SEPARATION AXIOMS BY SIMPLE EXTENSIONS (*)

Cosimo GUIDO (**)

Sunto. Nella prima parte studieremo, per \( i = 0,1,2,2_{\frac{1}{2}} \), gli spazi topologici, che chiameremo QT\( _i \), che verificano gli assiomi di separazione T\( j \) per \( j < i \) e che hanno un'estensione semplice T\( _i \).

Nella seconda parte prenderemo in considerazione topologie, su un sostegno fissato S, che sono T\( j \), \( j < i \), e non-T\( _i \) massimali (MNT\( _i \).

Daremos alcuni esempi e provremo alcune proprietà delle topologie QT\( _i \) e delle topologie MNT\( _i \).

1. Given a topological space \((S, \tau)\) and a subset \( X \subseteq S \), we shall denote by \( \tau(X) = \{ A \cup (B \cap X) / A, \text{B} \in \tau \} \) the simple extension of \( \tau \) by \( X \).

We shall denote by \( cl_X(int_X) \) the closure (the interior) of \( X \) in \((S, \tau)\) and by \( cl_{\tau'}(X) \) the closure (the interior) of \( X \) with respect to any other topology \( \tau' \) on \( S \). \( \mathcal{T}(x) (\tau(x)) \) will mean the family of (open) neighbourhoods of \( x \in S \) in the topology \( \tau \). We shall call fundamental neighbourhood system of \( x \) in the topological space \( (***) \) Dipartimento di Matematica dell'Università, 73100 Lecce.

(*) Subject classification AMS 1980 54A10, 54D10.
(S,\tau) a basis of the filter \mathcal{F}(x).

If X \subset S, CX will be the complement of X in S while we shall denote by CYX or Y\setminus X the complement of X in Y when X \subset Y \subset S.

The topological definitions we need follow W.J. Thron's Topological structures [6] except that we shall denote by T_{2*4} the T_{2u} axiom.

Let R be a topological property which is preserved under expansions, i.e. such that (S,\tau') is R whenever (S,\tau) is and \tau' \supseteq \tau.

Definition 1. We shall call a topological space (S,\tau) quasi-R if there exists a subset X \subset S which determines a R simple extension (S,\tau(X)).

Of course an expansion of a quasi-R topology on a set S is a quasi-R topology on S too, because \tau(X) \subset \tau'(X) if \tau \subset \tau' are topologies on S and X \subset S.

Now we are going to investigate the existence and some properties of quasi-T_{i} spaces, i = 0,1,2,2_{4}, providing examples and stating relations with separation axioms T_{0},T_{1},T_{2},T_{2_{4}}.

The following examples show the existence of non-quasi-T_{0} and of quasi-T_{1} spaces.

Example 1. The indiscrete topology \omega on a set S with three or more points is not quasi-T_{0}; trivially \omega(X) = \{\emptyset, X, S\} fails to be T_{0} if X \subset S.

Example 2. Let S = \{x,x',y,y'\} and \tau = \{\emptyset,\{x,x'\},\{y,y'\},S\}. (S,\tau) is not a T_{0} space but \tau' = \tau(\{x,y\}) is a T_{0} topology on S, so (S,\tau') is quasi-T_{0}.
Furthermore $\tau'(\{x',y'\})$ is the discrete topology on $S$ and then $(S,\tau')$ is quasi-$T_i$, $i = 1,2,2\frac{1}{4}$.

**Proposition 1.** If $(S,\tau)$ is quasi-$T_i$ then $(S,\tau)$ is $T_0$.

**Proof.** Let $\tau' = \tau(X)$ be a $T_i$ simple extension of $\tau$; trivially $\{A \cap X/A \in \tau(a)\}$ is a fundamental neighbourhood system of a point $a \in X$ while $\tau(b)$ is a fundamental neighbourhood system of a point $b \not\in X$.

Now let $x \neq y$ be two points of $S$; if $x \not\in X$, an open neighbourhood $U \in \tau(x)$ exists such that $y \not\in U$; if $x, y \in X$, $A \cap X \in \tau'(x)$ and $y \not\in A \cap X$, then $A \in \tau(x)$ and $y \not\in A$.

While every quasi-$T_i$ space, $i = 1,2,2\frac{1}{4}$, is $T_0$, example 2 shows that a quasi-$T_i$ space is not necessarily $T_1$ even for $i > 0$.

**Definition 2.** Let $i,j = 0,1,2,2\frac{1}{4}$. We say that $(S,\tau)$ is $QT_i$ if it is quasi-$T_i$ and it is $T_j$ if $j < i$.

Trivially $(S,\tau)$ is $QT_0$ ($QT_1$) iff it is quasi-$T_0$ (quasi-$T_1$).

If we denote by $T_i$ ($QT_i$), $i = 0,1,2,2\frac{1}{4}$, the class of $T_i$ ($QT_i$) to pological spaces, then the following proper inclusions hold

$$QT_0 \supset T_0 \supset QT_1 \supset T_1 \supset QT_2 \supset T_2 \supset QT_{2\frac{1}{4}} \supset T_{2\frac{1}{4}}.$$ 

Example 2 proves the existence of non-$T_0$ $QT_0$ spaces and non-$T_1$ $QT_1$ spaces. Further examples provide us the truth of other strict inclusions.

**Example 3.** If $S = \{a,b,c\}$ and $\tau = \{\emptyset,\{a\},\{a,b\},S\}$, then $(S,\tau)$ is a $T_0$ but not a $QT_1$ space.
Example 4. We remark that the particular point topology and the excluded point topology on a set \( S \) belong to \( QT_1 \) but not to \( T_1 \).

In the first case we obtain a \( T_1 \) simple extension by the complement of the particular point; in the second case we obtain the same result by the set containing the only excluded point.

Nevertheless non-\( T_1 \), \( QT_1 \) spaces having non-discrete \( T_1 \) simple extensions exist: if fact let us consider an infinite set \( S \), take two distinct points \( a \neq a' \) in \( S \) and call \( \bar{A} \subseteq S \) open iff \( CA \) if finite and \( \bar{a} \in A \Rightarrow \bar{a}' \in A \); the simple extension of such a topology on \( S \) by the subset \( C(a') \) is then the cofinite topology on \( S \).

Example 5. Let \( \tau \) be the cofinite topology on the infinite set \( S \). We shall prove that \( (S, \tau) \) is a non-\( QT_2 \) \( T_1 \) space.

If \( X \subseteq S \), \( X \notin \tau \) (i.e. \( S-X \) is finite) and \( \tau' = \tau(X) \) let us take two distinct points \( x \neq y \) in \( S-X \) and two open neighbourhoods \( U \in \tau'(x), U = A \cup (B \cap X) \) and \( V \in \tau'(y), V = A' \cup (B' \cap X) \); we have \( x \in A \in \tau \) and \( y \in A' \in \tau \) so \( U \cap V \supseteq A \cap A' \neq \emptyset \) and \( \tau(X) \) is not a \( T_2 \) topology.

Example 6. (Modified Fort Space, see [4]). Let \( S \) be the union of an infinite set \( N \) with a set having only two distinct points \( x \neq y \) not belonging to \( N \); consider the topology \( \tau \) on \( S \) whose open sets are the cofinite sets in \( S \) and the subsets of \( N \).

Trivially \( (S, \tau) \) is a \( T_1 \) but not a \( T_2 \) spaces; furthermore we see that \( \tau(\{x,y\}) \) is the discrete topology and consequently \( (S, \tau) \) belong to \( QT_2 \).

Example 7. (Relatively Prime integer topology, see [4]). Let \( S = \mathbb{Z} \) be the set of positive integers and, if \( a, b \in S, (a, b) = 1 \) let us
consider \( U_\alpha(b)=\{b+na \in S/n \in \mathbb{Z}\} \); the family \( \mathcal{G} = \{U_\alpha(b)/a, b \in S, (a,b)=1\} \) is a basis of a topology \( \tau \) on \( S \).

It is well-know that \((S, \tau)\) is a \( T_2 \) but not a \( T_{2\frac{1}{2}} \) space; we shall now prove that \( \tau \) is not a \( QT_{2\frac{1}{2}} \) topology on \( S \).

Let \( \tau' = \tau(X) \) be a simple extension of \( \tau \). We recall that the family \( \{U_p(x) \cap X/(p,x)=1, p \in \mathbb{Z}^+\} \) is a fundamental neighbourhood system of \( x \in X \) in \( \tau' \) as well as \( \{U_p(y)/(p,y) = 1, p \in \mathbb{Z}^+\} \) is a fundamental neighbourhood system of \( y \in S-X \) in \( \tau' \).

First we suppose that for each \( x \in X \) a positive integer \( h \) exists such that \( h x \in S-X \), and we consider two distinct points \( x \neq y \) in \( S-X \) and two fundamental neighbourhoods \( U_p(x), U_p(y) \) of \( x, y \) respectively. Then we have

\[
\text{cl}_{\tau'}(U_p(x)) = \text{cl}(U_p(x)) \cap (C_x U \text{cl}(U_p(x) \cap X))
\]

\[
\text{cl}_{\tau'}(U_q(y)) = \text{cl}(U_q(y)) \cap (C_y U \text{cl}(U_q(y) \cap X)).
\]

As the closure in \( \tau \) of \( U_p(x) \) (of \( U_q(y) \)) contains all multiples of \( p \) (of \( q \)) in \( S \) and \( C_x \) contains at least one multiple of \([p,q]\), we have \( \text{cl}_{\tau'} U_p(x) \cap \text{cl}_{\tau'} U_q(y) \neq \emptyset \) and consequently \( \tau(X) \) is not a \( T_2 \) topology on \( S \).

Now we suppose that each positive integer multiple of \( x \) is in \( X \) for some \( x \) in \( X \). We consider such an \( x \) and we observe that if \( k \in \mathbb{Z}^+, p \in \mathbb{Z}^+ \) and \((p,kx)=1\), then

\[
U_p(kx) \cap X \supseteq \{(k-mp)x \in S/m \in \mathbb{Z}\} = A_{pk}.
\]
It is easily seen that \( \{ ptx/te^Z^+ \} \in \text{cl} A_{pk} \); in fact given any integer a prime with ptx we have

\[
U_a(ptx) \cap A_{pk} \neq \emptyset \iff \exists n, m \in Z \text{ such that } an + pxm + (pt-k)x = 0.
\]

This is surely true since a and px are relatively prime. If we take \( h \neq k \) in \( Z^+ \) and \( p, q \) in \( Z^+ \) prime with \( kx, hx \) respectively then the closures

\[
\text{cl}_{\tau'}(U_p(kx) \cap X) = \text{cl}(U_p(kx) \cap X) \supseteq A_{pk}U\{ptx \in S/te^Z\}
\]

\[
\text{cl}_{\tau'}(U_q(hx) \cap X) = \text{cl}(U_q(hx) \cap X) \supseteq A_{qh}U\{qx \in S/te^Z\}
\]

have a non-empty intersection and \( \tau(X) \) is not \( T_{2\frac{1}{2}} \) in this case too.

We remark that also the coarser prime integer topology (see [4] again) is a \( T_2 \) but not a \( QT_{2\frac{1}{2}} \) topology.

**Example 8.** The Double Origin Topology, the Simplified Arens Square and the Minimal Hausdorff Topology considered in [4] provide examples of \( QT_{2\frac{1}{2}} \) spaces. We give a detailed description of the third case only.

Let us consider the topological product of \( A=\{1, 2, 3, \ldots, \omega, \ldots, -3, -2, -1\} \) linearly ordered with the interval topology and the discrete topological space \( (Z^+, \alpha) \); let \( (S, \tau) \) be obtained from such topological product by adding two ideal points \( a \) and \( -a \) whose fundamental neighbourhoods are of the kind

\[
\{a\} \cup \{(i, j)/i < \omega, j > n\} \text{ and } \{-a\} \cup \{(i, j)/i > \omega, j > m\}
\]

respectively.

\((S, \tau)\) is a minimal Hausdorff non-\( T_{2\frac{1}{2}} \) space while the simple exten-
sion of \( \tau \) by \( X = \{a\} \cup \{(i,j)/ i \leq \omega, j > 0\} \) is a \( T_{\frac{1}{2}} \) topology on \( S \).

In order to give a characterization of \( QT_1 \) topological spaces we shall denote by \( T \) the set of those points which are in the closure of some other distinct point in the space \((S, \tau)\) i.e.

\[
T = \bigcup_{x \in S} ((\text{cl}(x)) - \{x\}).
\]

Trivially \((S, \tau) \in T_1 \) iff \( T = \emptyset \).

**Lemma 1.** \( X \subseteq S, (S, \tau(X)) \in T_1 \implies T \subseteq X. \)

**Proof.** If there were two distinct points \( x \neq y \) such that \( y \in \text{cl}(x) \) and \( y \not\in X \), then \( \tau(y) \) would be a fundamental neighbourhood system of \( y \) in \( \tau' = \tau(X) \) and it would follow from \( y \in \text{cl}(x) \) that every neighbourhood in \( \tau'(y) \) contains \( x \).

**Proposition 2.** \((S, \tau) \in QT_1 \iff (T, \tau|_T) \in T_1 \).

**Proof.** \( \implies \) Let \( X \) be a subset of \( S \) such that \((S, \tau(X)) \in T_1 \); it follows from Lemma 1 that \( T \subseteq X \) and consequently \( \tau|_T = \tau(X)|_T \); so \((X, \tau|_T) \in T_1 \).

\( \Longleftarrow \) It is easily seen that \((S, \tau') \in T_1 \), if \( \tau' = \tau(T) \).

First we consider \( a, b \in T \), \( a \neq b \); as each open set in \( \tau|_T \) belongs to \( \tau' \) and \( \tau|_T \) is a \( T_1 \) topology, then \( U \in \tau'(a) \) and \( V \in \tau'(b) \) exist such that \( b \not\in U \) and \( a \notin V \).

If \( a \in T \) and \( b \in S - T \), we have \( b \notin (\text{cl}(y)) - \{y\} \) for each \( y \in S \);
in particular \( b \notin (\text{cl}\{a\}) - \{a\} \) and, of course, \( b \notin \text{cl}\{a\}; U \in \tau(b) \subset \tau'(b) \) exists such that \( a \notin U \). On the other hand we can consider \( A \in \tau(a) \) and the open set \((A \cap T) \in \tau'\) is an open neighbourhood of \( a \) in \( \tau' \) which does not contain \( b \).

The case \( a, b \in S - T, \ a \neq b \), is trivial.

**Remarks.** If the set \( T \) defined above is a non-trivial open set in \((S, \tau)\), then \((S, \tau)\) cannot be a \( QT_1 \) space.

The set \( T \) in the spaces of example 4 is closed. This is not true for all spaces. For instance let us consider an infinite set \( S \) and fix \( a \in S \); let the open sets in \( \tau \) be the cofinite subsets of \( S \) containing \( a \). \((S, \tau)\) is a \( T_0 \) space and its subspace \( T = S - \{a\} \) is a \( T_1 \) space; \((S, \tau)\) is then a \( QT_1 \) space and \( T = S - \{a\} \) is not a closed subset.

**Proposition 3.** If \( \{S, \tau\} \) is a \( QT_0 \) regular space which has a \( T_0 \) regular simple extension, then \((S, \tau)\) is a \( T_2 \) space.

**Proof.** Let \((S, \tau(X))\) be a \( T_0 \) regular simple extension; then \((S, \tau(X))\) is a \( T_2 \) space and consequently \((S, \tau) \in QT_2\). If follows from proposition 1 that \((S, \tau)\) must be \( T_0 \) and the assertion is true.

**Remark.** The existence of a \( T_0 \) regular simple extension is an essential hypothesis in proposition 3; the space \( S = \{a, b\} \) with the indiscrete topology is a \( QT_0 \) regular non-\( T_2 \) space.

2. We shall denote by \( \mathcal{L}(S) \) the complete lattice of topologies on a given set \( S \) under set inclusion. The least element is the indiscrete topology and the greatest element is the discrete topology.

In agreement with [2] we shall call antiatom of \( \mathcal{L}(S) \) any topo-
logy on \( S \) which is coarser than itself and the discrete topology only.

It is well known that the antiatoms are the ultratopologies already described in [2].

**Definition 3.** If \( R \) is a topological property which is preserved under expansions, we shall say that \((S, \tau)\) is a maximal non-\( R \) (MNR) space if it is not a \( R \) space but every proper expansion of \( \tau \) is a \( R \) topology.

**Remarks.** \((S, \tau)\) is MNR iff \( \tau \) is not \( R \) but \( \tau(X) \) is \( R \) for each \( X \not\in \tau \). Furthermore it is obvious that a MNR space is a quasi-\( R \) space. Finally we remark that the spaces given in examples 2, 4, 6, 8 are QT \(_i\) but are not MNT \(_i\) spaces, \( i = 0, 1, 2, 2 \frac{1}{2} \) respectively.

MNT \(_1\) spaces, \( i = 0, 1, 2, 3 \), were considered by Thomas in [5]; here we shall give more detailed characterizations and properties of MNT \(_1\) and MNT \(_2\) spaces and a proof that no \( T_2 \) MNT \(_{2 \frac{1}{2}}\) space exists.

If follows trivially from proposition 1 that every MNT \(_1\), \( i > 0 \), space is a \( T_0 \) space. Furthermore the following results are easily seen to be true and can be found in [2] and [5].

Let \((S, \tau)\) be a non-\( T_1 \) topological space. Then \((S, \tau)\) is MNT \(_1\) if
\[(S, \tau)\] is an antiatom in \( \mathcal{P}(S) \).

If \((S, \tau)\) is MNT \(_1\) then \((S, \tau)\) is MNT \(_2\), MNT \(_{2 \frac{1}{2}}\), MNT \(_3\).

Every regular antiatom in \( \mathcal{P}(S) \) is a \( T_1 \) space.

Thomas proved in [5] that every non-\( T_0 \) (non-\( T_1 \), non-\( T_2 \)) topology is coarser than a MNT \(_0\) (MNT \(_1\), MNT \(_2\)) topology; moreover each \( T_1 \) non-\( T_2 \) topology is included in a MNT \(_2\) topology which is a MNT \(_3\) topology too.
We shall prove a characterization of \( T_1 \) \( MNT_2 \) spaces and obtain as an immediate consequence (example 9) the construction of a \( T_1 \) \( MNT_2 \) space already given by Thomas in [5] theorem 1.

Next we realize that all \( T_1 \) \( MNT_2 \) topologies are the ones we describe in example 9.

**PROPOSITION 4.** Let \( (S, \tau) \) be a \( T_1 \) space. \( (S, \tau) \) is \( MNT_2 \) iff the following conditions hold

1) Only two distinct points \( x \neq y \) exist in \( S \) such that each neighbourhood of \( x \) intersect each neighbourhood of \( y \).

2) The subspace \( S - \{x, y\} \) has the discrete topology.

3) \( X \subseteq S, X \notin \tau, x \in X (y \in X) \Rightarrow CX \cup \{y\} \in \tau \). (\( CX \cup \{x\} \in \tau \)).

**Proof.** The given conditions are necessary.

Trivially distinct points \( x \neq y \) exist which verify the first condition; if there were another point \( z, z \neq x \) and \( z \neq y \), such that the neighbourhoods of \( z \) and those of \( y \) intersect each other, then \( y \) and \( z \) would be non-separated points in \( \tau(\{x\}) \), which contradicts the hypothesis since \( \{x\} \) is a non-open set.

Every point distinct from \( x \) and \( y \) must be open; otherwise \( x \) and \( y \) should be non-separated points in \( \tau(\{a\}) \), if \( \{a\} \) were a non-open subset of \( S \).

In order to verify the third condition, we first prove that if \( X \subseteq S \) and \( X \notin \tau \) then \( CX \in \tau \).

Indeed let \( x \subseteq S \) be neither open nor closed so that \( \tau(X) \) and \( \tau(CX) \) are \( T_2 \) topologies; by 1) and 2) \( X \) contains only one (say \( x \)) of the points \( x, y \) and consequently we can find \( A \in \tau(x), B \in \tau(y) \)
such that \((A \cap X) \cap B = \emptyset\) i.e. \(A \cap B \subseteq CX\). Then if \(U \in \tau(x)\), \(V \in \tau(y)\) we have \((U \cap V) \cap CX \neq \emptyset\); otherwise we should have \(U \cap V \subseteq X\) and the open neighbourhoods \((U \cap A) \in \tau(x)\), \((V \cap B) \in \tau(y)\) would separate \(x\) and \(y\) in \(\tau\). Eventually we obtain, from the assumption that \(X\) is neither open nor closed, that for every \(U \in \tau(x)\) and \(V \in \tau(y)\) one must have \(U \cap (V \cap CX) = (U \cap V) \cap CX \neq \emptyset\) which contradicts the Hausdorff character of the topology \(\tau(CX)\).

Finally we suppose that \(x \not\in S, x \not\in \tau, x \in X\). If \(y \not\in X\) be condition 3) is trivial; if \(y \in X\) and \(CX \cup \{y\} \not\in \tau\) we contradict the hypothesis by finding two disjoint neighbourhoods of \(x\) in \(\tau\) in the following way: we consider \(U \in \tau(x)\) and \(V \in \tau(y)\) such that \(U \cap X\) and \(V\) separate \(x\) and \(y\) in the \(T_2\) topology \(\tau(X)\); then \((U \cap V) \cap X = \emptyset\); since \((X \setminus \{y\}) \in \tau(x)\) we have \(U' = (X \setminus \{y\}) \cap U \in \tau(x), V \in \tau(y)\) and \(U' \cap V = \emptyset\).

The conditions are sufficient.

By 1) \((S, \tau)\) cannot be a \(T_2\) space.

Now let \(X\) be a non-open subset of \(S\) containing \(x\); \(CX \cup \{y\}\) is an open neighbourhood of \(y\) in \(\tau\) which does not intersect the open neighbourhood \(X \setminus \{y\}\) of \(x\) in \(\tau(X)\); hence \(\tau(X)\) is a \(T_2\) topology.

Remark. The \(T_1\) axiom is only requested to show that conditions 1), 2), 3) in proposition 4 are sufficient to have a \(MNT_2\) space.

Example 9. Let \(S = \{x, y\} \cup N\) where \(x, y \not\in N\) and \(N\) is an infinite set. Furthermore let \(\phi\) be an ultrafilter on \(N\) such that \(\bigcap_{F \in \phi} F = \emptyset\).

Then \(T = \{A \subseteq S/A \cap N \neq \emptyset \implies (A \cap N \in \phi)\}\) is a \(T_1\) but not a \(T_2\) topology on \(S\). The conditions of proposition 4 are easily verified; hence \((S, \tau)\) is a \(MNT_2\) space.

Corollary 1. The spaces described in example 9 are the only
$T_1 MNT_2$ spaces.

Proof. Let $(S, τ)$ be a $T_1 MNT_2$ space and consider an ultrafilter $υ$ with two distinct limit points $x ≠ y$ in $S$; of course $x, y$ are the points of conditions 1), 2) in proposition 4.

Let $N$ be the complement of $(x, y)$ in $S$. It is easily proved by condition 3) that $τ(x) ∩ N = τ(y) ∩ N = υ ∩ N$ which complete the proof.

COROLLARY 2. Each $MNT_2$ space is a $MNT_{21}$ space too.

Proof. It follows trivially from [5] (theorem 1) and corollary 1.

COROLLARY 3. The bicompact subsets $^{(1)}$ of a $T_1 MNT_2$ space are closed.

Proof. Let $S = \{x, y\} \cup N$ be a $T_1 MNT_2$ space with the notations of example 9. If $A \subseteq S$ is an infinite subset of $S$, then $A ∩ N$ is an infinite subset too; we can consider $U = ϕ$ such that $A-U$ is infinite; the open cover $\{\{a \ /	ext{aeA-U}\} \cup \{Uu(x, y)\}\text{ of } A$ has no finite subcover and consequently $A$ is not compact. The compact subset of $S$ are therefore the finite subsets of $S$ which are closed of course.

Remark. The existence in $(S, τ)$ of only two distinct non-separated points is not a sufficient condition in order to prove that all bicompact subsets of $S$ are closed. For instance if we consider the space of example 6 we can see that $P = \{x\} \cup N$ is a bicompact non-closed subset of $S$.

$^{(1)}$ Here bicompact means that each open cover has a finite subcover.
No $T_1$ MNT$_2$ space exists which is a bicompact space.

We are now going to verify that each $T_2$ non-$T_{2rac{1}{2}}$ topology on a set $S$ has a non-$T_{2rac{1}{2}}$ simple extension i.e. that there exists no MNT$_{2rac{1}{2}}$ space which is a $T_2$ space too.

Let $(S, \tau)$ be a $T_2$ non-$T_{2rac{1}{2}}$ space and let $x \neq y$ be two distinct points of $S$ such that $clU \cap clV \neq \emptyset \quad \forall U \in \tau(x), V \in \tau(y)$. Then we have the following

**Lemma 2.** The family $\mathcal{B} = \{P \subseteq S/ \exists U \in \tau(x), V \in \tau(y) : U \cap V = \emptyset$ and $clU \cap clV = P\}$ is a basis of a filter $\phi$ on $S$ that has no cluster point.

**Proof.** Trivially $\emptyset \notin \mathcal{B}$; if $U, U' \in \tau(x), V, V' \in \tau(y)$ and $clU \cap clV = P \in \mathcal{B}$, $clU' \cap clV' = P' \in \mathcal{B}$, then $U'' = U \cap U' \in \tau(x)$, $V'' = V \cap V' \in \tau(y)$ and $P'' = clU'' \cap clV'' \subseteq P \cap P'$ is an element of $\mathcal{B}$.

Now let us call $\phi$ the filter which has $\mathcal{B}$ as a basis.

If $z$ were a cluster point of $\phi$, $z \neq x$, then $z$ would belong to the closure of every neighbourhood of $x$, which contradicts the assumption that $(S, \tau)$ is a $T_2$ space; the same argument shows that there exists no cluster point distinct from $y$.

**Remark.** Since $\phi$ is a closed filter, we must have $\bigcap_{F \in \phi} F = \emptyset$ from lemma 1 and consequently every element of $\phi$ has more than one point.

Now let us consider $M \in \tau(x)$ such that the family

$$\mathcal{V} = \{V \in \tau(y)/M \cap V = \emptyset\}$$
is non-empty.

Trivially $C(c1M)$ belongs to $\gamma'$ and then it is easily seen that

$$\bigcup_{V \in \gamma'} c1M \cap c1V = c1M \text{-} \text{int}(c1M) = P \in \phi$$

**PROPOSITION 5.** With the notations given above and assuming that $(S, \tau)$ is a $T_2$ non-$T_{2\frac{1}{2}}$ space, then $(S, \tau)$ has a non-$T_{2\frac{1}{2}}$ simple extension.

**Proof.** We shall denote again by $x$ and $y$ two distinct points such that $c1U \cap c1V \neq \emptyset \forall U \in \tau(x), V \in \tau(y)$.

First we see that the subset $P \in \phi$ already considered above is not open in $\tau$. Indeed if $P$ were an open subset in $\tau$, then we should have $\text{int}(c1M) \cup P \in \tau(x)$ and, since $\text{int}(c1M) \cup P = c1M$, $c1M$ and $N = C(c1M)$ would be clopen neighbourhoods of $x, y$ respectively, which is an absurd.

Now we consider a non-open subset $\{z\} \not\subset P$ and we prove that $\tau' = \tau(\{z\})$ is not a $T_{2\frac{1}{2}}$ topology.

In fact note that $z \notin \{x, y\}$, recall $\tau(x), \tau(y)$ are fundamental neighbourhood systems of $x, y$ in $\tau'$ and consider $A \in \tau(x), B \in \tau(y)$; since the elements of $\phi$ have more than one point we have

$$\text{cl}_{\tau'} A \cap \text{cl}_{\tau'} B = \text{cl}_{\tau'}(A \cap \{z\} \cup \text{cl}(A \cap \{z\})) \cap \text{cl}_{\tau'}(B \cap \{z\}) \supset \supset (\text{cl1A} \cap \text{cl1B}) - \{z\} \neq \emptyset.$$ 

It is proved in [5] that the $MNT_2$ spaces are precisely the $MNT_3$ spaces. A similar result for $MNT_2$ and $MNT_{2\frac{1}{2}}$ spaces follows trivially from proposition 5 and corollary 2.
COROLLARY 4. The MNT\textsubscript{2} spaces are precisely the MNT\textsubscript{2\frac{1}{4}} spaces.

REFERENCES


