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Complex geometry of real hyperboloids

Simon Gindikin
Departm. of Math., Hill Center, Rutgers University
110 Frelinghysen Road, Piscataway, NJ 08854, U.S.A.
gindikin@math.rutgers.edu

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Abstract. We investigate, using horospheres, the spectrum of the Laplace-Beltrami operator on pseudo hyperbolic spaces (hyperboloids), which, in general, is ultra hyperbolic. Real horospheres define only the continuous spectrum, but the discrete spectrum is connected with complex horospheres. We connect with them the horospherical Cauchy transform which acts at $\bar{\partial}$-cohomology. Correspondingly, for continuous spectrum we have an analogue of the Poisson problem with data on the real boundary but for the discrete spectrum - with cohomological data on the complex boundary.

Keywords: pseudo hyperbolic spaces, Laplace-Beltrami operetor, horospheres, horospherical space, complex horospheres, degenerated horospheres, horospherical transform, horospherical Cauchy transform, $\bar{\partial}$-cohomology

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Let $X = X_{p,q}$ be the hyperboloid of the signature $(p, q), p + q = n + 1$ at $\mathbb{R}^{n+1}$:

$$\Box(x) = x_1^2 + x_2^2 + \cdots - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1.$$

On X the group $SO(p; q)$ acts and, in general, relative to this action it will be the pseudo hyperbolic space with the canonical invariant pseudo Riemannian metric. For 2 cases we have the Riemannian metric: for $q = 0$ it is the sphere with the spherical geometry and for $p = 1$ we have the two-sheeted hyperboloid with the hyperbolic metric. Let us consider the Laplace-Beltrami differential operator. It is possible to write down it explicitly but we will not do so since it does not help our aims. There is a possibility to construct its spectral decomposition using the theory of representations and hypergeometrical functions, but we will operate here with the geometrical language of horospheres.

Horospherical transform at hyperbolic case ($p = 1$)**.** At the affine space there is a classical method of the decomposition of functions on plane waves, which can be expressed in the form of the Radon transform: functions transform at the integrals along hyperplanes and functions can be reconstructed through these integrals. For the hyperbolic space this construction has an analogue where hyperplanes are replaced by horospheres. Let us remind that in hyperbolic geometry there are 2 analogues of affine hyperplanes: geodesic hyperbolic hyperplanes and horospheres - limits of spheres when their centers and radiuses tend to infinity.

The cone without the vertex

$$\Xi = \{\xi \in \mathbb{R}^{n+1}; \Box(\xi) = 0, \xi \neq 0\}$$

parameterizes horospheres

$$E(\xi) = \{x \in X; < \xi, x >= 1\},$$

where the dot-product $<,>$ corresponds the quadratic form $\Box(x)$. So the horospheres are sections of the hyperboloid by isotropic sections $< \xi, x >= r, \xi \in \Xi, r \neq 0$. We can only take $r = 1$. For $f(x) \in C_0^\infty$ we define the *horospherical Radon transform*

$$\hat{f}(\xi) = \int_X \delta(< \xi, x > -1) f(x) \mu(dx),$$

where μ is the invariant measure $(d\Box(x) \rfloor dx)$ on X. So we integrate f along the horospheres $E(\xi)$. The cone Ξ can be considered as the asymptotic cone of the hyperboloid X and its projectivization as the boundary of X.

The central fact is that *under this correspondence (generalized) eigen functions of the Laplace-Beltrami operator correspond to homogeneous functions of different degrees of homogeneity on* Ξ. So degrees of homogeneity parameterize (continuous) eigen values of the Laplace-Beltrami operator. The decomposition on homogeneous functions at Ξ-picture is just the Mellin transform. If we have an inversion of the horospherical Radon transform then we can transform this decomposition at the spectral decomposition of the Laplace-Beltrami operator. Also for each degree of homogeneity we obtain the expression of eigen functions through corresponding homogeneous functions on Ξ - analogues of the Poisson formula. Homogeneous functions on Ξ are defined by restrictions on any section of the cone Ξ transversal to generators which can be identified with the sphere and interpreted as the boundary of X. This transformation gives the familiar structure of the classical Poisson formula.

A remarkable fact is that the inversion for the horospherical Radon transform has the same structure as Radon's inversion formula at the affine space: we apply to $\hat{f}$ some operator along the family of parallel horospheres (they correspond to proportional ξ) and take the average along horospheres passing through point x in which we reconstruct f. This operator is very different for even and odd n: for odd n it is just the differentiation of the order $n-1$, but for even n it is non local (this difference has an important connection with the Huygens principle). There is a modification of the transform which gives a possibility to write the inversion independent of the parity of n. We choose this construction since it is a bridge to our basic construction in the general case.

Let us call *the horospherical Cauchy transform*

$$\mathcal{C}f(\xi,r)=\int_X \frac{f(x)}{r-<\xi,x>}\mu(dx), \xi\in\Xi, Im(r)<0.$$

For real r we take the boundary values $\mathcal{C}f(\xi,r-i0)$ (in the sense of distributions). We have

$$f(x)=c\int_\Sigma \{Cf(\xi;<\xi,x>-i0)\}_r^{(n-1)}\nu(d\xi),$$

where we take $(n-1)$-th derivative on r, substitute $r=<\xi,x>-i0$ and integrate over any compact section of Ξ transversal to generators on the invariant measure. Here and below we do not give values of constants. Horospherical Radon and Cauchy transforms can be expressed one through the other using elementary formulas for distributions.

Complex horospheres. In the general case (if $p\neq 1$) we also can consider horospheres $E(\xi)$ parameterized by points of the corresponding cone Ξ, but the horospherical transform will have a kernel and, as a result, can not be inverted. This is connected with the fact that in these cases the Laplace-Beltrami operator has not only continuous spectrum, but also discrete spectrum, which just corresponds to the kernel of the horospherical transform. For a long time only one example was known where this gap was corrected. Gelfand and Graev [1] considered the case $p=3, q=1$ corresponding to one-sheeted hyperboloid at $\mathbb{R}^4$. They considered not only integrals along the horospheres (2-dimensional at this case) but also along one-dimensional linear generators on this hyperboloid which they called *degenerated horospheres*. It is possible to reconstruct any function through these two types of integrals. Correspondingly, the inversion formula gives the projection on the subspace of continuous spectrum through the horospherical transform and the subspace of discrete spectrum through the degenerate horospherical transform.

The natural conjecture that in the general case we need to extend the integration along horospheres by the integration along linear generators does not work: for $p=q=2$ the integrals along the generators have the same kernel as the integrals along horospheres and it was already unknown how to treat the case $q=1$ for $p\geq 3$. Gelfand several times payed attention to the case $p=q=2$ which is equivalent to the group $SL(2;\mathbb{R})$.

What is the difference in the geometry of horospheres in the hyperbolic case ($p=1$) and other signatures when the horospherical transform has a non trivial kernel, connected with the discrete spectrum? In the last cases there are not enough horospheres! If we were to take $x\in X$ in the case when X is the hyperbolic space then for any hyperplane at the tangent space T_xX there is a horosphere which is tangent to it. In the non hyperbolic case there is an open set of such hyperplanes for which tangent horospheres do not exist and there are non trivial functions for which "horospherical waves" are trivial. In [2] it was suggested that if there is not sufficient amount of real horospheres then let us consider complex ones: for any tangent hyperplane at a point x there is a tangent complex horosphere. More exact, the complexification of this hyperplane is tangent for a complex horosphere. The crucial moment is how to define a version of the horospherical transform for some complex horospheres and functions on a real hyperboloid?

The idea is to replace the δ-function in the definition of the horospherical Radon transform on a Cauchy kernel. The central idea: *it is possible to do this for complex horospheres without real points.* Let $Z \supset X, \Xi_{\mathbb{C}} \supset \Xi$ are the complexifications of the hyperboloid X and the cone Ξ at $\mathbb{C}^{n+1}$. Let $E(\zeta, r), \zeta \in \Xi_{\mathbb{C}}, r \neq 0 \in \mathbb{C}$, be the complex horosphere - the section of Z by the isotropic hyperplane $< \zeta, z >= r$. *Let the horosphere $E(\zeta, r)$ has no real points (it does not intersect X). Then we define the horospherical Cauchy transform*

$$\mathcal{C}f(\zeta, r) = \int_X \frac{f(x)}{< \zeta, x > -r} \mu(dx).$$

If the horosphere $E(\zeta, r)$ has no real points the integrand has no singularities and the integral is convergent. The transform $\mathcal{C}f$ is homogeneous of degree -1 on (ζ, r); so it is often convenient to put $r = 1$ and we will write just $\mathcal{C}f(\zeta)$ for $\mathcal{C}f(\zeta, 1)$. Let us remark that this definition of the horospherical Cauchy transform is compatible with above definition for real $\xi \in \Xi$ which is possible to connect with the complex horospheres $E(\xi; r), r \in \mathbb{C}, Im(r) \neq 0$.

This definition of the horospherical Cauchy transform has a special interest for the cases when there is an open set of complex horospheres. Unfortunately, it happens just for 2 occasions: $(2, q)$ and $(p, 0)$. Let us consider each of these cases.

The case $(2, q)$**.** We already mentioned that this case for $q = 2$ contains the group $SL(2; \mathbb{R})$. In this case the Laplace-Beltrami operator is hyperbolic. For $p = 1$ or $q = 0$ it is elliptic. In other cases it is ultrahyperbolic. It is instructive to compare Poisson's problem in all these cases. For $p = 2$ we have both continuous and discrete spectrums and the spectral problem has 2 steps: to construct projectors on subspaces of continuous and discrete spectrums and then to construct spectral decompositions of each of these subspaces.

The continuous spectrum is connected with real horospheres or their complexifications and the discrete spectrum with an open set at $\Xi_{\mathbb{C}}$ parameterizing complex horospheres $E(\zeta)$ without real points. Let us describe this set. It is an elementary geometrical computation using the action of the pseudo orthogonal group $SO(2;q)$ [2]. Let $\zeta = \xi + i\eta \in \Xi_{\mathbb{C}}$. Then

$$\Box(\xi) = \Box(\eta), < \xi, \eta >= 0$$

and *the set*

$$\tilde{\Xi} = \{\zeta; \Box(\xi) = \Box(\eta) > 1\}$$

is the maximal open set, parameterizing complex horospheres $E(\zeta)$ without real points. This set has 2 connected components $\Xi_{\pm}$: for a fixed ξ possible η lie on two-sheeted hyperboloid. At homogeneous coordinates (ζ, r) the condition is $\Box(\zeta) > |r|^2$.

There are a few degenerated cases when horospheres also have no real points. For us it is important that the considered above complex horospheres $E(\xi, r), \Box(\xi) = 0, Im(r) < 0$ are extensions of real horospheres. It turns out that if one was to combine these 3 types of the horospherical Cauchy transform: $\mathcal{C}f(\xi, r), \mathcal{C}f(\zeta), \zeta \in \Xi_{\pm}$ then any function $f(x)$ can be reconstructed through this combination. Let us remark that $\mathcal{C}f(\zeta)$ will be holomorphic at $\Xi_{\pm}$. Let us describe the inversion formulas. They are connected with the boundary values of $\mathcal{C}f$ in the sense of distributions: $\mathcal{C}f(\xi, r - i0), r \in \mathbb{R}$ and $\mathcal{C}f(\zeta), \zeta \in \partial\Xi_{\pm}$. If we were to take $x \in X$ then through this point the horospheres $E(\xi), \Box(\xi) = 0, < \xi, x >= 1$, are passing. If we were to put $\xi = x + \lambda$ then $< \lambda, x >= 0, \Box(\lambda) = -1$ since $\Box(x) = 1$.

The horosphere $E(\xi + i\eta) \in \partial\tilde{\Xi}$ is passing through a point $x \in X$ if and only if

$$\xi = x, < \eta, x >= 0, \Box(\eta) = 1$$

and then $E(\zeta)$ intersects X just at the point x. Using $SO(2, q)$-action we can consider just $x = (1, 0, 0, \dots, 0)$. Then $< \zeta, x >= 1$ gives $\xi_1 = 1, \eta_1 = 0$. We have then $(\eta_2)^2 - (\eta_3)^2 - \dots - (\eta_{n+1})^2 = 1$ and again using the invariance we can consider only $\eta = (0, \pm 1, 0, \dots, 0)$ and from the orthogonality conditions we have $< \eta, \xi >=< \eta, x >= 0$ and $\xi_2 = 0$. As a consequence we have $\xi_3 = \dots = \xi_{n+1} = 0$ (the other way $\Box(\xi)$ would be more than 1). So $\xi = (1, 0, 0, \dots, 0)$. Similarly, if $< \zeta, x >= 1$ for some $x \in X$ then $x = (1, 0, 0, \dots, 0)$ so boundary horospheres intersect X at unique points. We can see that in this proof it is crucial that p can not be more than 2.

So we have 2 types of horospheres passing through a point $x \in X$. Let us look at their intersections with the tangent plane T_x and again it is possible to suggest that $x = (1, 0, \dots, 0)$. Then the tangent plane is $\{x_1 = 0\}$. Let $y = (x_2, x_3, \dots, x_{n+1}), \lambda = (0, \tilde{\lambda}), \eta = (0, \tilde{\eta})$ where λ, η correspond to 2 types of the horospheres under consideration. These horospheres at the intersections with the tangent space give hyperplanes of real type

$$< \lambda, y >= 0, \Box(\tilde{\lambda}) = -1; < \tilde{\eta}, y >= 0, \Box(\tilde{\eta}) = 1.$$

We see that for almost all infinitesimal directions there are either real horospheres or complex horospheres without real points.

Let us describe the analytic picture corresponding to this geometrical picture. For $f(x) \in C_0^\infty(X)$ we have 3 components of the horospherical Cauchy transform with values on $\Xi, \partial\tilde{\Xi}_\pm$. The inversion formula has the same structure as in affine and hyperbolic spaces. Let for $x \in X$

$$\Lambda(x) = \{\lambda; <\lambda, x>= 0, \Box(\lambda) = -1\}; \Pi(x) = \{\eta; <\eta, x>= 0, \Box(\eta) = 1\}$$

and $\Pi_\pm(x)$ are the components of two-sheeted hyperboloid $\Pi(x)$. For a function $F(\zeta, r)$ let $F_r^{(n-1)}(\zeta, r) = \partial^{(n-1)}F(\zeta, r)/\partial r^{n-1}$ and $F_r^{(n-1)}(\zeta)$ be its restriction for $r = 1$. Using the homogeneity of $F(\zeta, r)$ it is possible to replace the differentiation on r by a differentiation on ζ.

Then we have the reconstruction formula

$$f(x) = c(\int_{\Lambda(x)} \mathcal{C}f_r^{(n-1)}(x + \lambda - i0)\mu(d\lambda) + \int_{\Pi_\pm(x)} \mathcal{C}f_r^{(n-1)}(x + i\eta)\mu(d\eta)),$$

where we integrate on the invariant measures on the hyperboloids; we do not give as before an explicit formula for the constant c. In this formula we integrate on the set of boundary horospheres passing through the point x using their parametrization above.

The inversion formula has 3 terms which we denote as $\mathcal{P}_0 f, \mathcal{P}_\pm f$ and consider as operators on f. Then $\mathcal{P}_0$ is the projection on the subspace of continuous spectrum and $\mathcal{P}_\pm$ on two components of the discrete spectrum. In the language of representations they correspond to continuous, holomorphic and antiholomorphic discrete series of representations.

On eigen spaces of the Laplace-Beltrami operator the horospherical transform can not be defined by the integral which does not exist, but can be defined it as a distribution. If $F(\xi)$ is a homogeneous function on Ξ of homogeneity α then $\int_\Lambda F_r^{(n-1)}(\xi)\mu(d\xi)$ gives a (generalized) eigen function with an eigen value depending of α. The differentiation reduces to the multiplication on (α^{n-1}). We obtain an analogue formula Poisson for eigenfunctions from the continuous spectrum. To transform this Poisson integral at more standard form we remark that we integrate along the section Ξ_x of the cone Ξ by the hyperplane $< x, \xi >= 1$. We can fix a point (let $x = (1, 0, \ldots, 0)$) and let Ξ_0 the corresponding section of Ξ. This section Ξ_0 is a projectivization of Ξ and can be interpret as the boundary of X. Using the homogeneity we can express the integrand on Ξ_x throuh its restriction $F(\xi)$ on Ξ_0. Then at this expression will appear a factor $P_\alpha(x; \xi), x \in X, \xi \in \Xi_0$ which we can interpret as the Poisson kernel.

For the unitary case we consider degrees $is, s \in \mathbb{R}$; so homogeneous functions $F(u\xi) = u^{is}F(\xi)$. For $f \in L^2(X)$ we consider the decomposition of $\mathcal{C}f \in L^2(\Xi)$

at the Mellin integral

$$\mathcal{C}f(\xi) = \int_{-\infty}^{\infty} F_s(\xi)ds, \xi \in \Xi_0, s \in \mathbb{R}$$

where F_s are homogeneous of degree is. Then we apply the inversion formula:

$$f(x) = c\int_{-\infty}^{\infty} s^{n-1} f_s(x)ds,$$

where $F_s(\xi) = \mathcal{C}f_s(\xi)$ and f_s are eigen functions corresponding to the homogeneous components F_s in the decomposition of $\mathcal{C}f$ - "projections" of f on the eigen spaces of the Laplace-Beltrami operator. At this decomposition it is appear the Plancherel measure s^{n-1} which has the same form for even and odd n as a result that we work with the horospherical Cauchy transform rather than with the horospherical Radon transform.

The case of discrete spectrum (2 other terms at the inversion formula) is considering similarly but with some changes reflecting that we integrate boundary values of holomorphic functions at the domains $\Xi_\pm$. So for the reconstruction at $x \in X$ we integrate along sections of the boundaries of $\Xi_\pm$ by the hyperplanes $< x, \zeta >= 1$. There $\zeta = x + i\eta, \eta \in \Pi_\pm(x)$ (the sheets of two-sheeted hyperboloid $\Pi(x)$). As for continuous spectrum eigen functions correspond to some homogeneous functions on $\Xi_\pm$. The complex cone $\Xi_{\mathbb{C}}$ is invariant relative to actions of the group $\mathbb{C} - 0$ as dilations which commutates with the action of the group $SO(n, \mathbb{C})$. It corresponds to fibering on the generators $\{\mathbb{C} - 0\}$. The set $\tilde{\Xi}$ is invariant relative the diletions $\zeta \mapsto \exp(i\theta)\zeta$ what can can be verified directly. Moreover the domains $\Xi_\pm$ are invariant relative to the semigroup of multiplications on $\rho\exp(i\theta), \rho > 1$.As a result intersections of $\Xi_\pm$ with generators are the exteriors of circles which are the intersections of boundaries of $\Xi_\pm$. So we have fibering of $\Xi_\pm$ on such one-dimensional domains. We consider homogeneous holomorphic functions on some integer degree n. The substitution at the inversion formula gives Poisson type representation of eigen functions for some discrete eigen values defining by k through boundary values of holomorphic homogeneous functions on $\Xi_\pm$. In this case we can not separate Poison kernel and define the homogeneous functions by some holomorphic functions on the projectivization $P(\Xi_\pm)$ (which we can represent by a section). The projectivization $P(\Xi_{\mathbb{C}})$ can be interpret as the boundary of Z - the complexification of X. We define eigen functions $f^k(x)$ for discrete eigen values through sections of line bundles on $P(\Xi_\pm) \subset P(\Xi_{\mathbb{C}})$ - their (generalized) horospherical Cauchy transform.

At L^2-situation degrees of homogeneity $k \leq 0$ since must be holomorphic at the exterior of the circle. They correspond to holomorphic and anti holomorphic discrete series of representation. We can decompose $\mathcal{C}f_{\pm}(\zeta), \zeta \in \Xi_{\pm}$, at the Fourier (Taylor) series relative to dilations

$$\mathcal{C}f(\zeta) = \sum_{k\leq 0} F^k_{\pm}(\zeta), \zeta \in P(\Xi_{\pm})$$

where F_k are homogeneous. Then

$$f_{\pm} = c\sum_{k\leq 0} k^{n-1} f^k_{\pm}(x), F^k_{\pm} = \mathcal{C}f^k_{\pm}.$$

The case $q = 0$ (the sphere). It is the another case (along with the two-sheeted hyperboloid) when the metric is Riemannian and the spherical operator Laplace-Beltrami is elliptic [3]. On the sphere $\Box(x) = (x_1)^2 + \cdots + (x_{n+1})^2 = 1$ the compact group $SO(n+1)$ acts and there are no (real) horospheres at all (Ξ is empty). As the result there is continuous spectrum. For the investigation of the discrete spectrum we consider complex horospheres $E(\zeta), \Box(\zeta) = (\zeta_1)^2 + \cdots + (\zeta_{n+1})^2 = 0, < \zeta, x >= 1$. *Horospheres $E(\zeta)$ have no real points if and only if*

$$\Box(\xi) = \Box(\eta) < 1.$$

To see it we remark that by the action of $SO(n+1)$ we can transform ζ to the form $(\lambda, i\lambda, 0, \ldots, 0)$. Then the condition that the corresponding horosphere does not intersect the sphere is $|\lambda| < 1$ which gives the above condition on ζ. We denote this domain as $\tilde{\Xi} \subset \Xi_{\mathbb{C}}$.

Boundary horospheres

$$E(\zeta), \Box(\xi) = \Box(\eta) = 1$$

intersect the sphere X. *Such horosphere is passing trough a point $x \in X$ if and only if*

$$\zeta = x + i\eta, < x, \eta >= 0.$$

Denote this set of η through $\Pi(x)$; it is $n-1$-dimensional sphere. Using the invariancy it is sufficient to consider $x = (1, 0, \cdots, 0)$. This point lies at the horosphere if $\xi_1 = 1, \eta_1 = 0$. Then $x_2 = \cdots = x_{n+1} = 0$. As the result we have the fibering of the boundary $\partial\tilde{\Xi}$ over $X(\ \zeta = \xi + i\eta \mapsto \xi)$ on the spheres $\Pi(x)$.

For functions $f(x)$ on X we define the horospherical Cauchy transform $\mathcal{C}f(\zeta)$ holomorphic at $\zeta \in \tilde{\Xi}$. Since the sphere X is compact we have no problems for broad classes of functions and distributions. The most broad class is the class of hyperfunctions which maps on the space of all holomorphic functions on $\tilde{\Xi}$. The eigenfunctions of the spherical Laplacian are spherical polynomials. For them the horospherical Cauchy transform is well defined. It turns out that *functions* *on the sphere are spherical polynomials if and only if their horospherical Cauchy transforms are homogeneous polynomials on* $\tilde{\Xi}$ *and then on* $\Xi_{\mathbb{C}}$. The degrees of polynomials defines the eigen values. Maxwell was the first who connected real spherical polynomials with homogeneous polynomials on the complex cone $\Xi_{\mathbb{C}}$. However, he did not interpret their connection as a variant of Poisson's formula.

The inversion of the horospherical transform has a familiar structure

$$f(x) = c\int_{\Pi(x)} \mathcal{C}f(x + i\eta)\mu(d\eta).$$

In the case when $\mathcal{C}f$ is a homogeneous polynomial of a degree k we obtain an analogue of Poisson formula for spherical polynomials. On $\Xi_{\mathbb{C}}$ we have dilations of $\zeta \mapsto u\zeta, u \in \mathbb{C}-0$, which commutate with the action of $SO(n,\mathbb{C})$. The projectivization of $\Xi_{\mathbb{C}}$ relative to these dilations can be interpreted as the (complex) boundary of the complex hyperboloid $Z \supset X$. We talk about homogeneity relative to these dilations. Then the domain $\tilde{\Xi}$ will be invariant for dilations with $|u| \leq 1$ and its boundary - for $|u| = 1$. So $\tilde{\Xi}$ will fiber on discs and its boundary on circles. We mentioned that holomorphic functions on the complex cone extend at the vertex. So homogeneous functions are polynomials. At the Poisson integral we can for all x integrate over a section of $\partial\tilde{\Xi}$ for a fixed x^0 which we can interpret as a domain at the complex boundary. The restriction on sections for other x we can compute using homogeneity for the dilations with $u = \exp(i\theta)$. Such dilations are dependent on $x^0, x, \eta \in \Pi(x^0)$. As a result in the integrand will appear a factor $P_k(x^0, x; \eta)$ which we can consider as an analogue of the Poisson kernel.

Let us decompose the holomorphic function $\mathcal{C}f(\zeta), \zeta \in \tilde{\Xi}$ at the Taylor series relative to the dilations (the fibering on discs):

$$\mathcal{C}f(\zeta) = \sum_{k\geq 0} F_k(\zeta)$$

where F_k are homogeneous polynomials. Then

$$f(x) = \sum_{k\geq 0} k^{n-1} f_k(x), f_k(x) = (\mathcal{C})^{-1}F_k(x)$$

where f_k are eigen functions of the spherical Laplacian (spherical polynomials). It gives the decomposition on spherical polynomials for a broad class of functional spaces on X.

General case. We will not give a complete picture in the general case since it requires several technical tools which are outside of the scope of this paper. We give only a general flavor of the picture. We can suppose that $p > 1$ since

$p=0$ is impossible and $p=1$ is the hyperbolic case. The continuous spectrum can be considered in the same way, therefore we need to consider only the case of discrete spectrum and correspondingly complex horospheres. Firstly, as it was at considered examples, *for any $x \in X$ and almost all infinitesimal directions there is a horosphere $E(\xi+i\eta)$ which passes x in this direction and either $\eta=0$ or $\Box(\xi)=\Box(\eta)=1$.* So this condition plays the same role as in the above examples.

However besides the case $q=0$ and $p=2$ these horospheres always have real points. For this reason we will consider degenerated complex horospheres without real points. They are intersections of some number horospheres $E(\zeta^i)$ with orthogonal ζ^i. Specifically we will define them in the following way. Let $\xi^{[I]}=\{\xi^1,\ldots,\xi^\kappa\}$ be a system of orthogonal (relative $<,>$) vectors $\xi^i \in \Xi$ and $X(\xi^{[I]})$ be the result of the intersection of X by the isotropic planes

$$< \xi^{[I]}, x > = < \xi^1, z > = \cdots = < \xi^\kappa, x > = 0.$$

Let $Z(\xi^{[I]})$ be the complexification of this intersection. Let us remark that the isotropic hyperplanes $< \xi^i, x > = 0$ are not horospheres, but they are boundary points of Ξ and the whole κ-plane generated by $\xi^{[I]}$ lies on this boundary. We choose maximal κ such that $\Xi(\xi^{[I]}$ has complex degenerated horospheres $E(\xi^{[I]},\zeta)$:

$$< \xi^{[I]}, z > = < \zeta, z > = 0, < \xi^{[I]}, \zeta > = 0.$$

without real points.

There are 2 cases:

$$(i) p-2 \geq q;$$

$$(ii) q > p-2.$$

The index $\kappa=q$ at (i) and $p-2$ at (ii). The case (i) contains the sphere for $q=0$ and (ii) has the special case $p=2$. In a sense inside each of these classes we reduce the general case to these special cases. Namely $X(\xi^{[I]})$ will be fibering over the special hyperboloids on isotropic κ-planes.

If $q=1$ then take $z_p - z_{p+1} = 0$ and the intersection will be fibering over $(p-1,0)$-hyperboloid (sphere) on lines. Recalling our results about complex horospheres for the spheres we can choose ζ such that the degenerated horospheres (of the dimension $n-2$) will have no real points. We can take $\zeta=(\lambda, i\lambda, 0,\ldots,0)$ So complex horospheres without real points on the bases induce similar degenerated (of the codimension $\kappa+1$) horospheres for $X(\xi^{[I]})$. Let us remark that in the case (3,1) of Gelfand- Graev [1] the base is the circle and horocycles are points. For this reason they could avoid the consideration of complex horospheres.

We have enough of degenerated horospheres to construct an invertible version of the horospherical transform. Let us define a mixed horospherical Cauchy transform. For the degenerated horosphere $E(\xi^{[I]}, \zeta)$ we define

$$\mathcal{C}f(\xi^{[I]}, \zeta) = \int_{X(\xi^{[I]})} f(x)/(<\zeta, x> -1)\mu(dx).$$

From our construction it is clear that on $X(\xi^{[I]})$ the integrand has no singularities. This definition mixes Radon's type construction on ξ and Cauchy's type - on ζ. So the result depends on real variables ξ^i and complex variables ζ on which it is holomorphic. Lets give a geometrical interpretation. Let us denote through $\tilde{\Xi} \subset \Xi_{\mathbb{C}}$ the domain $\Box(\xi) = \Box(\eta) < 1$ at the case (i) and $\Box(\xi) = \Box(\eta) > 1$ in the case (ii). Besides the special cases this domain is not Stein one. *If a horosphere $E(\zeta)$ contains one of constructed degenerated horospheres without real points then and only then*

$$\zeta \in \tilde{\Xi}.$$

For $\xi^{[I]}$ let $\Xi(\xi^{[I]})$ be the set of such $\zeta \in \tilde{\Xi}$ that $E(\zeta)$ contains a degenerated horosphere in $X(\xi^{[I]}))$. These subvarieties are Stein and we have the covering of $\tilde{X}$ by them. $\Xi(\xi^{[I]}))$ is fibering on the sets $\hat{E}(\xi^{[I]}, \zeta)$ containing fixed degenerate horospheres $E(\xi^{[I]}, \zeta)$. Let us call them *dual horospheres.* Let us translate $\mathcal{C}f(\xi^{[I]}, \zeta)$ for each $\xi^{[I]}$ as a holomorphic function on $\Xi(\xi^{[I]})$ which is constant on dual horospheres. It turns out that functions $f(X)$ can be reconstructed through $\mathcal{C}f(\xi^{[I]}, \zeta)$. Correspondingly any function can be reconstructed through $\mathcal{C}f(\xi)$ and $\mathcal{C}f(\xi^{[I]}, \zeta)$. These formulas are similar to formulas for reconstructions for the Radon-John transform when we integrate functions on planes of a codimension greater than 1. As at this case $\mathcal{C}f$ satisfy some differential equations (John's system) which are equivalent to closeness of some forms. Their integration on different cycles (non unique) gives the inversion. We do not give these formulas here.

It turns out that functions $F(\xi^{[I]}, \zeta)$ holomorphic on $\zeta \in \Xi(\xi^{[I]})$ and satisfying to John's system can be interpreted as $\bar{\partial}$-cohomology $H^{(\kappa)}(\tilde{\Xi}, \mathcal{O})$. To see it we need to use a special language for $\bar{\partial}$-cohomology - smoothly parameterized Cech cohomology [4].

In such a way, *for general (p,q) the horospherical Cauchy transform takes values at $\bar{\partial}$-cohomology of the degree κ.*

The domain $\tilde{\Xi}$, as we saw above, is invariant relative to the action of the circle (along generators of $\Xi_{\mathbb{C}}$) and, moreover, a semigroup bounded by this circle. Taking functions $F(\xi^{[I]}, \zeta)$, homogeneous relative of this action, and the corresponding cohomology we obtain images of eigen spaces with discrete eigen values. We can interpret this cohomology as cohomology on the projectivization

$P(\tilde{\Xi})$ with coefficients at line bundles $\mathcal{O}^{(k)}$. We also can consider $P(\tilde{\Xi})$ as a domain on the complex boundary $b(Z)$. The inversion formulas give as Poisson's type formulas for eigen functions for discrete spectrum through this cohomology. Finally, we can define L^2-norm of Hardy type for cohomology on $\tilde{\Xi}$, decompose it at Taylor series and produce the Placherel formula on $L^2(X)$.

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