Congruences for (2, 3)-regular partition with designated summands

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Abstract. Let $P_{D_{2,3}}(n)$ count the number of partitions of $n$ with designated summands in which parts are not multiples of 2 or 3. In this work, we establish congruences modulo powers of 2 and 3 for $P_{D_{2,3}}(n)$. For example, for each $n \geq 0$ and $\alpha \geq 0$,

\[ P_{D_{2,3}}(6 \cdot 4^\alpha + 2n + 5 \cdot 4^{\alpha+1}) \equiv 0 \pmod{2^4} \]
\[ P_{D_{2,3}}(4 \cdot 3^{\alpha+3}n + 10 \cdot 3^{\alpha+2}) \equiv 0 \pmod{3} \]

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1 Introduction

A partition of a positive integer $n$ is a non-increasing sequence of positive integers whose sum is $n$. A partition is (2, 3)-regular partition of $n$ if none of the parts are divisible by 2 or 3.

Andrews, Lewis and Lovejoy [1] have investigated a new class of partition with designated summands are constructed by taking ordinary partitions and tagging exactly one of each part size. The total number of partitions of $n$ with designated summands is denoted by $PD(n)$. Hence $PD(4) = 10$, namely

$4', 3'+1', 2'+2, 2+2', 2'+1'+1, 2'+1+1', 1'+1+1+1, 1+1'+1+1, 1+1+1+1+1, 1+1+1+1+1$.

Andrews et al. [1] have derived the following generating function of $PD(n)$, namely

\[ \sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1f_2f_3}, \]

\[ f_1 = (2q, q^2, q^4, \ldots), \quad f_2 = (q, q^2, \ldots), \quad f_3 = (q, q^4, \ldots) \]

\[ f_6 = f_2^3 \]

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where

\[ f_n := \prod_{j=1}^{\infty} (1 - q^{nj}), n \geq 1. \]

(2)

Andrews et al. [1] and N. D. Baruah and K. K. Ojah [3] have also studied PDO \((n)\), the number of partitions of \(n\) with designated summands in which all parts are odd. The generating function of PDO \((n)\) is given by

\[ \sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}. \]

(3)

Mahadeva Naika et al. [12] have studied PD\(_3\) \((n)\), the number of partitions of \(n\) with designated summands whose parts not divisible by 3 and the generating function is given by

\[ \sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_2^2 f_9}{f_1 f_2 f_{18}}. \]

(4)

In [13] Mahadeva Naika et al. have established many congruences for PD\(_2\) \((n)\), the number of bipartitions of \(n\) with designated summands and the generating function is given by

\[ \sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_2^2 f_1^4 f_2^2 f_{18}}{f_1 f_2 f_3^2}. \]

(5)

Motivated by the above works, in this paper, we defined PD\(_{2,3}\) \((n)\), the number of partitions of \(n\) with designated summands in which parts are not multiples of 2 or 3. For example PD\(_{2,3}\) \((4)\) = 4, namely

\[ 1' + 1 + 1 + 1, \quad 1 + 1' + 1 + 1, \quad 1 + 1 + 1' + 1, \quad 1 + 1 + 1 + 1'. \]

The generating function of PD\(_{2,3}\) \((n)\) is given by

\[ \sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12} f_{18}}. \]

(6)

Following Ramanujan, for \(|ab| < 1\), we define his general theta function \(f(a,b)\) as

\[ f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} p^{n(n-1)/2}. \]

(7)

The important special cases of \(f(a,b)\) are

\[ \varphi(q) := f(q,q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \frac{1}{f_1 f_4}. \]

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\[ \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1} \]  
(9)
and

\[ f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1, \]  
(10)

where the product representations arise from famous Jacobi’s triple product identity [5, p. 35, Entry 19]

\[ f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \]  
(11)

In this paper, we list few formulas which helps to prove our main results in section 2. In section 3, we obtain several congruences modulo powers of 2 and congruences modulo 3 in section 4.

## 2 Preliminary results

We list few dissection formulas to prove our main results.

**Lemma 1.** [14, p. 212] We have the following 5-dissection

\[ f_1 = f_{25} (a(q^5) - q - q^2/a(q^5)), \]  
(12)

where

\[ a := a(q) := (q^2, q^3, q^5)_\infty \frac{(q, q^4, q^5)_\infty}{(q, q^4, q^5)_\infty}. \]  
(13)

**Lemma 2.** The following 2-dissection holds:

\[ \frac{f_9}{f_1} = \frac{f_{18} f_2^3}{f_{36} f_6 f_2^2} + q \frac{f_{36} f_6 f_2^2}{f_{12} f_2^3}. \]  
(14)

Identity (2) is nothing but Lemma 3.5 in [16].

**Lemma 3.** The following 3-dissection holds:

\[ \frac{f_4}{f_1} = \frac{f_{12} f_4^4}{f_3^2 f_9^2 f_6} + q \frac{f_2^2 f_3 f_9}{f_3^2 f_9^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^2}. \]  
(15)

Identity (3) is nothing but Lemma 2.6 in [3].

**Lemma 4.** The following 3-dissection holds:

\[ \frac{f_2}{f_1} = \frac{f_4 f_6^4}{f_3^2 f_9^2} + 2q^2 \frac{f_2 f_3}{f_3^2} + 4q^2 \frac{f_6 f_{18} f_9}{f_3^2}. \]  
(16)
Equation (16) was proved by Hirschhorn and Sellers [10].

**Lemma 5.** [5, p. 49] We have

\[
\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}), \tag{17}
\]
\[
\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{18}
\]

**Lemma 6.** The following 2-dissections holds:

\[
f_1^2 = \frac{f_2f_8^5}{f_2^2f_16^2} - 2q \frac{f_2f_{16}^2}{f_8^2}, \tag{19}
\]
\[
\frac{1}{f_1^2} = \frac{f_8^3}{f_2^2f_{16}^2} + 2q \frac{f_4f_{16}^2}{f_8^2f_2^2}, \tag{20}
\]
\[
f_4^1 = \frac{f_{10}^4}{f_2^2f_8^4} - 4q \frac{f_2f_1^4}{f_2^4}, \tag{21}
\]
\[
\frac{1}{f_4^1} = \frac{f_{14}^4}{f_2^2f_8^4} + 4q \frac{f_2f_1^4}{f_2^4}. \tag{22}
\]

Lemma (6) is a consequence of dissection formulas of Ramanujan, collected in Berndt’s book [5, p. 40, Entry 25].

**Lemma 7.** The following 2-dissection holds:

\[
\frac{f_3}{f_1} = \frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}} + q \frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}. \tag{23}
\]

Xia and Yao [18] gave a proof of Lemma (7).

**Lemma 8.** The following 2-dissections holds:

\[
\frac{f_3^2}{f_1^2} = \frac{f_4f_6f_{12}}{f_2^2f_8f_{24}} + 2q \frac{f_4f_8^2f_{24}}{f_2^2f_{12}^2}, \tag{24}
\]
\[
\frac{f_3^2}{f_2^2} = \frac{f_2f_4f_{12}}{f_6^2f_8f_{24}} - 2q \frac{f_2f_8f_{12}f_{24}}{f_4f_6^2}. \tag{25}
\]

Xia and Yao [17] proved (24) by employing an addition formula for theta functions. Replacing \( q \) by \( -q \) in (20) and then using the fact that \(( -q; q)_{\infty} = \frac{f_3}{f_1f_4}\), we obtain (25).
Lemma 9. The following 2-dissections holds:
\[
\begin{align*}
\frac{f_3^3}{f_1} &= \frac{f_3^4 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}, \\
\frac{f_3^3}{f_3} &= \frac{f_3^4}{f_{12}} - 3q \frac{f_3^4 f_{12}^3}{f_4 f_6^2}, \\
\frac{f_3}{f_1} &= \frac{f_3^6 f_6^3}{f_2^2 f_{12}^2} + 3q \frac{f_3^3 f_6 f_{12}^2}{f_4^2}.
\end{align*}
\]

(26)

Hirschhorn, Garvan and Borwein [8] proved (26) and (27). For proof of (28), see [4].

Lemma 10. The following 2-dissections holds:
\[
\begin{align*}
\frac{1}{f_1 f_3} &= \frac{f_2^6 f_{12}^3}{f_2^2 f_4 f_6 f_{24}} + q \frac{f_4^3 f_{24}^2}{f_2^2 f_6 f_8 f_{12}}, \\
f_1 f_3 &= \frac{f_2^6 f_{12}^3}{f_2^2 f_4 f_6 f_{24}} - q \frac{f_4^3 f_6 f_{24}^2}{f_2^2 f_8 f_{12}}.
\end{align*}
\]

(29)

(30)

Equation (29) was proved by Baruah and Ojah [3]. Replacing \( q \) by \(-q\) in (29) and using the fact that \((-q; -q)_\infty = \frac{f_2^3}{f_1 f_4}\), we get (30).

Lemma 11. The following 3-dissection holds:
\[
f_1 f_2 = \frac{f_6 f_4^3}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 f_3 f_{18}^4 - \frac{f_6 f_4^2}{f_3 f_{18}}.
\]

(31)

One can see this identity in [9].

Lemma 12. (Cui and Gu [7, Theorem 2.2]). For any prime \( p \geq 5 \),
\[
f_1 = \sum_{k=\frac{1-p}{2} \atop k \neq \frac{p-1}{6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2 + k}{2}} f \left( -q^{\frac{3p^2 + (6k+1)p}{2}}, q^{\frac{-3p^2 - (6k+1)p}{2}} \right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2 - 1}{24}} f p^2,
\]

(32)

where
\[
\frac{\pm p-1}{6} := \begin{cases} 
\frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}, \\
\frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}.
\end{cases}
\]

Furthermore, for \( \frac{-(p-1)}{2} \leq k \leq \frac{p-1}{2} \) and \( k \neq \frac{\pm p-1}{6} \),
\[
\frac{3k^2 + k}{2} \neq \frac{p^2 - 1}{24} \pmod{p}.
\]
3 Congruences Modulo Powers of 2.

Theorem 1. For $n \geq 1$ and $\alpha \geq 0$, then

$$PD_{2,3}(18n) \equiv 0 \pmod{4},$$

$$PD_{2,3}(2 \cdot 3^{\alpha+3}n) \equiv 0 \pmod{4}.$$  

Proof. We have

$$\sum_{n=0}^\infty PD_{2,3}(n)q^n = \frac{f_4 f_6 f_9 f_{36}}{f_1 f_{12} f_{18}}.$$  

(35)

Substituting (14) into (35), we obtain

$$\sum_{n=0}^\infty PD_{2,3}(n)q^n = \frac{f_4 f_6 f_{12}}{f_2 f_{18}} + q \frac{f_4^3 f_6^3 f_{36}}{f_2^3 f_{12} f_{18}}.$$  

(36)

Extracting the even terms in the above equation

$$\sum_{n=0}^\infty PD_{2,3}(2n)q^n = \frac{f_2 f_3 f_6}{f_1 f_9}.$$  

(37)

Substituting (16) into (37), we find

$$\sum_{n=0}^\infty PD_{2,3}(6n)q^n = \frac{f_2^5 f_3^5}{f_1 f_9^3}.$$  

(39)

By the binomial theorem, it is easy to see that for positive integers $k$ and $m$,

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod{4}$$  

(40)

and

$$f_{2k}^{4m} \equiv f_k^{8m} \pmod{8}.$$  

(41)

Invoking (41) into (39), we find that

$$\sum_{n=0}^\infty PD_{2,3}(6n)q^n = \frac{f_1 f_2 f_3^5}{f_6^3} \pmod{8}.$$  

(42)
Employing (31) into (42), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n)q^n = \frac{f_3^4 f_6^4}{f_2^6 f_{18}^2} - q \frac{f_3^5 f_9 f_{18}}{f_2^4} - 2q^2 \frac{f_3^6 f_9^2}{f_2^6 f_{18}^2} \quad (\text{mod 8}). \] (43)

Extracting the terms involving \(q^{3n}\) from both sides of (43) and replacing \(q^3\) by \(q\), we have
\[ \sum_{n=0}^{\infty} PD_{2,3}(18n)q^n = \frac{f_3^4 f_6^4}{f_2^6 f_6^2} \quad (\text{mod 8}). \] (44)

Congruence (33) follow from (40) and (44).

Equation (44) can be rewritten as
\[ \sum_{n=0}^{\infty} PD_{2,3}(18n)q^n = \frac{f_3^4}{f_6^2} \left( \frac{f_6^2}{f_2} \right)^2 \quad (\text{mod 8}). \] (45)

Replacing \(q\) by \(-q\) in (17) and using the fact that \(\phi(-q) = \frac{f_2^4}{f_2}\), we find that
\[ \frac{f_6^2}{f_2} = \frac{f_6^2}{f_{18}} - 2q^2 \frac{f_3^6 f_9 f_{18}}{f_6 f_{18}^2}. \] (46)

Employing (47) into (45), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(18n)q^n = \frac{f_3^4 f_6^4}{f_2^6 f_{18}^2} + 4q^2 \frac{f_3^6 f_9^4}{f_2^4 f_6^2} - 4q \frac{f_3^5 f_9 f_{18}}{f_2^6 f_{18}^2} \quad (\text{mod 8}). \] (48)

Extracting the terms involving \(q^{3n}\) from both sides of (48) and replacing \(q^3\) by \(q\), we obtain
\[ \sum_{n=0}^{\infty} PD_{2,3}(54n)q^n = \frac{f_3^4 f_6^4}{f_2^6 f_6^2} \quad (\text{mod 8}). \] (49)

In view of the congruences (44) and (49), we get
\[ PD_{2,3}(54n) \equiv PD_{2,3}(18n) \quad (\text{mod 8}). \] (50)

Utilizing (50) and by mathematical induction on \(\alpha\), we arrive
\[ PD_{2,3}(2 \cdot 3^{\alpha+3}n) \equiv PD_{2,3}(18n) \quad (\text{mod 8}). \] (51)

Using (33) into (51), we get (34).
Theorem 2. For $n \geq 0$ and $\alpha \geq 0$, we have

\begin{align*}
PD_{2,3}(72n + 42) &\equiv 0 \pmod{4}, \\
PD_{2,3}(36n + 30) &\equiv 0 \pmod{4}, \\
PD_{2,3}(144n + 120) &\equiv 0 \pmod{4}, \\
PD_{2,3}(9 \cdot 4^{\alpha+3}n + 30 \cdot 4^{\alpha+2}) &\equiv 0 \pmod{4}, \\
PD_{2,3}(54n + 18) &\equiv 4 \cdot PD_{2,3}(18n + 6) \pmod{8}, \\
PD_{2,3}(54n + 36) &\equiv 2 \cdot PD_{2,3}(18n + 12) \pmod{8}, \\
PD_{2,3}(36n + 30) &\equiv 2 \cdot PD_{2,3}(72n + 60) \pmod{8}.
\end{align*}

Proof. Extracting the terms involving $q^{3n+1}$ from (48), dividing by $q$ and then replacing $q^3$ by $q$, we have

\[\sum_{n=0}^{\infty} PD_{2,3}(54n + 18)q^n \equiv -4 \frac{f_1^5 f_3 f_6}{f_2^3} \pmod{8}.\]  

(59)

Extracting the terms involving $q^{3n+1}$ from (43), dividing by $q$ and then replacing $q^3$ by $q$, we obtain

\[\sum_{n=0}^{\infty} PD_{2,3}(18n + 6)q^n \equiv -\frac{f_1^5 f_3 f_6}{f_2^3} \pmod{8}.\]  

(60)

From (59) and (60), we arrive at (56).

Extracting the terms involving $q^{3n+2}$ from (48), dividing by $q^2$ and then replacing $q^3$ by $q$, we find

\[\sum_{n=0}^{\infty} PD_{2,3}(54n + 36)q^n \equiv 4 \frac{f_1^6 f_4^4}{f_2^4 f_3^4} \pmod{8}.\]  

(61)

Extracting the terms involving $q^{3n+2}$ from (43), dividing by $q^2$ and then replacing $q^3$ by $q$, we obtain

\[\sum_{n=0}^{\infty} PD_{2,3}(18n + 12)q^n \equiv -2 \frac{f_1^6 f_4^4}{f_2^4 f_3^4} \pmod{8}.\]  

(62)

In view of the congruences (61) and (62), we get (57).

From (60), we have

\[\sum_{n=0}^{\infty} PD_{2,3}(18n + 6)q^n \equiv 7 \frac{f_1^5 f_3 f_6}{f_2^3} \pmod{8}.\]  

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Invoking (41) into (63), we get

\[
\sum_{n=0}^{\infty} PD_{2,3}(18n + 6)q^n \equiv \frac{7f_2 f_3 f_6}{f_1} \pmod{8}. \tag{64}
\]

Employing (28) into (64), we obtain

\[
\sum_{n=0}^{\infty} PD_{2,3}(18n + 6)q^n \equiv \frac{7f_2 f_3 f_6}{f_1 f_4} + 21q \frac{f_2^2 f_3 f_6}{f_4^2} \pmod{8}. \tag{65}
\]

Extracting the terms involving \(q^{2n}\) from (65) and then replacing \(q^2\) by \(q\), we have

\[
\sum_{n=0}^{\infty} PD_{2,3}(36n + 6)q^n \equiv \frac{7f_2 f_3 f_6}{f_1 f_6} \pmod{8}. \tag{66}
\]

Invoking (40) into (66), we get

\[
\sum_{n=0}^{\infty} PD_{2,3}(36n + 6)q^n \equiv 3f_2^2 \pmod{4}. \tag{67}
\]

Extracting the terms involving \(q^{2n+1}\) from (67), we get (52).

Extracting the terms involving \(q^{2n+1}\) from (65), dividing by \(q\) and then replacing \(q^2\) by \(q\), we obtain

\[
\sum_{n=0}^{\infty} PD_{2,3}(36n + 24)q^n \equiv \frac{5f_2^2 f_3 f_6}{f_1^3} \pmod{8}. \tag{68}
\]

Invoking (41) into (68), we get

\[
\sum_{n=0}^{\infty} PD_{2,3}(36n + 24)q^n \equiv \frac{5f_2^2}{f_1^3} (f_1 f_3)^2 \pmod{8}. \tag{69}
\]

Employing (30) into (69), we obtain

\[
\sum_{n=0}^{\infty} PD_{2,3}(36n + 24)q^n \equiv 5 \frac{f_2^2 f_3^2}{f_1^3 f_4} + 5q^2 \frac{f_2^2 f_3^2 f_4}{f_1^3 f_4} - 10q \frac{f_2^2 f_3^2 f_6}{f_2^3} \pmod{8}. \tag{70}
\]

Extracting the terms involving \(q^{2n+1}\) from (70), dividing by \(q\) and then replacing \(q^2\) by \(q\), we get

\[
\sum_{n=0}^{\infty} PD_{2,3}(72n + 60)q^n \equiv 6 \frac{f_2^2 f_3^2 f_6}{f_1^3} \pmod{8}. \tag{71}
\]
Invoking (41) into equation (62), we find that
\[ \sum_{n=0}^{\infty} PD_{2,3}(18n + 12)q^n \equiv 6 \frac{f_6^5}{f_1^2 f_3^8} \pmod{8}. \] (72)

Invoking (40) into (72), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(18n + 12)q^n \equiv 2 \frac{f_6^3}{f_2^6} \pmod{4}. \] (73)

Congruence (53) fellow extracting the terms involving \( q^{2n+1} \) from (73).
Which implies that
\[ \sum_{n=0}^{\infty} PD_{2,3}(36n + 12)q^n \equiv 2 \frac{f_6^3}{f_1^2} \pmod{4}. \] (74)

Substituting (26) into (74), we have
\[ \sum_{n=0}^{\infty} PD_{2,3}(36n + 12)q^n \equiv 2 \frac{f_6^3 f_2^6}{f_2^6 f_1^2} + 2q \frac{f_6^3 f_2^6}{f_4^6} \pmod{4}. \] (75)

Which implies,
\[ \sum_{n=0}^{\infty} PD_{2,3}(72n + 48)q^n \equiv 2 \frac{f_6^3}{f_2^6} \pmod{4}. \] (76)

Congruence (54) fellow extracting the terms involving \( q^{2n+1} \) from (76).
From equation (76) and (73), we have
\[ PD_{2,3}(72n + 48) \equiv PD_{2,3}(18n + 12) \pmod{4}. \] (77)

By mathematical induction on \( \alpha \), we arrive at
\[ PD_{2,3}(18 \cdot 4^{\alpha+1} + 3 \cdot 4^{\alpha+2}) \equiv PD_{2,3}(18n + 12) \pmod{4}. \] (78)

Using (54) into (78), we get (55).
Equation (72) can rewritten as
\[ \sum_{n=0}^{\infty} PD_{2,3}(18n + 12)q^n \equiv 6 \left( \frac{f_3^3}{f_1^2} \right)^2 \pmod{8}. \] (79)

Employing (26) into (79), we obtain
\[ \sum_{n=0}^{\infty} PD_{2,3}(18n + 12)q^n \equiv 6 \frac{f_6^6 f_5^6}{f_2^6 f_1^2 f_3^{12}} + 6q^2 \frac{f_6^6}{f_4^6} + 12q \frac{f_4^6 f_6^2 f_2^2}{f_2^6} \pmod{8}. \] (80)
Extracting the terms involving $q^{2n+1}$ from (80), dividing by $q$ and then replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n + 30)q^n \equiv 12 \frac{f_2^3 f_3^2 f_6^2}{f_1^2} \pmod{8}. \quad (81)$$

From (71) and (81), we get (58). \quad \text{QED}

**Theorem 3.** For each $n \geq 0$ and $\alpha \geq 0$, we have

$$PD_{2,3}(72 \cdot 25^{\alpha+1}n + 6 \cdot 25^{\alpha+1}) \equiv PD_{2,3}(72n + 6) \pmod{4}, \quad (82)$$

$$PD_{2,3}(360(5n + i) + 150) \equiv 0 \pmod{4}, \quad (83)$$

where $i = 1, 2, 3, 4$.

**Proof.** From the equation (67), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(72n + 6)q^n \equiv 3f_1^2 \pmod{4}. \quad (84)$$

Employing (12) in the above equation, and then extracting the terms containing $q^{5n+2}$, dividing by $q^2$ and replacing $q^5$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(360n + 150)q^n \equiv 3f_5^2 \pmod{4}, \quad (85)$$

which yields

$$\sum_{n=0}^{\infty} PD_{2,3}(1800n + 150)q^n \equiv 3f_1^2 \equiv \sum_{n=0}^{\infty} PD_{2,3}(72n + 6)q^n \pmod{4}. \quad (86)$$

By induction on $\alpha$, we obtain (82). The congruence (83) follows by extracting the terms involving $q^{5n+i}$ for $i = 1, 2, 3, 4$ from both sides of (85). \quad \text{QED}

**Theorem 4.** For each $n \geq 0$ and $\alpha \geq 0$, we have

$$PD_{2,3}(24n + 20) \equiv 0 \pmod{16}, \quad (87)$$

$$PD_{2,3}(6 \cdot 4^{\alpha+2}n + 5 \cdot 4^{\alpha+2}) \equiv 0 \pmod{16}. \quad (88)$$
Proof. Extracting the terms involving $q^{3n+1}$ from (38), dividing by $q$ and then replacing $q^3$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n + 2)q^n = 2\frac{f_2^4 f_3^2}{f_1^4}.$$  \hspace{1cm} (89)$$

Invoking (41) into equation (89), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n + 2)q^n = 2(f_1 f_3)^2 \pmod{16}. \hspace{1cm} (90)$$

Substituting (30) into (90), we arrive

$$\sum_{n=0}^{\infty} PD_{2,3}(6n + 2)q^n \equiv 12(f_1 f_3)^2 \pmod{16}. \hspace{1cm} (91)$$

Extracting the terms involving $q^{2n+1}$ from (91), dividing by $q$ and then replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 8)q^n \equiv 12f_2^2 f_6^2 \pmod{16}. \hspace{1cm} (92)$$

Extracting the terms involving $q^{2n+1}$ from (92), we get (87).

Extracting the terms involving $q^{2n+1}$ from (92) and replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 8)q^n \equiv 12(f_1 f_3)^2 \pmod{16}. \hspace{1cm} (93)$$

In view of the congruences (90) and (93), we get

$$PD_{2,3}(24n + 8) \equiv 6 \cdot PD_{2,3}(6n + 2) \pmod{16}. \hspace{1cm} (94)$$

Utilizing (94) and by mathematical induction on $\alpha$, we arrive

$$PD_{2,3}(6 \cdot 4^{\alpha+1} \cdot 4^{\alpha+1}) \equiv 6^{\alpha+1} \cdot PD_{2,3}(6n + 2) \pmod{16}. \hspace{1cm} (95)$$

Using (87) into (95), we arrive (88). \hspace{1cm} \textbf{QED}

\textbf{Theorem 5.} For each $n \geq 0$ and $\alpha \geq 0$, we have

$$PD_{2,3}(6 \cdot 4^{\alpha+1} + 4^{\alpha+2}) \equiv PD_{2,3}(6n + 4) \pmod{32}. \hspace{1cm} (96)$$
Proof. Extracting the terms involving $q^{3n+2}$ from (38), dividing by $q^2$ and then replacing $q^3$ by $q$, we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n + 4)q^n = 4 \frac{f_2^3 f_6^3}{f_1^3 f_3} . \] (97)

Invoking (41) into (97), we arrive
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n + 4)q^n \equiv 4 \frac{f_2^3 f_6^3}{f_2 f_3} \quad (\text{mod } 32). \] (98)

Employing (27) into (98), we find
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n + 4)q^n \equiv 4 \frac{f_2^3 f_6^3}{f_2 f_3} - 12q \frac{f_2 f_6 f_{12}^3}{f_4} \quad (\text{mod } 32). \] (99)

Extracting the terms involving $q^{2n}$ from (99) and replacing $q^2$ by $q$, we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(12n + 4)q^n \equiv 4 \frac{f_2^3 f_6^3}{f_1 f_6} \quad (\text{mod } 32). \] (100)

Employing (26) into (100), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(12n + 4)q^n \equiv 4 \frac{f_2 f_6 f_{12}^3}{f_{12}} + 4q \frac{f_2 f_6 f_{12}^3}{f_4 f_6} \quad (\text{mod } 32). \] (101)

Extracting the terms involving $q^{2n+1}$ from (101), dividing by $q$ and then replacing $q^2$ by $q$, we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(24n + 16)q^n \equiv 4 \frac{f_2^3 f_6^3}{f_2 f_3} \quad (\text{mod } 32). \] (102)

In view of the congruences (98) and (102), we obtain
\[ PD_{2,3}(24n + 16) \equiv PD_{2,3}(6n + 4) \quad (\text{mod } 32). \] (103)

Utilizing (103) and by mathematical induction on $\alpha$, we get (96). \[ \square \]

**Theorem 6.** For $n \geq 0$, we have
\[ PD_{2,3}(48n + 34) \equiv 0 \quad (\text{mod } 8), \] (104)
\[ PD_{2,3}(48n + 46) \equiv 0 \quad (\text{mod } 8), \] (105)
\[ PD_{2,3}(96n + 52) \equiv 0 \quad (\text{mod } 8), \] (106)
\[ PD_{2,3}(96n + 76) \equiv 0 \quad (\text{mod } 8). \] (107)
Proof. Extracting the terms involving $q^{2n+1}$ from (99), dividing by $q$ and then replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 10)q^n \equiv 20 \frac{f_1 f_3 f_6^3}{f_2} \pmod{32}. \quad (108)$$

Substituting (30) into (108), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 10)q^n \equiv 20 \frac{f_2^6 f_3^2 f_6^4}{f_4 f_2^{12}} - 20q \frac{f_1^4 f_5^4}{f_2^2 f_3^2 f_6^2} \pmod{32}. \quad (109)$$

Extracting the terms involving $q^{2n}$ from (109) and replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 10)q^n \equiv 20 \frac{f_2^6 f_3^2 f_6^4}{f_4 f_2^{12}} \pmod{32}. \quad (110)$$

Invoking (40) into (110), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 10)q^n \equiv 4 f_2^4 f_6^2 \pmod{16}. \quad (111)$$

By the binomial theorem, it is easy to see that for positive integers $k$ and $m$,

$$f_{2k}^m \equiv f_k^{2m} \pmod{2} \quad (112)$$

Invoking (112) into (111), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 10)q^n \equiv 4 f_4 f_6 \pmod{8}. \quad (113)$$

Congruences (104) follows that extracting the terms involving $q^{2n+1}$ from (113).

Extracting the terms involving $q^{2n+1}$ from (109), dividing by $q$ and then replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 22)q^n \equiv 12 \frac{f_3^3 f_5^3 f_6^3}{f_1 f_4 f_6} \pmod{32}. \quad (114)$$

Invoking (40) into (114), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 22)q^n \equiv 12 \frac{f_3^4 f_6^2}{f_1^2} \pmod{16}. \quad (115)$$
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Invoking (112) into (115), we obtain

$$
\sum_{n=0}^{\infty} PD_{2,3}(24n + 22)q^n \equiv 4 \frac{f_2^2 f_{12}}{f_2} \pmod{8}.
$$

(116)

Extracting the terms involving $q^{2n+1}$ from (116), we get (105).

Extracting the terms involving $q^{2n}$ from (101) and replacing $q^2$ by $q$, we get

$$
\sum_{n=0}^{\infty} PD_{2,3}(24n + 4)q^n \equiv 4 \frac{f_1 f_3^2 f_3}{f_6} \pmod{32}.
$$

(117)

Substituting (30) into (117), we find

$$
\sum_{n=0}^{\infty} PD_{2,3}(24n + 4)q^n \equiv 4 \frac{f_2 f_3^2 f_3}{f_6 f_4} - 4q \frac{f_2^3 f_4^2 f_2}{f_6 f_4^2 f_2} \pmod{32}.
$$

(118)

Extracting the terms involving $q^{2n}$ from (118) and replacing $q^2$ by $q$, we get

$$
\sum_{n=0}^{\infty} PD_{2,3}(48n + 4)q^n \equiv 4 \frac{f_2 f_3^4 f_4}{f_2 f_3 f_4 f_6} \pmod{32}.
$$

(119)

Invoking (40) into (119), we have

$$
\sum_{n=0}^{\infty} PD_{2,3}(48n + 4)q^n \equiv 4 \frac{f_3^2}{f_3} \pmod{16}.
$$

(120)

Invoking (112) into (120), we obtain

$$
\sum_{n=0}^{\infty} PD_{2,3}(48n + 4)q^n \equiv 4 \frac{f_6}{f_6} \pmod{8}.
$$

(121)

Congruences (106) obtained by extracting the term involving $q^{2n+1}$ from (121).

Extracting the terms involving $q^{2n+1}$ from (118), dividing by $q$ and then replacing $q^2$ by $q$, we get

$$
\sum_{n=0}^{\infty} PD_{2,3}(48n + 28)q^n \equiv 28 \frac{f_1 f_2 f_4^2}{f_4 f_6} \pmod{32}.
$$

(122)

Invoking (40) into (122), we have

$$
\sum_{n=0}^{\infty} PD_{2,3}(48n + 28)q^n \equiv 12 \frac{f_1^2 f_2^2}{f_6} \pmod{16}.
$$

(123)
Invoking (112) into (123), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n + 28)q^n \equiv 4f_2f_{12} \pmod{8}. \quad (124)$$

Extracting the terms involving $q^{2n+1}$ from (124), we get (107).

**Theorem 7.** For any prime $p \equiv 5, \alpha \geq 1$ and $n \geq 0$, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48p^{2\alpha}n + 10p^{2\alpha})q^n \equiv 4f_2f_3 \pmod{8}. \quad (125)$$

**Proof.** Extracting the terms involving $q^{2n}$ from (113) and replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n + 10)q^n \equiv 4f_2f_3 \pmod{8}. \quad (126)$$

Define

$$\sum_{n=0}^{\infty} f(n)q^n = f_2f_3 \pmod{8}. \quad (127)$$

Combining (126) and (127), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(48n + 10)q^n \equiv 4 \sum_{n=0}^{\infty} f(n)q^n \pmod{8}. \quad (128)$$

Now, we consider the congruence equation

$$2 \cdot \frac{3k^2 + k}{2} + 3 \cdot \frac{3m^2 + m}{2} \equiv \frac{5p^2 - 5}{24} \pmod{p}, \quad (129)$$

which is equivalent to

$$(2 \cdot (6k + 1))^2 + 6 \cdot (6m + 1)^2 \equiv 0 \pmod{p},$$

where $\frac{-5}{2}$, $m \leq \frac{p-1}{2}$, and $p$ is a prime such that $\frac{-6}{p} = -1$. Since $\frac{-6}{p} = -1$ for $p \equiv 5 \pmod{6}$, the congruence relation (129) holds if and only if both $k = m = \frac{3\sqrt{p^2 - 1}}{2}$. Therefore, if we substitute (32) into (127) and then extracting the terms in which the powers of $q$ are congruent to $5 \cdot \frac{p^2 - 1}{24}$ modulo $p$ and then divide by $q^{5 \cdot \frac{p^2 - 1}{24}}$, we find that

$$\sum_{n=0}^{\infty} f \left( pn + 5 \cdot \frac{p^2 - 1}{24} \right) q^n = f_{2p}f_{3p},$$
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which implies

$$\sum_{n=0}^{\infty} f\left(p^2n + 5 \cdot \frac{p^2-1}{24}\right) q^n = f_2 f_3$$

(130)

and for $n \geq 0$,

$$f\left(p^2n + pi + 5 \cdot \frac{p^2-1}{24}\right) = 0,$$

(131)

where $i$ is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha}n + 5 \cdot \frac{p^{2\alpha}-1}{24}\right) = f(n).$$

(132)

Replacing $n$ by $p^{2\alpha}n + 5 \cdot \frac{p^{2\alpha}-1}{24}$ in (128), we arrive at (125).

Corollary 1. For each $n \geq 0$ and $\alpha \geq 0$, we have

$$PD_{2,3}(3 \cdot 4^{\alpha+3}n + 34 \cdot 4^{\alpha+1}) \equiv 0 \pmod{8},$$

(133)

$$PD_{2,3}(3 \cdot 4^{\alpha+3}n + 46 \cdot 4^{\alpha+1}) \equiv 0 \pmod{8},$$

(134)

$$PD_{2,3}(6 \cdot 4^{\alpha+3}n + 13 \cdot 4^{\alpha+2}) \equiv 0 \pmod{8},$$

(135)

$$PD_{2,3}(6 \cdot 4^{\alpha+3}n + 19 \cdot 4^{\alpha+2}) \equiv 0 \pmod{8}.$$  

(136)

Proof. Corollary (1) follows from the Theorem (5) and Theorem (6).

Theorem 8. For $n \geq 0$, we have

$$PD_{2,3}(12n + 11) \equiv 0 \pmod{4},$$

(137)

$$PD_{2,3}(24n + 19) \equiv 0 \pmod{4},$$

(138)

$$PD_{2,3}(24n + 17) \equiv 0 \pmod{4},$$

(139)

$$PD_{2,3}(108n + 63) \equiv 0 \pmod{4},$$

(140)

$$PD_{2,3}(108n + 99) \equiv 0 \pmod{4},$$

(141)

$$PD_{2,3}(216n + 27)q^n \equiv 2\psi(q) \pmod{4},$$

(142)

$$PD_{2,3}(72n + 6) \equiv PD_{2,3}(36n + 3) \pmod{4},$$

(143)

$$PD_{2,3}(96n + 28) \equiv 2 \cdot PD_{2,3}(24n + 7) \pmod{4}.$$  

(144)

Proof. Extracting the odd terms in (36), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n = \frac{f_2^3 f_3^3 f_5^2}{f_1^3 f_2 f_6}.$$  

(145)
Invoking (40) into (145), we obtain
\[ \sum_{n=0}^{\infty} PD_{2,3}(2n + 1)q^n = \frac{f_1 f_2 f_3 f_6^2}{f_6^3} \quad (\text{mod 4}). \] (146)

Substituting (31) into (146), we find that
\[ \sum_{n=0}^{\infty} PD_{2,3}(2n + 1)q^n = \frac{f_2^3 f_6^6}{f_6^3 f_{18}} - q \frac{f_3^3 f_6^3 f_{18}}{f_6^3} - 2q^2 \frac{f_3^3 f_{18}^4}{f_6^3} \quad (\text{mod 4}). \] (147)

Extracting the terms involving \( q^{3n} \) from (147) and replacing \( q^3 \) by \( q \), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n + 1)q^n = \frac{f_2^6 f_6^3}{f_2^3 f_6^3} \quad (\text{mod 4}). \] (148)

Invoking (40) into (148), we have
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n + 1)q^n = \frac{f_2^6}{f_2^3} \quad (\text{mod 4}). \] (149)

Employing (24) into (149), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n + 1)q^n = \frac{f_4^6 f_2^3 f_6^3}{f_2^3 f_6^3 f_{24}} + 2q \frac{f_4 f_6^3 f_{24} f_{12}}{f_2^3 f_{12}} \quad (\text{mod 4}). \] (150)

Extracting the terms involving \( q^{2n+1} \) from (150), dividing by \( q \) and then replacing \( q^2 \) by \( q \), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(12n + 7)q^n = 2 \frac{f_2^6 f_6^3 f_{12}}{f_2^3 f_6} \quad (\text{mod 4}). \] (151)

Invoking (112) into (151), we obtain
\[ \sum_{n=0}^{\infty} PD_{2,3}(12n + 7)q^n = 2f_2 f_{12} \quad (\text{mod 4}). \] (152)

Extracting the terms involving \( q^{2n+1} \) from (152), we obtain (137).

Extracting the terms involving \( q^{2n} \) from (152), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(24n + 7)q^n = 2f_1 f_6 \quad (\text{mod 4}). \] (153)
Extracting the terms involving \( q^{2n} \) from (124) and replacing \( q^2 \) by \( q \), we get
\[
\sum_{n=0}^{\infty} PD_{2,3}(96n + 28)q^n \equiv 4f_1f_6 \pmod{8}. \tag{154}
\]
In view of congruences (154) and (153), we obtain (144).

Extracting the terms involving \( q^{3n+1} \) from (147), dividing by \( q \) and then replacing \( q^3 \) by \( q \), we have
\[
\sum_{n=0}^{\infty} PD_{2,3}(6n + 3)q^n \equiv 3\frac{f_3^3 f_6}{f_1 f_2} \pmod{4}. \tag{155}
\]
Invoking (40) into (155), we find
\[
\sum_{n=0}^{\infty} PD_{2,3}(6n + 3)q^n \equiv \frac{f_3^3 f_6}{f_1 f_2} \pmod{4}. \tag{156}
\]
Employing (26) into (156), we get
\[
\sum_{n=0}^{\infty} PD_{2,3}(6n + 3)q^n \equiv 3\frac{f_3^3 f_6}{f_1 f_2} + 3q\frac{f_6 f_{12}^3}{f_2 f_4} \pmod{4}. \tag{157}
\]
Extracting the terms involving \( q^{2n} \) from (157) and replacing \( q^2 \) by \( q \), we obtain
\[
\sum_{n=0}^{\infty} PD_{2,3}(12n + 3)q^n \equiv 3\frac{f_3^3 f_6}{f_1 f_2} \pmod{4}. \tag{158}
\]
Invoking (40) into (158), we have
\[
\sum_{n=0}^{\infty} PD_{2,3}(12n + 3)q^n \equiv \frac{f_1 f_2 f_3^3}{f_6} \pmod{4}. \tag{159}
\]
Substituting (31) into (159), we find
\[
\sum_{n=0}^{\infty} PD_{2,3}(12n + 3)q^n \equiv 3\frac{f_3^3 f_6}{f_1 f_2} - 3q\frac{f_3 f_0 f_{18}}{f_6} - 6q^2\frac{f_3 f_{18}^4}{f_6 f_9} \pmod{4}. \tag{160}
\]
Extracting the terms involving \( q^{3n} \) from (160) and replacing \( q^3 \) by \( q \), we obtain
\[
\sum_{n=0}^{\infty} PD_{2,3}(36n + 3)q^n \equiv 3\frac{f_3^2 f_4}{f_6^2} \pmod{4}. \tag{161}
\]
Invoking (40) into (161), we have
\[ \sum_{n=0}^{\infty} PD_{2,3}(36n + 3)q^n \equiv 3f_1^2 \pmod{4}. \] (162)

Extracting the terms involving \( q^{2n} \) from (67) and replacing \( q^2 \) by \( q \), we obtain
\[ \sum_{n=0}^{\infty} PD_{2,3}(72n + 6)q^n \equiv 3f_1^2 \pmod{4}. \] (163)

In view of congruences (163) and (162), we obtain (143).

Extracting the terms involving \( q^{3n+2} \) from (160), dividing by \( q^2 \) and then replacing \( q^3 \) by \( q \), we have
\[ \sum_{n=0}^{\infty} PD_{2,3}(36n + 27)q^n \equiv 2f_4^4 f_6^4 f_2^2 f_3^2 \pmod{4}. \] (164)

Invoking (40) into (164), we have
\[ \sum_{n=0}^{\infty} PD_{2,3}(36n + 27)q^n \equiv 2f_4^4 f_6^4 f_2^2 f_3^2 \pmod{4}. \] (165)

Congruences (140) and (141) follows extracting the terms involving \( q^{3n+1} \) and \( q^{3n+2} \) from (165).

Invoking (112) into (165), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(36n + 27)q^n \equiv 2f_4^2 f_6^2 \pmod{4}. \] (166)

Extracting the terms involving \( q^{6n} \) from (166) and replacing \( q^6 \) by \( q \), we get (142).

Extracting the terms involving \( q^{3n+2} \) from (147), dividing by \( q^2 \) and then replacing \( q^3 \) by \( q \), we have
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n + 5)q^n \equiv 2f_4^4 f_6^4 f_2^2 f_3^2 \pmod{4}. \] (167)

Invoking (40) into (167), we have
\[ \sum_{n=0}^{\infty} PD_{2,3}(6n + 5)q^n \equiv 2f_4^4 f_6^4 f_2^2 f_3^2 \pmod{4}. \] (168)
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Congruences (137) follows that extracting the terms involving \(q^{2n+1}\) from (168).

Extracting the terms involving \(q^{2n}\) from (168) and replacing \(q^2\) by \(q\), we get

\[
\sum_{n=0}^{\infty} PD_{2,3}(12n + 5)q^n \equiv 2\frac{f_4^4}{f_1^2} \pmod{4}.
\] (169)

Substitute (26) and (23) in (169)

\[
\sum_{n=0}^{\infty} PD_{2,3}(12n + 5)q^n \\
= 2\frac{f_1^4 f_6^6 f_{16}^2 f_{24}^2}{f_2^2 f_8 f_{12}^2 f_{18}^2} + 2q^2\frac{f_1^4 f_6^6 f_8^2 f_{48}^2}{f_2^2 f_8^2 f_{16}^2 f_{24}^2} + 2q^4\frac{f_6^6 f_8^2 f_{12}^2 f_{48}^2}{f_2^2 f_{16}^2 f_{24}^2} \pmod{4}.
\] (170)

Extracting the terms involving \(q^{2n+1}\) from (170), dividing by \(q\) and then replacing \(q^2\) by \(q\), we have

\[
\sum_{n=0}^{\infty} PD_{2,3}(24n + 17)q^n \equiv 2f_2 f_3 f_{12} + 2f_2 f_3 f_{12} \pmod{4},
\] (171)

Invoking (112) into (171), we get

\[
\sum_{n=0}^{\infty} PD_{2,3}(24n + 17)q^n \equiv 2f_2 f_3 f_{12} + 2f_2 f_3 f_{12} \pmod{4},
\] (172)

which implies (139).

**Theorem 9.** For \(n \geq 0,\ \alpha \geq 0\)

\[
PD_{2,3}(648n + 459) \equiv 0 \pmod{4},
\] (173)

\[
PD_{2,3}(8 \cdot 9^{\alpha+3}n + 51 \cdot 9^{\alpha+2}) \equiv 0 \pmod{4}.
\] (174)

**Proof.** Employing (18) into (142), we get

\[
PD_{2,3}(216n + 27)q^n \equiv 2f(q^3, q^6) + 2q^3(q^9) \pmod{4}.
\] (175)

Congruences (173) follows extracting the terms involving \(q^{3n+2}\) from (175).

Extracting the terms involving \(q^{3n+1}\) from (175), dividing by \(q\) and then replacing \(q^3\) by \(q\), we have

\[
PD_{2,3}(648n + 243)q^n \equiv 2q^3 \pmod{4}.
\] (176)
Extracting the terms involving $q^{3n}$ from (176) and replacing $q^3$ by $q$, we obtain

$$PD_{2,3}(1944n + 243)q^n \equiv 2\psi(q) \pmod{4}. \quad (177)$$

In view of congruences (142) and (177), we have

$$PD_{2,3}(1944n + 243) \equiv PD_{2,3}(216n + 27) \pmod{4}. \quad (178)$$

Utilizing (178) and by mathematical induction on $\alpha$, we get

$$PD_{2,3}(24 \cdot 9^{\alpha + 2} n + 3 \cdot 9^{\alpha + 2}) \equiv PD_{2,3}(216n + 27) \pmod{4}. \quad (179)$$

Using (173) into (179), we obtain (174).

4 Congruences Modulo 3.

Theorem 10. For $n \geq 0$ and $\alpha \geq 0$, then

$$PD_{2,3}(6n + 3) \equiv 0 \pmod{3}, \quad (180)$$
$$PD_{2,3}(6n + 5) \equiv 0 \pmod{3}, \quad (181)$$
$$PD_{2,3}(36n + 30) \equiv 0 \pmod{3}, \quad (182)$$
$$PD_{2,3}(4 \cdot 3^{\alpha + 3} n + 10 \cdot 3^{\alpha + 2}) \equiv 0 \pmod{3}. \quad (183)$$

Proof. Substituting (15) into (35), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_6^2 f_9 f_{18}^2}{f_{12}^2 f_{36}} + q \frac{f_6^2 f_9 f_{18}^2}{f_{12}^2 f_{36}^2} + 2q^2 \frac{f_6^2 f_9 f_{18}^2}{f_{12}^2 f_{36}^2}. \quad (184)$$

Extracting the terms involving $q^{3n}$ from (184) and replacing $q^3$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(3n)q^n = \frac{f_6^2 f_9 f_6^2}{f_1^2 f_4 f_{12}}. \quad (185)$$

By the binomial theorem, it is easy to see that for positive integers $k$ and $m$,

$$f_{3k}^m \equiv f_k^{3m} \pmod{3}. \quad (186)$$

Invoking (186) into (185), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(3n)q^n \equiv \frac{f_2^8}{f_4^2} \pmod{3}. \quad (187)$$
Extracting the terms involving $q^{2n}$ from (187) and replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1^8}{f_2^4} \pmod{3}. \quad (188)$$

But

$$\frac{f_1^8}{f_2^4} = \frac{f_1^4 f_2^2}{f_2^4}. \quad (189)$$

Substituting (30) into (190), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1^2 f_2^2}{f_2^4} \pmod{3}. \quad (190)$$

Extracting the terms involving $q^{2n+1}$ from (191), dividing by $q$ and then replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 6)q^n \equiv \frac{f_2^2 f_6^2}{f_1^4} \pmod{3}. \quad (192)$$

Invoking (186) into (192), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 6)q^n \equiv \frac{f_2^2 f_6^2}{f_1^4 f_3} \pmod{3}, \quad (193)$$

which implies

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 6)q^n \equiv \psi(q)\psi(q^3) \pmod{3}. \quad (194)$$

Employing (18) into (194), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 6)q^n \equiv \psi(q^3)f(q, q^6) + q^3\psi(q^9) \pmod{3}. \quad (195)$$

Congruences (182) follows by extracting the terms involving $q^{3n+2}$ from (195).

Extracting the terms involving $q^{3n+1}$ from (195), dividing by $q$ and then replacing $q^3$ by $q$, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n + 18)q^n \equiv \psi(q)\psi(q^3) \pmod{3}. \quad (196)$$
In view of congruences (194) and (196), we obtain
\[ PD_{2,3}(36n + 18)q^n \equiv PD_{2,3}(12n + 6) \pmod{3}. \] (197)
Utilizing (197) and by mathematical induction on \( \alpha \), we get
\[ PD_{2,3}(4 \cdot 3^{\alpha+2}n + 2 \cdot 3^{\alpha+2}) \equiv PD_{2,3}(12n + 6) \pmod{3}. \] (198)
Using (182) into (198), we get (183).
Invoking (186) into (145), we get
\[ \sum_{n=0}^{\infty} PD_{2,3}(2n + 1)q^n \equiv \frac{f_4}{f_3} \pmod{3}. \] (199)
Congruences (180) and (181) follows by extracting the terms involving \( q^{3n+1} \)
and \( q^{3n+2} \) from (199).

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References
Congruences for (2, 3)-regular partition with designated summands


