ \dot{\lambda}_y &= -\frac{\partial h}{\partial y} = -\frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial y} = \frac{1}{2}(\lambda_a - \lambda_b)u_1 + (\lambda_c)u_2) \\ \dot{\lambda}_d &= -\frac{\partial h}{\partial d} = -\frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial d} = \frac{1}{2}(-\lambda_b u_1 - (\lambda_c)u_2) \end{aligned}$$

$$\begin{aligned} \dot{\lambda}_a &= 0 & \dot{a} &= -\frac{y}{2}u_1 + \frac{x}{2}u_2 \\ \dot{\lambda}_b &= 0 & \dot{b} &= -(\frac{y}{2} + d)u_1 - \frac{x}{2}u_2 \\ \dot{\lambda}_c &= 0 & \dot{c} &= -\frac{x}{2}u_1 - (\frac{y}{2} - d)u_2 \end{aligned}$$

A function $a \in C^\infty(T^*M)$ is a constant of the motion of the Hamiltonian system associated with $h \in C^\infty(T^*M)$ if and only if $\{h, a\} = 0$. If $\gamma : [0, T] \to M$ is a length minimizer on sub-Riemannian manifold, associated with a control $u(\cdot)$, then due the Pontryagin maximum principle there exists $\lambda_0 \in T^*_{\gamma(0)}M$ such that defining

$$\lambda(t) = (P_{0,t}^{-1})^* \lambda_0, \ \ \lambda(t) \in T^*_{\gamma(t)}M,$$

where $P_{0,t}$ is the flow of the nonautonomous vector field

$$X_{u(t)} = \sum_{i=1}^{m} u_i(t) X_i(\gamma(t))$$

and one of the following condition is satisfied

$$\begin{aligned} (N) \ \ u_i(t) &= \langle \lambda(t), X_i(\gamma(t)) \rangle, \ \ \forall i = 1, \ldots, m, \\ (A) \qquad 0 &= \langle \lambda(t), X_i(\gamma(t)) \rangle, \ \ \forall i = 1, \ldots, m. \end{aligned}$$

If $\lambda(t)$ satisfies (N) then it is called normal extremal (and $\gamma(t)$ a normal extremal trajectory). If $\lambda(t)$ satisfies (A) then it is called abnormal extremal (and $\gamma(t)$ a abnormal extremal trajectory).

4 Normal extremals

Following [1], every normal extremal is a solution of the Hamiltonian system $\dot{\lambda} = \vec{h}(\lambda(t))$, i.e. in our case of the system

$$\dot{\lambda} = \vec{h}(\lambda(t)) = \frac{1}{2}(h_1^2(\lambda) + h_2^2(\lambda) + h_3^2(\lambda)), \ \ h_i(\lambda) = \langle \lambda, X_i(q) \rangle,$$

and let us introduce

$$\begin{aligned} h_4(\lambda) &:= \langle \lambda, X_4(q) \rangle, \\ h_5(\lambda) &:= \langle \lambda, X_5(q) \rangle, \\ h_6(\lambda) &:= \langle \lambda, X_6(q) \rangle. \end{aligned}$$

Since $X_1, \ldots, X_6$ are linearly independent then $\{h_1, \ldots, h_6\}$ defines a system coordinates on T^*M and thus we can consider λ to be parametrized by h_i

consequently, we are looking for a solution of the system $\dot{h}_i = \{h, h_i\}$, i.e.

$$\dot{h}_1 = \{h, h_1(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_1\} = \{h_2, h_1\}h_2 + \{h_3, h_1\}h_3 = -h_4h_2 - h_5h_3$$
$$\dot{h}_2 = \{h, h_2(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_2\} = \{h_1, h_2\}h_1 + \{h_3, h_2\}h_3 = h_4h_1 - h_6h_3$$
$$\dot{h}_3 = \{h, h_3(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_3\} = \{h_1, h_3\}h_1 + \{h_2, h_3\}h_2 = h_5h_1 + h_6h_2$$
$$\dot{h}_4 = \vec{h}(h_4(t)) = \{h, h_4(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_4\} = 0$$
$$\dot{h}_5 = \vec{h}(h_5(t)) = \{h, h_5(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_5\} = 0$$
$$\dot{h}_6 = \vec{h}(h_6(t)) = \{h, h_4(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_6\} = 0$$

To analyze the system of ordinary differential equations above we have the following assignments:

$$\dot{h}_4 = 0 \rightsquigarrow h_4 \text{ is a constant function} \rightsquigarrow h_4 = k_1,$$
$$\dot{h}_5 = 0 \rightsquigarrow h_5 \text{ is a constant function} \rightsquigarrow h_5 = k_2,$$
$$\dot{h}_6 = 0 \rightsquigarrow h_6 \text{ is a constant function} \rightsquigarrow h_6 = k_3$$

and the system of linear ordinary differential equations, which can be written in the matrix form as

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{pmatrix} = \begin{pmatrix} 0 & -k_1 & k_2 \\ k_1 & 0 & -k_3 \\ -k_2 & k_3 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

The main tool to analyze the systems of linear ODEs, is eigenvalues and eigenvectors computations. In our case, the straightforward computation

$$\begin{vmatrix} -\lambda & -k_1 & k_2 \\ k_1 & -\lambda & -k_3 \\ -k_2 & k_3 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda(k_1^2 + k_2^2 + k_3^2) = -\lambda(\lambda^2 + (k_1^2 + k_2^2 + k_3^2)) = 0$$

leads to one real and two complex conjugated eigenvalues:

$$\lambda_1 = 0, \ \ \lambda_{\pm i} = \pm i\sqrt{k_1^2 + k_2^2 + k_3^2},$$

where the eigenvectors are

$$v_0 := \begin{pmatrix} k_3 \\ k_2 \\ k_1 \end{pmatrix}, \ \ v_i := \begin{pmatrix} k_1^2 + k_2^2 \\ -i\sqrt{k_1^2 + k_2^2 + k_3^2}k_1 + k_2k_3 \\ i\sqrt{k_1^2 + k_2^2 + k_3^2}k_2 - k_1k_3 \end{pmatrix},$$
$$v_{-i} := \begin{pmatrix} k_1^2 + k_2^2 \\ i\sqrt{k_1^2 + k_2^2 + k_3^2}k_1 - k_2k_3 \\ -i\sqrt{k_1^2 + k_2^2 + k_3^2}k_2 - k_1k_3 \end{pmatrix},$$

respectively. From the theory of linear systems of ODEs in matrix form, the solution can be described as a linear combination of real and imaginary part of $u = v_o \exp(0t) + v_i \exp(it) + v_{-i} \exp(-it)$. The straightforward computation leads to

$$\begin{aligned}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} v_0^x \\ v_0^y \\ v_0^d \end{pmatrix} + \begin{pmatrix} v_i^x \exp(it) \\ v_i^y \exp(it) \\ v_i^d \exp(it) \end{pmatrix} + \begin{pmatrix} v_{-i}^x \exp(-it) \\ v_{-i}^y \exp(-it) \\ v_{-i}^d \exp(-it) \end{pmatrix} \\
&= \begin{pmatrix} k_3 \\ k_2 \\ k_1 \end{pmatrix} + \begin{pmatrix} (k_1^2 + k_2^2)(\cos(t) + i\sin(t)) \\ (-ikk_1 + k_2 k_3)(\cos(t) + i\sin(t)) \\ (ikk_2 - k_1 k_3)(\cos(t) + i\sin(t)) \end{pmatrix} \\
&+ \begin{pmatrix} (k_1^2 + k_2^2)(\cos(t) - i\sin(t)) \\ (ikk_1 - k_2 k_3)(\cos(t) - i\sin(t)) \\ (-ikk_2 - k_1 k_3)(\cos(t) - i\sin(t)) \end{pmatrix} \\
&= \begin{pmatrix} k_3 \\ k_2 \\ k_1 \end{pmatrix} + \begin{pmatrix} 2(k_1^2 + k_2^2)\cos(t) \\ 2kk_1 \sin(t) \\ -2k_1 k_3 \cos(t) - 2kk_2 \sin(t) \end{pmatrix} + i \begin{pmatrix} 0 \\ 2k_2 k_3 \sin(t) \\ 0 \end{pmatrix},
\end{aligned}$$

where $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$.

5 Local controllability

To perform the Lie bracket motions we apply a periodic input, i.e. for the vector fields $X_4 = [X_1, X_2], X_5 = [X_1, X_3], X_6 = [X_2, X_3]$, respectively, the input

$$\begin{aligned}
v_1(t) &= (-A\omega \sin \omega t, A\omega \cos \omega t, 0), \\
v_2(t) &= (0, -A\omega \sin \omega t, A\omega \cos \omega t), \\
v_3(t) &= (-A\omega \sin \omega t, 0, A\omega \cos \omega t)
\end{aligned}$$

is applied, because, according to [6], the Lie bracket of a pair of vector fields corresponds to the direction of a displacement in the state space as a result of a periodic input with sufficiently small amplitude A, i.e. the bracket motions are generated by periodic combination of the vector controlling fields. The theoretic approach above leads to four new control sequences with respect to the parameters k_1, k_2 and k_3. These control sequences are the following:

$$\begin{aligned}
e_1(t) &= (2k_1^2 \cos(t), 2kk_1 \sin(t), -2k_1 k_3 \cos(t)), \\
e_2(t) &= ((k_1^2 + k_2^2)\cos(t), 2k_1 k_3 \sin(t), -2kk_2 \sin(t)), \\
e_{3A}(t) &= (2k_2^2 \cos(t), 0, -2kk_2 \sin(t)), \\
e_{3B}(t) &= (0, 2k_1 k_3 \sin(t), 0).
\end{aligned}$$

It is easy to see that by appropriate choice of coordinates k_1, k_2 and k_3 we can get classical periodic inputs as a normal extremal of the underlying sub–Riemannian structure. For example, the choice $k_1 \mapsto -\sqrt{A\omega}, k_2 = k_3 = 0$ and $t \mapsto \omega t$ leads to identification $e_1(t) \cong -v_1(t)$.

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