ABOUT THE EXISTENCE OF MULTIPLE SOLUTIONS FOR A CLASS OF BOUNDARY VALUE PROBLEMS WITH DISCONTINUOUS NONLINEARITIES

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Sunto. In questo lavoro si estende al caso in cui la nonlinearietà è discontinua alcuni recenti risultati sull'esistenza di soluzioni multiple per il problema al contorno (di Dirichlet o di Neumann)

$$Au(x) = f(u(x)) + k(x) + t \Psi(x) \quad \text{in } \Omega, \quad Bu = 0 \quad \text{su } \partial \Omega,$$

in funzione del parametro reale t.

Si prova anche con un esempio che tale estensione non si ha se non si intende la soluzione in un opportuno senso multivoco, precisamente nel senso che

$$Au(x) - k(x) - t \Psi(x) \in \hat{f}(u(x)),$$

dove $\hat{f}(s)$ è l'intervallo chiuso avente per estremi i limiti di f per $c \to s$ da destra e da sinistra.

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INTRODUCTION AND STATEMENT OF THE MAIN RESULTS.

Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^n \) with sufficiently smooth boundary \( \partial \Omega \), (for example of class \( \mathcal{C}^2 \)), and let \( A \) be a linear uniformly elliptic differential operator of the second order formally defined by

\[
A(u) = \sum_{i,j} a_{ij} D_i D_j u + \sum_i a_{i} u + a_0 u.
\]

where \( a_0 \in L^\infty(\Omega) \) and the coefficients \( a_{ij} = a_{ji} \) are continuously differentiable in \( \Omega \). Let us denote by \( B \) either the Dirichlet or the Neumann boundary operator.

Moreover let \( f : \mathbb{R} \to \mathbb{R} \) be a mapping of bounded variation on all compact intervals such that \( f(s) \) belongs to the interval whose extremes are \( f(s-) = \lim_{\varepsilon \to 0^+} f(s-\varepsilon) \) and \( f(s+) = \lim_{\varepsilon \to 0^+} f(s+\varepsilon) \) for all \( s \in \mathbb{R} \).

For a fixed \( k \in L^p(\Omega) \) with \( p > n \), and for a fixed \( \psi \in C(\overline{\Omega}), \psi(x) > 0 \) in \( \Omega \), we are interested in the solvability of the boundary value problem

\[
\begin{cases}
Au(x) - k(x) - t\psi(x) \in \bar{f}(u(x)) & \text{in } \Omega \\
Bu = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where we have put \( \bar{f}(s) = \{f(s)\} \) if \( f(s-) \leq f(s+) \) and \( \bar{f}(s) = [f(s+), f(s-)] \) if \( f(s+) < f(s-) \).

In this frame, following an approach due to Kazdan and Warner, [7], and a result of Stuart, [10], [11], we have the following

**THEOREM 1.** Assume that

i) \( \limsup_{s \to -\infty} f(s)/s < \lambda_1 < \liminf_{s \to \infty} f(s)/s \).
where $\lambda_1$ is the first eigenvalue of $Au = \lambda u$ in $\Omega$, $Bu = 0$ on $\partial \Omega$. Then there exists $t_0$ such that $(\tilde{P})_t$ has no solution for $t > t_0$ and at least one solution for $t < t_0$.

In the case when $f$ is continuously differentiable and satisfies i) and

\[ \limsup_{s \to +\infty} \frac{f(s)}{s} < +\infty, \]

then a multiplicity result holds, in the sense that the boundary value problem $(\tilde{P})_t$ has at least two solutions for $t < t_0$, and at least one solution for $t = t_0$ (see Dancer, [5], Amann and Hess, [3], and their remarks concerning related researches).

However such a multiplicity result does not hold in the discontinuous case as we show in Section 2. Therefore, if we wish to give a result of the Amann-Hess type, we have to consider a weaker notion of solution, which is obtained by filling up all the jumps of $f$ rather than only the downward ones. In other words we study the boundary value problem

\[
\begin{cases}
Au(x) - k(x) - t \Psi(x) \in \hat{f}(u(x)) & \text{in } \Omega \\
Bu = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where we have denoted by $\hat{f}(s)$ the convex hull of $\{f(s-), f(s+)\}$ for all $s \in \mathbb{R}$.

Then we have the following

THEOREM 2. Assume that $f$ has only finitely many discontinuities in any compact interval and that all the jumps are upward. Moreover assume that $f$ satisfies conditions i) , ii) and
iii) for some $M, r > 0$ the mapping $f(s) + Ms$ is increasing on $[-r, r]$.

Then there exists $t_0$ such that the Dirichlet boundary value problem

$$\begin{cases}
Au(x) - k(x) - t\psi(x) \in \hat{f}(u(x)) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

has no solution for $t > t_0$, at least one solution for $t = t_0$ and at least two solutions for $t < t_0$.

The same result holds in the Neumann case if we replace iii) with the stronger assumption

iii)' for all $r > 0$ there exists $M_r > 0$ such that $f(s) + M_r s$ is increasing in the interval $[-r, r]$.

Section 1. PROOF OF THEOREM 1.

First of all notice that, by replacing (if necessary) $A$ with $A + \lambda I$ and $f$ with $f + \lambda I$, we can always assume that $\lambda_1 > 0$.

For all $k, q$ let us denote by $\| \cdot \|_{k, q}$ the usual norm in the Sobolev space $W^{k, q}(\Omega)$, and by $\| \cdot \|_q$ the usual norm in $L^q(\Omega)$. Moreover $\| \cdot \|$ will denote the norm in $C^1(\overline{\Omega})$. Since in the sequel we shall always assume that $p > n$, we have the compact imbedding of $W^{2, p}(\Omega)$ into $C^1(\overline{\Omega})$.

By a subsolution (supersolution) of $(\tilde{P})_t$ we mean an element of $W^{2, p}(\Omega)$ such that

$$\begin{align*}
Au(x) - (k + t\psi)(x) \leq & \max \tilde{f}(u(x)) \quad \text{(respectively } \geq \min \tilde{f}(u(x))) \quad \text{in } \Omega \\
Bu \leq & 0 \quad \text{(respectively } \geq 0) \quad \text{on } \partial\Omega.
\end{align*}$$
Evidently \( u \in W^2, p(\Omega) \) is a solution of \((\bar{P})_t\) if and only if \( u \) is a sub- and a supersolution for \((\bar{P})_t\).

The assertion of Theorem 1 easily follows from the following three lemmas.

**Lemma 1.1.** Let \( I = \{ t \mid (\bar{P})_t \text{ has a supersolution} \} \). Then \( I \) is a nonempty interval which is unbounded from below and is bounded from above.

**Lemma 1.2.** For all \( K, t \in \mathbb{R} \), there exists a subsolution \( v \) of \((\bar{P})_t\) such that \( v(x) \leq K \) in \( \Omega \).

**Lemma 1.3.** If \( v \) and \( w \) are a sub- and a supersolution for \((\bar{P})_t\) and \( v \leq w \), then \((\bar{P})_t\) has a solution \( u \) with \( v \leq u \leq w \).

In fact put \( t_0 = \sup I \). Then for every \( t > t_0 \) the boundary value problem \((\bar{P})_t\) has no supersolution and therefore no solution. On the other hand for \( t < t_0 \), there exists a supersolution \( w \) of \((\bar{P})_t\), and by Lemma 1.2 there exists a subsolution \( v \) of \((\bar{P})_t\) with \( v \leq -\|w\| \leq w \). Hence by Lemma 1.3 there exists a solution \( u \) of \((\bar{P})_t\) with \( v \leq u \leq w \).

The proof of Lemma 1.1 and Lemma 1.2 is essentially contained in [7], whereas Lemma 1.3 is a result due to C.Stuart,[10], [11]. (Notice that Stuart considers only the case when \( B \) is the Dirichlet boundary operator, but his arguments can be easily extended to the Neumann case).

Section 2.

We give now an example of boundary value problems of type \((\bar{P})_t\)', which shows that the multiplicity result of Amann and Hess,[3], cannot be extended to this kind of problems.

Consider the Neumann boundary value problem
\[(\tilde{N})_t \quad -u''(x) - t \in \tilde{f}(u(x)) \quad \text{for} \quad x \in [0,1] \quad u'(0) = u'(1) = 0,\]

where \( f : \mathbb{R} \to \mathbb{R} \) is defined by \( f(s) = s+1 \) for \( s > 0 \), \( f(s) = -s-1 \) for \( s < 0 \) and \( f(0) = c \in [-1,1] \).

Evidently \( f \) is a mapping of bounded variation on compact intervals such that conditions i) and ii) hold and \( f(s) \) belongs to the interval of extremes \( f(s^-), f(s^+) \) for all \( s \in \mathbb{R} \). Then by Theorem 1 there exists \( t_0 \) such that \((\tilde{N})_t \) is solvable for \( t < t_0 \) and is not solvable for \( t > t_0 \). We have \( t_0 \geq 1 \), since the constant mapping \( u(x) = t - 1 \) is a solution of \((\tilde{N})_t \) for all \( t \leq 1 \). We shall prove that \((\tilde{N})_t \) has a unique solution for \( t = -1 \), which proves our statement.

In fact let \( u \in W^{2,1}([0,1]) \subseteq C^1([0,1]) \) be a solution of \((\tilde{N})_{-1} \), i.e. such that

\[
-u''(x) + 1 =
\begin{cases}
  u(x) + 1 & \text{in } \{ x \in [0,1] | u(x) > 0 \}, \\
  c & \text{in } \{ x \in [0,1] | u(x) = 0 \}, \\
  -u(x) - 1 & \text{in } \{ x \in [0,1] | u(x) < 0 \}.
\end{cases}
\]

We shall prove that \( u(x) = -2 \). To do this it is sufficient to prove that \( u(x) \leq 0 \) in \([0,1]\). In fact one has that \( \{ x \in [0,1] | u(x) = 0 \} = 0 \), (since \( c \neq 1 \)), and therefore \( u \) is the unique solution of the boundary value problem \( u''(x) - u(x) - 2 = 0 \) in \([0,1]\), \( u'(0) = u'(1) = 0 \), and this is just the constant mapping \( u(x) = -2 \).

To prove that \( u(x) \leq 0 \) in \([0,1]\), assume on the contrary that \( u(x) > 0 \) for some \( x \in [0,1] \). Then there exist \( 0 \leq x_0 < x_1 < x_2 \leq 1 \) such that \( u(x) > 0 \).
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in \( \bigcup x_0' \bigcup x_1' \), and \( u(x_0') = 0 \) (respectively \( u(x_1') = 0 \)), if \( x_0 > 0 \), (respectively \( x_1 < 1 \)). Then \( u \) is a solution in \( \bigcup x_0' \bigcup x_1' \) of \( u'' + u = 0 \), i.e. \( u(x) = c_1 \cos x + c_2 \sin x \) where the constants \( c_1, c_2 \) are such that

\[
\begin{align*}
  u'(0) &= u'(1) = 0 & \text{if } x_0 = 0, x_1 = 1, \\
v'(0) &= u(x_1') = 0 & \text{if } x_0 = 0, x_1 < 1, \\
v(x_0') &= 0, u(x_1) = 0 & \text{if } 0 < x_0', x_1 < 1, \\
v(x_0') &= 0, u'(1) = 0 & \text{if } 0 < x_0', x_1 = 1. 
\end{align*}
\]

But since \( 0 < x_1' - x_0' < 1 < \pi/2 \), it is easy to see that \( c_1 = c_2 = 0 \) in all cases, so that \( u(x) = 0 \) in \( \bigcup x_0', x_1' \), contradicting our assumption.

Section 3. - PROOF OF THEOREM 2.

Without loss of generality we can assume that \( f(s) = f(s-) = \min \hat{f}(s) \) for all \( s \in \mathbb{R} \). Then if we put \( t_0 = \sup \{ t \mid (\hat{P})_t \text{ has a supersolution} \} \), since any solution of \( (\hat{P})_t \) is a solution of \( (\hat{P})_t \) and any solution of \( (\hat{P})_t \) is a supersolution for \( (\hat{P})_t \), from Theorem 1 it follows that \( (\hat{P})_t \) is solvable for \( t < t_0 \), and not solvable for \( t > t_0 \). Hence we have only to prove that

1. \( (\hat{P})_t \) is solvable for \( t = t_0 \).
2. \( (\hat{P})_t \) has at least two solutions for \( t < t_0 \).

The proof of (1) is essentially contained in [3], and we shall not reproduce it here.

To prove (1) assume on the contrary that for some \( t^* < t_0 \) the problem \( (\hat{P})_t^* \) has a unique solution \( u^* \). We shall reach a contradiction through several steps.
First of all notice that by assumptions i) and ii), and by the fact that \( f \) is of bounded variation on compact intervals, there exist \( \lambda' \), \( \lambda'' \), \( c \) such that \( 0 < \lambda' < \lambda_1 < \lambda'' \) and

\[
(3.1) \quad f(s) > \max (\lambda's + c, \lambda''s + c) \quad \text{for all } s \in \mathbb{R}.
\]

**Lemma 3.1.** We can assume that there exists \( K \in \mathbb{R} \) such that \( f \) is continuous in \( K \) and \( f(s) = \text{const.} = f(K) \) for all \( s \leq K \).

**Proof.** Let \( u_1 \) be the solution of \( Au(x) - \lambda'u(x) = c + k(x) + (t^*-1)\psi(x) \) in \( \Omega \), \( \partial u = 0 \), and let \( K < -\| u_1 \| \) such that \( f \) is continuous in \( K \). Then we consider the boundary value problem

\[
\begin{cases}
Au(x) - k(x) - t\psi(x) \in f^*(u(x)) & \text{in } \Omega \\
Bu = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( f^* \) is defined by \( f^*(s) = f(s) \) for \( s > K \) and \( f^*(s) = f(K) \) for \( s \leq K \). By (3.1) and by the maximum principle one easily proves that for \( t > t^*-1 \) any solution of \( (\hat{P})_t \) is a solution of \( (\hat{P}^*)_t \) and viceversa, (see [4] section 3, for a similar situation). Since in the sequel we shall be interested in the solvability of \( (\hat{P})_t \) for \( t^*-1 < t < t_0^* + 1 \), we can replace \( f \) with \( f^* \) and the lemma is proved.

Now, since \( f \) is of bounded variation on compact intervals and all the jumps are upward, \( f \) has the form \( f = g - h \), where \( g \) and \( h \) are strictly increasing and \( h \) is continuous, (see [10], [11]).

**Lemma 3.2.** There exist a subsolution \( v \) and a supersolution \( w \) of \( (\hat{P})_t* \) such that \( v(x) \leq \min(K, w(x)) \) and \( u^* \) is interior to the set
\[ X = \{ u \in C^1(\Omega) \mid Bu = 0, \, v \leq u \leq w \} . \]

**Proof.** For an \( \varepsilon \) sufficiently small by Theorem 1 there exists a solution \( v \) of \( (\tilde{P})_{t^*+\varepsilon} \); then \( w \) is a supersolution of \( (\tilde{P})_{t^*} \). Then by Lemma 1.2 there exists a subsolution \( v \) of \( (\tilde{P})_{t^*} \) such that \( v \leq \min(K,- \| w \|, - \| u^* \| - 1) \) and therefore \( v \leq w \). Since \( v \leq w \) and \( u^* \) is the unique solution of \( (\tilde{P})_{t^*} \), we have that \( v \leq u^* \leq w \), and therefore \( u^* \in X \). Moreover, since \( v \leq - \| u^* \| - 1 \), one has \( v(x) < u^*(x) \) in \( \Omega \); hence for some \( r > 0 \) we have that

\[ (3.2) \quad z \in C^1(\Omega), \quad \| z - (u^* - v) \| < r \Rightarrow z \geq 0. \]

Therefore in order to prove that \( u^* \) is interior to \( X \) it is sufficient to prove that there exists \( r > 0 \) such that

\[ (3.3) \quad z \in C^1(\Omega), \quad Bz = 0, \quad \| z - (w - u^*) \| < r \Rightarrow z \geq 0. \]

We prove first that \( u^*(x) < w(x) \) in \( \Omega \). Assume on the contrary that \( u^*(x_0) = w(x_0) \) for some \( x_0 \in \Omega \). By the continuity of \( h, u^*, w \) and \( \psi \) there exists \( r > 0 \) such that \( B(x_0, r) \subset \Omega \) and \( h(u^*(x)) - h(w(x)) + \varepsilon \psi(x) > 0 \) for all \( x \in B(x_0, r) \). Now, since \( u^* \) and \( w \) are solutions of \( (\tilde{P})_{t^*} \) and \( (\tilde{P})_{t^*+\varepsilon} \) respectively and \( f \) has no downward jump, then we have \( (u^*-w)(x) \leq 0 \) on \( \partial B(x_0, r) \) and \( A(u^*-w)(x) = f(u^*(x)) - f(w(x)) - \varepsilon \psi(x) = g(u^*(x)) - g(w(x)) - h(u^*(x)) + h(w(x)) - \varepsilon \psi(x) = 0 \) in \( B(x_0, r) \). From this, by the maximum principle, it follows \( u^*(x) < w(x) \) in \( B(x_0, r) \) contrarily to our assumption.

We distinguish now the case that \( B \) is the Dirichlet operator from the case when \( B \) is the Neumann one. In the first case we have \( u^*(x) = \)
\( w(x) = 0 \) on \( \partial \Omega \). Then with the same argument of Lemma 1 of [4] one proves that \( \partial (u^* - w)/\partial v > 0 \) on \( \partial \Omega \), and by Lemma 2 of [4] one has that (3.3) holds for some \( r > 0 \).

In the case that \( B\delta u = \partial u/\partial v \), then we prove that \( u^*(x) < w(x) \) in \( \Omega \), and from this (3.3) easily follows. In fact let \( r > 0 \) such that \( u^*(x) \leq r \) and \( |w(x)| \leq r \) for all \( x \in \Omega \). Then we have by (ii) that \( f(u^*(x)) - f(w(x)) \leq -M_r(u^*(x) - w(x)) \) and therefore \( A(u^*-w)(x) + M_r(u^*-w)(x) \leq A(u^*-w)(x) - f(u^*(x)) + f(w(x)) = -\varepsilon \psi(x) < 0 \) for all \( x \in \Omega \). From this, since we have \( \partial (u-w)/\partial v = 0 \) on \( \partial \Omega \), by the strong maximum principle, (see [2], Lemma 1), it follows that \( u^*(x) < w(x) \) in \( \Omega \), as desired.

**Lemma 3.3.** There exists a sequence \( (f_n) \) of continuous mappings such that:

\[
\begin{align*}
(3.4) & \quad f_n(s) \leq f(s-) = f(s), \\
(3.5) & \quad f_n(s) \geq \max(\lambda's+c, \lambda''s+c), \\
(3.6) & \quad \gamma_n = f_n + h \text{ is increasing,} \\
(3.7) & \quad s \to s = \left( \liminf_{n \to +\infty} f_n(s), \limsup_{n \to +\infty} f_n(s) \right) \subset f(s).
\end{align*}
\]

**Proof.** Let \( z_1 < z_2 < \ldots < z_i < \ldots \) be the sequence of the points where \( f \) is discontinuous; (obviously \( z_i < z_1 \)). Let us fix \( i \) and put for all \( n \in \mathbb{N} \) and for all \( s \in [z_i, z_{i+1}] \), (with the convention that \( z_{i+1} = z_i + 1 \)) if \( f' \) is continuous on \( [z_i, z_{i+1}] \):

\[
\gamma_{i,n}(s) = (2n/(z_{i+1}-z_i))(f(z_{i+1}+(z_{i+1}-z_i)/2n)-f(z_i))(s-z_i)+f(z_i)
\]

\[
\delta_{i,n} = \sup\{z \in [z_i, z_{i+1}] \} \left| \gamma_{i,n}(s) \leq f(s) \text{ for } s \in [z_i, z_{i+1}] \right|
\]
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\[ f_{i,n}(s) = \begin{cases} 
\gamma_{i,n}(s) & \text{for } s \in [z_i, \delta_{i,n}]
\end{cases} \]

\[ f(s) \quad \text{for } s \in [\delta_{i,n}, z_{i+1}]. \]

It is easily seen that \( f_{i,n} \) is continuous in \([z_i, z_{i+1}]\). On the other hand since \( f \) is continuous on the left, one has \( f_{i,n}(z_{i+1}) = f(z_{i+1}) = z_{i+1} \).

Hence the mapping \( f_n \) defined by \( f_n(s) = f(s) \) if \( s < z_i \) and \( f_n(s) = z_{i+1} \) if \( s \in [z_i, z_{i+1}] \), is continuous. Moreover one easily proves that \( f_n \) satisfies conditions (3.4), (3.5) and (3.6). Finally assume that \( s_n \to s \). If \( s \neq z_i \) for all \( i \), we have \( f_n(s_n) = f_n(s) \) for \( n \) great and therefore \( f_n(s_n) \to f(s) \in f(s) \). Let now \( s = z_i \) for some \( i \) and let \( (f_n(s_i)) \) be a converging subsequence. If the set \( \{ k \mid s_{n_k} \leq z_i \} \) is infinite, then one has \( \lim_{k \to \infty} f_n(s_{n_k}) = f(z_i) \in f(s) \). If the set \( \{ k \mid s_{n_k} \leq z_i \} \) is finite, then for \( k \) great one has \( z_i < s_{n_k} \), and therefore by (3.4), (3.6) one has

\[ f(z_i) = f_n(z_i) = g_{n_n}(z_i) - h(z_i) < g_{n_n}(s_{n_k}) - h(z_i) = f_n(s_{n_k}) + h(s_{n_k}) - h(z_i) \leq f(s_{n_k}) + h(s_{n_k}) - h(z_i). \]

It follows that \( f(s) \leq \lim_{n \to \infty} f_n(s_{n_k}) \leq f(s) \), which proves the assertion.

**Lemma 3.4.** Put for convenience \( f_0 = f \) and consider for all \( n \in \mathbb{N} \cup \{0\}, t \in \mathbb{R} \) the boundary value problem

\[ \hat{P}_{n,t} \quad \begin{array}{l}
\text{in } \Omega, \quad Bu = 0 \quad \text{on } \partial \Omega.
\end{array} \]
Then there exists $\rho_0 > 0$ such that for any solution $u$ of $\hat{P}_n$, one has $\|u\|_{2,p} \leq \rho_0$.

Proof. Assume on the contrary that for all $j \in \mathbb{N}$ there exist $t_j \in [t_j^* - 1, t_j^* + 1]$, $n_j \in \mathbb{N} \cup \{0\}$ such that $\hat{P}_{n_j}^{t_j}$ has a solution $u_j$ with $\|u_j\|_{2,p} > j$. Since $f_n(s) \leq \lambda s + c$ for all $s \in \mathbb{R}$, by the maximum principle one has again $u_j \geq K$, for all $j \in \mathbb{N}$. On the other hand, by ii) and the fact that $f$ is of bounded variation on compact intervals, there exist $\lambda > c', c' > 0$ such that $f(s) < \lambda s + c'$ for all $s > K$. From this and from (3.4) it follows that $f_n(s) \leq \lambda s + c'$ for all $s > K$, and therefore

$$\lambda'' u_j(x) + c \leq f_n(u_j(x)) \leq f_n(u_j(x)) \leq \lambda u_j(x) + c'$$

for all $j \in \mathbb{N}$. Hence, if we put $\hat{u}_j = u_j/\|u_j\|_{2,p}$, $w_j = A\hat{u}_j^{(t_j, \psi + k)/\|u_j\|_{2,p}}$ we have

$$\lambda'' \hat{u}_j(x) + c \|u_j\|_{2,p} \leq w_j \|u_j\|_{2,p} \leq \lambda \hat{u}_j(x) + c'/\|u_j\|_{2,p}.$$  

From this it follows that $(\Lambda u_j)_{j \in \mathbb{N}}$ is bounded in $L^p(\Omega)$. Since the inverse $\Lambda^{-1}$ is compact from $L^p(\Omega)$ to $\{u \in C^1(\Omega)! Bu = 0\}$, we can assume that $(\hat{u}_j)_{j \in \mathbb{N}}$ converges to $\hat{u}$ in $C^1(\overline{\Omega})$. Therefore by the maximum principle $\Lambda^{-1}$ is increasing. Hence by (3.9) one has

$$\Lambda^{-1}(\lambda'' \hat{u}_j + c' \|u_j\|_{2,p}) \leq \Lambda^{-1}(w_j) = \hat{u}_j - \Lambda^{-1}(t_j \psi + k)/\|u_j\|_{2,p}.$$  

By passing to the limit one has $\Lambda^{-1}(\lambda'' \hat{u}) \leq \hat{u}$ i.e. $(1 - \lambda'' \Lambda^{-1})\hat{u} \geq 0$, where $\hat{u} \geq 0$, since $\|\hat{u}\|_{2,p} = \lim \|\hat{u}_j\|_{2,p} = 1$, and $\hat{u}(x) = \lim u_j(x)/\|u_j\|_{2,p}$, for
all \( x \in \Omega \). This is impossible since \( A^{-1} \) is a strongly positive compact endomorphism on the space \( \{ u \in C^{1}(\Omega) \mid \text{Bu} = 0 \} \), and \( \lambda'' > \lambda_1 \), (see [1], Theorem 3.2).

**Lemma 3.5.** Let \( t \in [t^{*}-1, t^{*}+1] \), and let \( Y \) be a closed subset of the space \( \{ u \in C^{1}(\Omega) \mid \text{Bu}=0 \} \). For all \( j \) let \( u_j \in Y \) be a solution of \( (\hat{P})_j \) where \( (n_j)_{j \in \mathbb{N}} \) is a strictly increasing sequence of natural numbers. Then \( (\hat{P})_j \) has a solution in \( Y \).

**Proof.** By Lemma 3.4 one has \( \| u_j \|_{2^*,P} \leq \theta_0 \) for all \( j \in \mathbb{N} \). Since \( W^{2,P}(\Omega) \) is reflexive and is compactly imbedded into \( C^{1}(\Omega) \), we can assume that \( (u_j)_j \) converges weakly in \( W^{2,P}(\Omega) \) and strongly in \( C^{1}(\Omega) \) to a mapping \( u \in W^{2,P}(\Omega) \). From this it follows that \( u \in Y \) and \( \text{Bu} = 0 \). On the other hand, by (3.8) we have that \( (f(u_j))_j \) is bounded in \( L^P(\Omega) \) and therefore we can assume that \( (f(u_j))_j \) is weakly convergent in \( L^P(\Omega) \) to a mapping \( \theta \). From this and from Fatou's lemma it follows that

\[
\liminf_{j \to \infty} f(u_j(x)) \leq \theta(x) \leq \limsup_{j \to \infty} f(u_j(x)),
\]

and therefore by (3.7) one has \( \theta(x) \in \hat{f}(u(x)) \).

The proof will be reached if we prove that \( Au(x) = t \psi(x) + k(x) + \theta(x) \).

In fact all \( \phi \in \tilde{W}^{1,P}(\Omega) \) or \( \phi \in W^{1,P}(\Omega) \) according to whether \( B \) is the Dirichlet or the Neumann boundary operator, one has

\[
\sum_{i} \int_{\Omega} a_{i} \nabla D_{i} u_{j} \cdot D \phi + \int_{\Omega} a_{0} u_{j} \phi = \int_{\Omega} (t \psi + k + f(u_j)) \phi
\]
for all \( j \in \mathbb{N} \). Passing to the limit as \( j \to \infty \), one has

\[
\sum_{i \in \mathcal{L}} a_i \epsilon_i D_i u_{\epsilon} \phi + \sum_{i \in \mathcal{L}} a_i (D_i u) \phi + \int_{\Omega} a_i u \phi = \int_{\Omega} (t_{\epsilon+} k + \theta) \phi.
\]

From this, since \( u \in H^2, p(\Omega) \) and \( Bu = 0 \), one has the assertion.

REMARK 3.1. By the embedding of \( H^2, p(\Omega) \) into \( C^1(\overline{\Omega}) \) and by lemma 3.4 there exists a constant, which will be denoted again by \( \rho_0 \), such that for any solution \( u \) of \( (\dot{\mathcal{P}}) \) with \( t \in [t^* - 1, t^* + 1] \) one has \( \|u\| \leq \rho_0 \).

In the sequel the balls considered have to be intended into the space \( \{u \in C^1(\overline{\Omega}) \mid 3u = 0\} \).

LEMMA 3.6. Let \( \rho_1 > 0 \) such that \( B(u^*, \rho_1) \subseteq X \), (see lemma 3.2). Then there exists \( n_0 \) such that for all \( n \geq n_0 \) one has:

1) \( (\dot{\mathcal{P}})_{n t^* + 1} \) has no solution,

11) every solution of \( (\dot{\mathcal{P}})_{n t^*} \) belongs to \( B(u^*, \rho_1) \).

Proof of 1). Assume on the contrary that there exists a strictly increasing sequence \( (n_j) \) of natural numbers such that \( (\dot{\mathcal{P}})_{n_j t^* + 1} \) has a solution \( u_j \) for all \( j \). Then by Remark 3.1 we have \( \|u_j\| \leq \rho_0 \); from this and from Lemma 3.5 it follows that \( (\dot{\mathcal{P}})_{t^* + 1} \) has a solution \( u \in B(0, \rho_0) \), contradicting the definition of \( t_0 \).

Proof of 11). Assume on the contrary that there exists a strictly increasing sequence \( (n_j) \in \mathbb{N} \) of natural numbers such that \( (\dot{\mathcal{P}})_{n_j t^*} \) has a solution \( u_j \) in \( B(0, \rho_0) \setminus B(u^*, \rho_1) \) for all \( j \). Then by Lemma 3.5 there exists a solution of
\((\hat{P})_{t\ast}\) in \(\overline{B}(0, \rho_0) \cap \mathcal{B}(u^\ast, \rho_1)\), contradicting the assumption that \(u^\ast\) is the unique solution of \((\hat{P})_{t\ast}\).

**Remark 3.2.** By modifying, (if necessary), the mapping \(g\) and \(h\) in \(\mathbb{R} \times [\rho, \rho]\), where \(\rho = \max(\rho_0, \|v\|, \|w\|)\), we can assume from now on that \(h\) is continuous, strictly increasing and bounded. (Obviously now \(g\) and \(g_n = f_n + h\) are increasing only into the interval \([-\rho, \rho]\). Then we have that the map \(A+H: \{u \in W^2, p(\Omega) \mid Bu = 0\} \rightarrow L^p(\Omega)\), (where \(H\) is the Nemitskii operator associated to \(h\)), is onto by a degree argument and is one-to-one by the maximum principle. Moreover, its inverse \((A+H)^{-1}\) is continuous from \(L^p(\Omega)\) to \(W^2, p(\Omega)\) and therefore completely continuous from \(L^p(\Omega)\) to the space \(E = \{u \in C^1(\Omega) \mid Bu = 0\}\).

In fact assume on the contrary that \((A+H)^{-1}\) is not continuous in some \(z \in L^p(\Omega)\). Then there exist \(\varepsilon > 0\) and a sequence \(z_n \in L^p(\Omega)\) such that \(z_n \rightarrow z\) as \(n \rightarrow \infty\) and \(\| (A+H)^{-1}(z_n) - (A+H)^{-1}z \|_{L^p} \geq \varepsilon\) for all \(n\). If we put \(u_n = (A+H)^{-1}(z_n)\), then we have that the sequence \(Au_n = H(u_n) + z_n\) is bounded in \(L^p(\Omega)\). Therefore by the compactness of \(H^{-1}: L^p(\Omega) \rightarrow E\), there exists a subsequence \(u_{n_k}\) which converges to some \(u \in E\).

Hence by the continuity of \(H\), we have that \(u_{n_k} \rightarrow z - H(u)\). From this by the continuity of \(A^{-1}\) from \(L^p(\Omega)\) to \(W^2, p(\Omega)\), it follows that \(u \in W^2, p(\Omega)\) and \(u_{n_k} \rightarrow u = A^{-1}(z - H(u))\) in \(W^2, p(\Omega)\). Hence one has \(u = (A+H)^{-1}(z)\) and \(u_{n_k} = (A+H)^{-1}(z_{n_k})\) converges to \(u\), contradicting the assumption \(\| (A+H)^{-1}(z_{n_k}) - u \|_{L^p} \geq \varepsilon\).

Finally notice that by using the maximum principle one easily has that \((A+H)^{-1}\) is an increasing mapping.
**Lemma 3.7.** For all $n \in \mathbb{N}$ and for all $t \in [t^*-1, t_0+1]$ put

$$G_n(t,u)(x) = g_n(u(x)) + t\psi(x) + k(x) \text{ for all } x \in \Omega.$$  

Then the mapping $(A+I)^{-1}G_n(t,\cdot)$ is completely continuous in $\Omega$.

Moreover for $n$ great one has

$$\deg(I-(A+I)^{-1}G_n(t,\cdot), B(u, \rho_n, 0)) = 1.$$  

**Proof.** The first part of the assertion follows from Remark 3.2 and from the continuity of $g_n$.

Now put $T_n = (A+I)^{-1}G_n(t^*, \cdot)$ for all $n$. By (3.4) and the fact that $f_n(s) = f(s)$ for $s \leq K$ and $v(x) \leq K$ in $\tilde{\Omega}$, we have that $w$ and $v$ are respectively a supersolution and a subsolution for $(\tilde{T}_n)^*\phi_n$ for all $n$, and therefore we have $v \leq T_n(v)$ and $T_n(w) \leq w$. On the other hand, since $g_n$ is increasing in $]-\rho_n, \rho_n[$, we have that $T_n$ is increasing in $X$. Therefore one has that $T_n(X) \subseteq X$. Finally since $X$ is bounded in $C(\tilde{\Omega})$, by the continuity of $g_n$ and the complete continuity of $(A+I)^{-1}G_n(t,\cdot)$, we have that $T_n(X)$ is relatively compact. Let $R_n > \rho_n$ such that $T_n(X) \subseteq B(u^*, R_n)$ and let $r$ be a retraction of $E$ on $X$, (see [6], Theorem 4.1). Then we have

$$u = \mu T_n(r(u)) + (1-\mu)u \text{ for all } u \in B(u, R_n), \mu \in [0, 1]$$

and therefore by the homotopy invariance property of the Leray-Schauder degree one has for all $n \in \mathbb{N}$

$$\deg(I-T_n r, B(u^*, R_n), 0) = \deg(I-u^*, B(u^*, R_n), 0) = 1.$$  

Now let $n \in \mathbb{N}$ and $u \in B(u^*, R_n)$ such that $u = T_n(r(u))$. Then $u \in X$ and
therefore \( u = T_n(u) \) since \( r(u) = u \). Hence by Lemma 3.6 one has \( u \in B(u^*, \rho_1) \) for \( n > n_0 \). From this, from the properties of the degree and from the fact that \( r(u) = u \) for all \( u \in B(u^*, \rho_1) \subset X \), one has for \( n > n_0 \)

\[
\deg(I - T_n, B(u^*, \rho_1), 0) = \deg(I - T_n r, B(u^*, \rho_1), 0) = \deg(I - T_n r, B(u^*, R_n), 0) = 1.
\]

We are now in a position of concluding the proof of Theorem 2. In fact by part (1) of Lemma 3.6 one has for all \( n > n_0 \)

\[
\deg(I - (A + H)^{-1} G_n(t_0 + 1, \ldots), B(0, \rho_0), 0) = 0,
\]

and therefore

\[
\deg(I - (A + H)^{-1} G_n(t^*, \ldots), B(0, \rho_0), 0) = 0
\]

by the homotopy invariance property. From this and from part (2) of Lemma 3.6 it follows that

\[
\deg(I - T_n, B(u^*, \rho_1), 0) = \deg(I - T_n, B(0, \rho_0), 0) = 0,
\]

contradicting the assertion of Lemma 3.7. Hence Theorem 2 is completely proved.
REFERENCES


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