Non-existence of smooth rational curves of degree $d = 13, 14, 15$ contained in a general quintic hypersurface of $\mathbb{P}^4$ and in some quadric hypersurface

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Abstract. Let $W \subset \mathbb{P}^4$ be a general quintic hypersurface. We prove that $W$ contains no smooth rational curve $C \subset \mathbb{P}^4$ with degree $d \in \{13, 14, 15\}$, $h^0(I_C(1)) = 0$ and $h^0(I_C(2)) > 0$.

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Introduction

For any positive integer $d$ let $M_d$ be the set of all smooth and rational curves $C \subset \mathbb{P}^4$ with $\deg(C) = d$. Let $\Gamma_d$ be the set of all non-degenerate $C \in M_d$ with $h^0(I_C(2)) > 0$. Clemens conjecture asks if for each $d$ a general quintic hypersurface $W \subset \mathbb{P}^4$ contains only finitely many elements of $M_d$ (a stronger form asks the same also for singular rational curves of degree $d > 5$) ([1], [2], [4], [12], [13], [14], [15], [19], [20], [24], [25]). For higher genera cases (and also for more general Calabi-Yau 3-folds), see [16], [17].

All the quoted finiteness results work for very low $d$, say $d \leq 12$. Here we add a very strong condition (to be contained in an integral quadric hypersurface) and prove the following result.

**Theorem 1.** If $13 \leq d \leq 15$, then a general quintic hypersurface of $\mathbb{P}^4$ contains no element of $\Gamma_d$.

The proof requires a result on the splitting type of the normal bundle of a smooth rational curve $C \subset \mathbb{P}^4$ ([3], [23]) and its use when $C$ is contained in quadric hypersurface.
Concerning elements of $C \in M_d$ contained in a hyperplane we prove the following result.

**Proposition 1.** Let $C \in M_d$ be a degenerate curve, say contained in a hyperplane $H$, and let $\alpha$ be the minimal degree of a surface of $H$ containing $C$. Assume that $C$ is contained in a general quintic hypersurface. If $13 \leq d \leq 17$, then $\alpha \in \{4, 5\}$. If $d \geq 18$, then $\alpha = 5$.

1 Preliminaries

Let $\mathcal{W}$ denote the set of all smooth quintic hypersurfaces $W \subset \mathbb{P}^4$ satisfying the thesis of [4]. In particular each $W \in \mathcal{W}$ contains only finitely many smooth rational curves $D$ of degree $\leq 11$ and all of them have as normal bundle $N_{D,W}$ the direct sum of two line bundles of degree $-1$, i.e. $h^i(N_{D,W}) = 0$, $i = 0, 1$.

For any scheme $A \subset \mathbb{P}^4$ let $I_A$ denote the ideal sheaf of $A$ in $\mathbb{P}^4$.

Let $X$ be any projective scheme, $N \subset X$ an effective Cartier divisor and $Z \subset X$ any closed subscheme. The residual scheme $\text{Res}_N(Z)$ of $Z$ with respect to $N$ is the closed subscheme of $X$ with $I_{Z,X} : I_{N,X}$ as its ideal sheaf. We always have $\text{Res}_N(Z) \subseteq Z$. If $Z$ is zero-dimensional, we have $\deg(Z) = \deg(Z \cap N) + \deg(\text{Res}_N(Z))$. For any line bundle $L$ on $X$ we have the exact sequence

$$0 \rightarrow I_{\text{Res}_N(Z),X} \otimes L(-N) \rightarrow I_{Z,X} \otimes L \rightarrow I_{Z \cap N,N} \otimes L|_N \rightarrow 0 \quad (1)$$

(the residual exact sequence of $N$ in $X$).

**Lemma 1.** Take any $C \in \Gamma_d$, $d \geq 6$, and any $Q \in |I_C(2)|$. Let $a_1 \geq a_2$ be the splitting type of the normal sheaf $N_{C,Q}$ of $C$ in $Q$. Then $a_1 \leq 3d - 8$.

**Proof.** Since $C$ is a smooth curve and $N_{C,Q}$ is the dual of the conormal sheaf of $C$ in $Q$, $N_{C,Q}$ is a rank 2 vector bundle and hence by the classification of vector bundles on $\mathbb{P}^1$ it has a splitting type. Let $b_1 \geq b_2 \geq b_3$ be the splitting type of the normal bundle $N_{C,\mathbb{P}^4}$ of $C$ in $\mathbb{P}^4$. We have $b_1 + b_2 + b_3 = 5d - 2$. By [3, case $r = 1$ of Lemma 4.3] we have $b_3 \geq d + 3$ and hence $b_1 \leq 3d - 8$. The injective map $N_{C,Q} \rightarrow N_{C,\mathbb{P}^4}$ gives $a_1 \leq 3d - 8$. \[QED\]

**Remark 1.** Obviously $\Gamma_d \neq \emptyset$ if and only if $d \geq 4$. The aim of this remark is to prove that $\dim \Gamma_d = 3d + 14$ and to prove a more precise result for the part associated to quadric hypersurfaces with a line as their singular locus. Let $Q \subset \mathbb{P}^4$ be an integral quadric and let $C \subset Q$ be a smooth and non-degenerate rational curve of degree $d$. Let $N_{C,Q}$ be the normal sheaf of $C$ in $Q$ and $N_{C,\mathbb{P}^4}$ the normal bundle of $C$ in $\mathbb{P}^4$. Since $C$ is a smooth curve and by its definition $N_{C,Q}$ is the dual of the conormal sheaf of $C$ in $Q$, $N_{C,Q}$ is locally free. Since $C$ is not contained in the singular locus of $Q$, $N_{C,Q}$ has rank 2. There is a natural
map \( j : N_{C,Q} \to N_{C,P^4} \), which is injective outside the finite set \( C \cap \text{Sing}(Q) \). Hence \( j \) is injective. We have an exact sequence

\[
0 \to N_{C,Q} \xrightarrow{j} N_{C,P^4} \xrightarrow{\omega} \mathcal{O}_C(2)
\]  

(2)

with \( \Delta := \text{coker}(u) \) supported on the finite set \( \text{Sing}(Q) \cap C \). Set \( e := \text{deg}(\Delta) \). If \( \text{Sing}(Q) \) is a point, \( o \), then blowing up it we get that \( e = 1 \) if \( o \in C \) and \( e = 0 \) if \( o \notin C \). Since \( Q \setminus \text{Sing}(Q) \) is homogeneous, \( N_{C,Q} \) is spanned. Since \( p_a(C) = 0 \), we get \( h^1(N_{C,Q}) = 0 \) and hence \( h^0(N_{C,Q}) = 3d + e \). Hence the subset of all \( \Gamma \) parametrizing curves contained either in smooth quadrics or in quadric cones with 0-dimensional vertex has dimension \( 3d + 14 \). Let \( \Gamma_{d,e} \) be the subset of all non-degenerate \( C \in M_d \) contained in some integral quadric hypersurface with singular locus a line \( R \) with \( \text{deg}(R \cap C) = e \). Now assume that \( Q \) has the line \( R \) as its singular locus. We consider only the part of the Hilbert scheme of \( Q \) formed by curves \( C' \) with \( \text{deg}(R \cap C') = e \) (it contains \( C \) by assumption). Let \( a_1 \geq a_2 \) be the splitting type of \( N_{C,Q} \). Since \( a_1 \leq 3d - 8 \) (Lemma 1), we have \( a_2 \geq e + 6 \). Hence \( h^1(N_{C,Q}(-Z)) = 0 \) and so \( \dim H(Q, Z, d) = 3d + e - 2e \). Since \( R \) has \( \infty^e \) subschemes of degree \( e \) and \( \mathbb{P}^4 \) has \( \infty^{11} \) rank 3 quadrics, we get that the part coming from quadrics with rank 3 has dimension \( \leq 3d + 11 \).

**Lemma 2.** Let \( \Gamma_{d,2} \) be the set of all non-degenerate \( C \in M_d, d > 12 \), with \( h^0(\mathcal{I}_C(2)) = 2 \) and contained in a smooth quintic hypersurface. Then \( \dim \Gamma_{d,2} \leq d + 25 \).

**Proof.** Fix \( C \in \Gamma_{d,2} \) and let \( T \subset \mathbb{P}^4 \) be the intersection of two different elements of \( |\mathcal{I}_C(2)| \). Let \( S \) be the irreducible component of \( T \) containing \( C \). Since \( C \) is non-degenerate, we have \( \text{deg}(S) \geq 3 \). Hence either \( \text{deg}(S) = 3 \) or \( S = T \) and \( T \) is irreducible.

(a) Assume \( \text{deg}(S) = 3 \). Since \( S \) spans \( \mathbb{P}^4 \), it is a minimal degree surface, i.e. either a cone over a rational normal curve of \( \mathbb{P}^3 \) or an embedding of the Hirzebruch surface \( F_1 \).

(a1) Assume that \( S \) is a cone with vertex \( o \) and let \( m : U \to S \) be its minimal desingularization. \( U \) is isomorphic to the Hirzebruch surface \( F_3 \) and \( m \) is induced by the complete linear system \( |\mathcal{O}_{F_3}(h + 3f)| \), where \( h \) is the section of the ruling of \( F_3 \) with negative self-intersection and \( f \) is a fiber of the ruling of \( F_3 \). We have \( j^2 = 0, f \cdot h = 1 \) and \( h^2 = -3 \). Let \( C' \) be the strict transform of \( C \) in \( U \) and take positive integers \( a, b \) with \( b \geq 3a \) and \( C' \in |ah + bf| \). Since \( m \) is induced by \( |h + 3f| \), we have \( b = d \). Since \( \omega_{F_3} \cong \mathcal{O}_{F_3}(-2h - 5f) \), the adjunction formula gives \( \omega_{C'} \cong \mathcal{O}_{C'}((a - 2)h + (d - 5)f) \). Since \( C \) is smooth, we have \( C' \cong C \) and in particular \( p_a(C') = 0 \). Hence \( -2 = (ah + df) \cdot ((a - 2)h + (d - 5)f) = (a - 2)(d - 3a) + a(d - 5) \). Hence \( a = 1 \). Since \( d \geq 7 \), the curve \( C = f(C') \) has a singular point at \( o \), a contradiction.
(a2) Assume $S \cong F_1$ and take integers $a, b$ with $b \geq a > 0$ and $C \in |ah + bf|$, where $h$ is the section of the ruling of $F_1$ with negative intersection and $f$ is a ruling of $F_1$. We have $|\mathcal{O}_{F_1}(1)| = |\mathcal{O}_{F_1}(h + 2f)|$ and $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h - 3f)$ and so $\omega_C \cong \mathcal{O}_C((a-2)h+(b-3)f)$. Hence $d = a+b$ and $-2 = (a-2)(b-a)+a(b-3)$. Hence $a = 1$ and $b = d - 1$. Since $d - 1 > 5$, every quintic hypersurface $W$ containing $C$ contains $S$. If $W$ is smooth, then its Picard group is generated by $\mathcal{O}_W(1)$, by the Lefschetz theorem and so it contains only surfaces whose degree is divisible by $5$. Hence $S \not\subseteq W$, a contradiction.

(b) Assume $S = T$, i.e. assume that $T$ is irreducible. For a general hyperplane $H \subset \mathbb{P}^4$, $T \cap H$ is an integral curve with $p_a(T \cap H) = 1$ and hence it has at most one singular point. Hence the one-dimensional part of $\text{Sing}(T)$ is either empty or a line.

(b1) Assume that $\text{Sing}(T)$ contains a line $L$. A general hyperplane section of $T$ is an irreducible and singular curve with arithmetic genus $1$. Hence if $T$ is a cone with vertex $o$, then $T$ is the image of a minimal degree cone $T'$ of $\mathbb{P}^5$ by a birational, but not isomorphic linear projection. If $T$ is not a cone, then it is the image of a minimal degree smooth surface $F$ of $\mathbb{P}^5$ by a birational, but not isomorphic linear projection ([8, Theorem 19.5]).

(b1.1) Assume that $T$ is the image of a minimal degree non-degenerate cone $T' \subset \mathbb{P}^5$ and let $u : U \to T'$ be its minimal desingularization. We have $U \cong F_4$ and $u$ is induced by the complete linear system $|\mathcal{O}_{F_4}(h + 4f)|$. Let $D \subset U$ be the strict transform of the curve, whose image in $\mathbb{P}^4$ is $C$. Write $D \in |ah + bf|$ with $b \geq 4a > 0$. As in step (a1) we first get $b = d$ and then $a = 1$. We get that $u(D)$ is singular and hence $C$ is singular, a contradiction.

(b1.2) Assume that $T$ is the image of a minimal degree smooth surface $F$ of $\mathbb{P}^5$ and let $D \subset F$ be the curve with image $C$. Since $C$ is smooth, $D$ is smooth. There is $e \in \{0, 2\}$ such that $F \cong F_e$ embedded by the complete linear system $|h + (e+1)f|$. Take positive integers $a, b$ such that $D \in |\mathcal{O}_{F_e}(ah + bf)|$ and $b \geq ea$. As in step (a) we first get $a = 1$ and then $b = d - 1$. If $e = 0$ we get that every quintic hypersurface containing $D$ contains $F$ and hence every quintic hypersurface containing $C$ contains $T$, contradicting the Lefschetz theorem as in step (a2). Now assume $e = 2$. $F_2$ has no smooth plane conic and its lines are either the elements of $|f|$ or $h$. Since $h \cdot (h + (d-1)f) = d - 3$, we have $\deg(L \cap C) = d - 3$. Since $3$ is a prime integer, the linear projection $\ell_L : \mathbb{P}^4 \setminus L \to \mathbb{P}^2$ maps $C$ birationally onto an integral plane cubic. Hence $C$ is contained in the intersection of $T$ with a cubic hypersurface, contradicting the assumption $d > 12$ by Bezout.

(b2) Assume that $\text{Sing}(T)$ is finite. Since $T$ is a complete intersection, it is a locally complete intersection. Hence $T$ is a normal Del Pezzo surface of degree 4. Let $u : V \to T$ be a minimal desingularization and $D$ the strict
Quartic hypersurface

transform of $C$ in $V$. Since $D$ is smooth and rational, the adjunction formula

gives $-2 = \omega_V \cdot D + D^2$. $V$ is rational and it is classified ([6]). Since $V$ is a
weak del Pezzo, $u$ is induced by the complete linear system $|\omega_V^\vee|$. Hence $d = O_T(C) \cdot O_T(1) = u^*(C) \cdot \omega_C^\vee$. Write $u^*(C) = D + \sum c_iD_i$ with $c_i \geq 0$ and
$D_i$ contracted by $u$. Since $\omega_V^\vee$ is spanned ([6, IV, §3, Théorème 1]), we get

$\omega_V^\vee \cdot D = 0$. Hence $\omega_V \cdot D = -d$. Hence $D^2 = d - 2$. Hence $h^0(O_D(D)) = d - 1$.

Thus the set of all $C \subset T$ depends on $d - 1$ parameters. Since the Grassmannian

$G(2, 15)$ of all lines of $|O_{\mathbb{P}^4}(2)|$ has dimension 26, this part of $\Gamma_{d,2}$ has dimension

at most $d + 25$.

Lemma 3. There is no non-degenerate $C \in \mathcal{M}_d$, $d > 12$, with $h^0(\mathcal{I}_C(2)) \geq 3$
and contained in a smooth quartic hypersurface.

Proof. Take a non-degenerate $C \in \mathcal{M}_d$, $d > 12$, with $h^0(\mathcal{I}_C(2)) \geq 3$. Let $T$ be
the intersection of two general elements of $|\mathcal{I}_C(2)|$ and let $S$ be the irreducible
component of $T$ containing $C$. Since $C$ is non-degenerate, we have $\deg(S) \geq 3$.

Hence either $\deg(S) = 3$ or $S = T$ and $T$ is irreducible. We exclude the case $S = T$, because $d > 8$ and $h^0(\mathcal{I}_T(2)) = 2$. We exclude the case $\deg(S) = 3$ as in
step (a) of the proof of Lemma 2.

Lemma 4. Let $\Delta(d)$ be the set of all $C \in \Gamma_d$ for which there exists a line
$L \subset \mathbb{P}^4$ with $\deg(L \cap C) \geq 5$. Then $\dim \Delta(d) \leq 12 + 3d$.

Proof. We take $C \in \Gamma_d$ and a line $L \subset \mathbb{P}^4$ such that $\deg(L \cap C) \geq 5$. Take
$Q \in |\mathcal{I}_C(2)|$. Bezout implies $L \subset Q$. If $Q$ has a line as its singular locus, then we
use Remark 1. Hence we may assume that either $Q$ is smooth or it is a cone with
vertex a single point, $o$. We write $e = 1$ if $Q$ is singular and $o \in C$ and $e = 0$
otherwise. Take $Z \subseteq C \cap L$ with $\deg(Z) = 5$. Let $a_1 \geq a_2$ be the splitting type of
$N_{C,Q}$. Since $a_1 \leq 3d - 8$ (Lemma 1), we have $a_2 \geq 4$. Hence $h^1(N_{C,Q}(-Z)) = 0$.

Use that $L$ has $\infty^5$ subschemes of degree 5 and that $Q$ has $\infty^3$ lines.

2 Proof of Theorem 1

Fix any non-degenerate $C \in \mathcal{M}_d$ and let $H \subset \mathbb{P}^4$ be any hyperplane. We
often use the exact sequence

$0 \to \mathcal{I}_C(t - 1) \to \mathcal{I}_C(t) \to \mathcal{I}_{C \cap H,H}(t) \to 0$ \hspace{1cm} (3)

Lemma 5. Let $Z \subset \mathbb{P}^3$ be a degree $d$ curvilinear scheme spanning $\mathbb{P}^3$. Assume
$d \leq 15$ and $h^1(\mathbb{P}^3, \mathcal{I}_Z(5)) > 0$. Then either there is a line $L \subset \mathbb{P}^3$ with
$\deg(L \cap Z) \geq 7$ or there is a conic $D$ with $\deg(D \cap C) \geq 12$. 

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Proof. Since $Z$ spans $\mathbb{P}^3$, we have $\deg(Z \cap N) \leq 14$ for every plane $N$. Assume for the moment the existence of a plane $N \subset \mathbb{P}^3$ such that $h^1(N, \mathcal{I}_{Z \cap N,N}(5)) > 0$, then $N$ contains either a line $L \subset \mathbb{P}^3$ with $\deg(L \cap Z) \geq 7$ or a conic $D$ with $\deg(D \cap C) \geq 12$ ([7, Corollaire 2]). Now assume $h^1(N, \mathcal{I}_{Z \cap C,N}(t)) = 0$ for all planes $N \subset \mathbb{P}^3$. We may assume $h^1(\mathcal{I}_Z(5)) = 0$ for all $Z' \subset Z$ (taking if necessary a smaller non-degenerate $Z$), because $h^1(N, \mathcal{I}_{Z \cap C,N}(t)) = 0$ for all planes $N$. Set $Z_0 := Z$. Let $N_1 \subset \mathbb{P}^3$ be a plane such that $e_1 := \deg(Z_0 \cap N_1)$ is maximal. Set $Z_1 := \text{Res}_{N_1}(Z_0)$. Define recursively for each integer $i \geq 2$ the plane $N_i \subset \mathbb{P}^3$, the integer $e_i$ and the scheme $Z_i$ in the following way. Let $N_i$ be any plane such that $e_i := \deg(Z_{i-1} \cap N_i)$ is maximal. Set $Z_i := \text{Res}_{N_i}(Z_{i-1})$. We have $e_i \leq e_{i-1}$ for all $i \geq 2$. For each $i \geq 1$ we have the exact sequence

$$0 \to \mathcal{I}_{Z_i}(5 - i) \to \mathcal{I}_{Z_{i-1}}(6 - i) \to \mathcal{I}_{Z_{i-1} \cap N_i,N_i}(6 - i) \to 0$$

(4)

If $e_i \leq 2$, then $Z_{i-1} \subset N_i$ and hence $Z_i = \emptyset$. Since $\deg(Z) \leq 15$, we get $\deg(Z_0) \leq 0$, i.e. $Z_0 = \emptyset$. Since $h^1(N_0, \mathcal{O}_{N_0}) = 0$, there is an integer $i$ such that $1 \leq i \leq 5$ and $h^1(\mathcal{I}_{Z_i \cap N_i,N_i}(6 - i)) > 0$. We call $f$ such a minimal integer. Since $h^1(N, \mathcal{I}_{Z \cap C,N}(5)) = 0$ for all planes $N$, we have $f \geq 2$. Hence $f \in \{2, 3, 4, 5\}$. We have $e_f \geq 8 - f$. Since the sequence $\{e_i\}$ is non-increasing, we get $f(8 - f) \leq 15$. Since $f \geq 2$, we get that $f \in \{2, 3, 5\}$.

(a) Assume $f = 3$. Since $e_1 \geq e_2 \geq e_3 \geq 5$, we get $e_1 = e_2 = e_3 = 5$. Since $e_3 \leq 7$ and $h^1(N_3, \mathcal{I}_{Z_2 \cap N_3,N_3}(3)) > 0$, there is a line $R \subset N_3$ with $\deg(R \cap Z_2) \geq 5$. Taking a plane $F$ containing $R$ and with maximal $\deg(M \cap Z_1)$ we get $e_2 \geq 6$, a contradiction.

(b) Assume $f = 2$. We have $e_2 \geq 6$. Since $e_1 \geq e_2$ and $e_1 + e_2 \leq 15$, we have $e_2 \leq 7$. Hence there is a line $R \subset N_2$ such that $\deg(R \cap Z_1) \geq 6$.

Assuming that $L$ does not exists, then $\deg(R \cap Z) = 6$. Let $M_1 \subset \mathbb{P}^3$ be a plane containing $R$ and with maximal $g_1 := \deg(M_1 \cap Z)$ among the planes containing $R$. Since $Z$ spans $\mathbb{P}^3$, we have $g_1 \geq 7$. Set $W_1 := \text{Res}_{M_1}(Z)$. By assumption $h^1(M_1, \mathcal{I}_{Z \cap M_1,M_1}(5)) = 0$. Hence the residual sequence of $M_1 \subset \mathbb{P}^3$ gives $h^1(\mathbb{P}^3, \mathcal{I}_{W_1}(4)) > 0$. Let $M_2 \subset \mathbb{P}^3$ be a plane with maximal $g_2 := \deg(W_1 \cap M_2)$. Set $W_2 := \text{Res}_{M_2}(W_1)$. Let $M_3 \subset \mathbb{P}^3$ be a plane with maximal $g_3 := \deg(W_2 \cap M_3)$. Set $W_3 := \text{Res}_{M_3}(W_2)$. In this way we get a non-decreasing sequence $\{g_i\}_{i \geq 2}$ with $\sum_{i \geq 2} g_i = d - g_1 \leq 8$. We get an integer $h \in \{2, 3\}$ with $h^1(M_h, \mathcal{I}_{M_h \cap W_{h-1},M_h}(6 - h)) > 0$ and $g_h \geq 8 - h$. As in step (a) we exclude the case $h = 3$. Hence $h = 2$. As in the first part of step (b) we get a line $D \subset \mathbb{P}^3$ such that $\deg(D \cap W_1) = 6$.

(b1) Assume $D \cap R = \emptyset$. Let $T \subset \mathbb{P}^3$ be a general quadric surface containing $D \cup R$. Since $\mathcal{I}_{D \cup R}(2)$ is spanned and $Z$ is curvilinear, $T$ is smooth and $T \cap Z = (D \cup R) \cap Z$ (as schemes). Hence $h^1(T, \mathcal{I}_{Z \cap T,T}(5)) = 0$. Since $\deg(\text{Res}_T(Z)) = d - 12 \leq 3$, we have $h^1(\mathcal{I}_{\text{Res}_T(Z)}(3)) = 0$. The residual sequence of $T$ gives a
contradiction.

(b2) Assume $D \cap R \neq \emptyset$ and $D \neq R$. Let $N$ be the plane spanned by $D \cup R$. Since $\deg(\Res_N(Z)) \leq d - 11$, we have $h^1(N, \mathcal{I}_{\Res_N(Z)}, N(4)) = 0$. The residual sequence of $N$ gives $h^1(N, \mathcal{I}_{Z \cap N, N(5)}) > 0$, contradicting one of our assumptions.

(b3) Assume $D = R$. Let $H, M \subset \mathbb{P}^3$ be general planes containing $R$. Since $\Res_{H \cup M}(Z) = \Res_H(\Res_M(Z))$, we have $\deg(\Res_{H \cup M}(Z)) \leq d - 12 \leq 3$. Hence $h^1(\mathcal{I}_{\Res_{H \cup M}(Z)}(3)) = 0$. The residual sequence of $H \cup M$ gives $h^1(H \cup M, \mathcal{I}_{Z \cap (H \cup M), H \cup M}(5)) > 0$. The minimality condition of $Z$ gives $Z \cap (H \cup R) = Z$. Hence $d = 12$. For any $q \in Z_{\text{red}}$ let $Z_q$ be the connected component of $Z$ containing $q$. Since $\Res_H(Z)$ has degree 6 and it is supported by $D$, we have $2 \deg(\Res_H(Z_q)) = \deg(Z_q)$ for all $q$. In particular we may take $q$ with $Z_q \not\subseteq R$. Since $Z$ is curvilinear, we may find a plane $N \supset R$ with $\deg(N \cap Z_q) > \deg(R \cap Z_q)$. Since $\deg(\Res_N(Z)) \leq 12 - 7$, we have $h^1(N, \mathcal{I}_{\Res_N(Z)}, N(4)) = 0$. The residual sequence of $N$ gives $h^1(N, \mathcal{I}_{Z \cap N, N(5)}) > 0$, contradicting one of our assumptions.

(c) Assume $f = 5$. Since $\deg(Z_{i-1}) \leq 4$, we get the existence of a line $R \subset N_3$ such that $\deg(R \cap Z_4) \geq 3$. Since $\deg(R \cap Z_3) \geq 3$, the maximality property of $N_4$ implies $e_4 \geq 4$. Hence $15 \geq 4 \cdot 4 + 3$, a contradiction.

Lemma 6. Fix a non-degenerate $C \in M_d$ contained in some $W \in \mathcal{W}$ and assume the existence of a conic $D \subset \mathbb{P}^4$ with $\deg(D \cap C) \geq 12$ and that $\deg(L \cap C) \leq 6$ for each line $L \subset D_{\text{red}}$. Then $D$ is smooth.

Proof. Take $W \in \mathcal{W}$ containing $C$. Let $N$ be the plane spanned by $D$. First assume that $D \subset N$ is a double line. Set $L := D_{\text{red}}$. Since $\deg(L \cap C) \leq 6$ by assumption, we have $\deg(L \cap C) = 6$. Bezout implies $L \subset W$. Since $W \in \mathcal{W}$, we have $N_{L, W} \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$. Bezout implies that $D \subset W \cap N$. Fix a general hyperplane $H \supset N$. Since $W$ is smooth $W \cap H$ has isolated singularities. We have an injective map $N_{L, H \cap W} \rightarrow N_{L, W}$, contradicting the inclusion $D \subset H \cap W$. Now assume that $D = R \cup L$ with $R, L$ lines and $R \neq L$. Since $\deg(L \cap C) \leq 6$ and $\deg(R \cap C) \leq 6$ by assumption, we have $\deg(L \cap C) = \deg(R \cap C) \leq 6$. Hence $L \cup R \subset W$, contradicting the fact that any two lines of $W$ are disjoint.

Lemma 7. Take a non-degenerate $C \in M_d$ contained in some $W \in \mathcal{W}$. Assume $h^1(\mathcal{I}_C(5)) > 0$ and that there is either a line $L$ with $\deg(L \cap C) \geq 7$ or a conic $D$ with $\deg(D \cap C) \geq 12$. Then $h^1(\mathcal{I}_C(4)) > h^1(\mathcal{I}_C(5))$.

Proof. Let $S_1$ be the set of all lines $L$ with $\deg(L \cap C) \geq 7$ and let $S_2$ be the set of all conics $D$ such that $\deg(D \cap C) \geq 10$. Assume for the moment that the sets $S_1$ and $S_2$ are finite. Let $N \subset \mathbb{P}^4$ be a general plane and let $M \subset \mathbb{P}^4$ be any hyperplane containing $N$. Set $V := H^0(\mathcal{I}_N(1))$. We have $\dim(V) = 2$. Since $S_1$
is finite and \( N \) is general, then \( N \cap L = \emptyset \) for all \( L \in S_1 \) and hence \( L \not\in M \) for all \( L \in S_2 \). Since \( S_2 \) is finite, then \( N \) contains a unique point of the plane spanned by any \( D \in S_2 \) and hence \( D \not\in M \). Lemma 5 gives \( h^1(M, \mathcal{I}_{C \cap M, M}(5)) = 0 \). Hence the bilinear map \( H^0(\mathcal{I}_C(5))^2 \times V \to H^0(\mathcal{I}_C(4))^2 \) is non-degenerate in the second variable. By the bilinear lemma we have \( h^1(\mathcal{I}_C(4)) \geq h^1(\mathcal{I}_C(5)) - 1 + \dim V \).

Now assume that \( S_1 \) is infinite and call \( \Delta \) an irreducible positive dimensional family of its elements. Take a general \((R, L) \in \Delta\). We have \( L \cap R = \emptyset \), unless either there is \( o \in \mathbb{P}^4 \) with \( o \in J \) for all \( J \in \Delta \) or there is a plane \( N \) with \( J \subset N \) for all \( J \in \Delta \). The second case is not possible, because \( C \not\in N \). The first case is excluded, because the linear projection from \( o \) would map \( C \) onto a non-degenerate curve of \( \mathbb{P}^3 \) with degree \( \leq (d - 1)/6 < 3 \).

Now assume that \( S_2 \) is infinite. Let \( S'_2 \) be the set of all \( D \in S_2 \) with \( D \) a smooth conic. As in the proof just given we find that the set of all lines \( R \) with \( \deg(R \cap C) = 6 \) and supporting a component of some \( D \in S_2 \) is finite. Hence it is sufficient to prove that \( S'_2 \) is finite. For each \( D \in S'_2 \) let \( \langle D \rangle \) be the plane spanned by \( D \). If \( D_1 \neq D_2 \), no hyperplane contains \( D_1 \cup D_2 \) by Bezout and hence \( \langle D_1 \rangle \cap \langle D_2 \rangle = \emptyset \). Since any two planes of \( \mathbb{P}^4 \) meet, we have \( \sharp(S'_2) \leq 1 \).

**Proof of Theorem 1:** Fix \( C \in M_d, d \leq 15 \).

By Remark 1 we may assume \( h^1(\mathcal{I}_C(5)) \geq 2d - 13 \).

(a) Assume \( h^0(\mathcal{I}_C(2)) = 1 \), say \( \{Q\} = [\mathcal{I}_C(2)] \). Fix a general hyperplane \( H \subset \mathbb{P}^4 \).

(a1) Assume that there is no line \( L \subset \mathbb{P}^4 \) with \( \deg(L \cap C) \geq 7 \) and no conic \( D \) with \( \deg(D \cap C) \geq 12 \). Lemma 5 gives \( h^1(H, \mathcal{I}_{C \cap H, H}(5)) = 0 \) for every hyperplane \( H \subset \mathbb{P}^4 \). Hence the bilinear lemma gives \( h^1(\mathcal{I}_C(4)) \geq h^1(\mathcal{I}_C(5)) + 4 \geq 2d - 9 \). Since \( C \cap H \) is in uniform position, we have \( h^1(H, \mathcal{I}_{C \cap H, H}(4)) \leq d - 13 \leq 2 \) ([10, Lemma 3.9]). By (3) we have \( h^1(\mathcal{I}_C(3)) \geq 2d - 11 \). Hence \( h^0(\mathcal{I}_C(3)) \geq 35 - 3d - 1 + 2d - 11 \geq 8 \). Since \( h^0(\mathcal{I}_C(2)) = 1 \), the general \( M \in [\mathcal{I}_C(3)] \) has not \( Q \) as a component. Set \( F := Q \cap M \). First assume that \( F \) is irreducible. The curve \( D := F \cap H \) is a complete intersection curve with degree 6 and arithmetic genus 4. In particular \( h^1(H, \mathcal{I}_{C \cap H, H}(3)) = 0 \). Thus \( h^1(H, \mathcal{I}_{C \cap H, H}(3)) = h^1(D, \mathcal{I}_{C \cap H, D}(3)) \). We have \( h^1(\mathcal{I}_{C \cap H, D}(3)) \leq 1 \), because \( \deg(\mathcal{I}_{C \cap H, D}(3)) = 18 - d \geq 3 \). Hence \( h^1(\mathcal{I}_C(3)) \leq 1 \). Since \( h^1(\mathcal{I}_C(2)) \geq 2d - 12 \), we have \( h^0(\mathcal{I}_C(2)) = 15 - 2d - 1 + h^1(\mathcal{I}_C(2)) \geq 2 \), contradicting the assumption of step (a).

Now assume that \( F \) is not irreducible. Call \( T \) the irreducible component of \( F \) containing \( C \). \( T \) is a non-degenerate surface and hence \( \deg(T) \geq 3 \). Since \( h^0(\mathcal{I}_C(2)) = 1 \), we have \( h^0(\mathcal{I}_T(2)) = 1 \) and hence neither \( \deg(T) = 3 \) nor \( T \) is the complete intersection of two quadrics.

Assume \( \deg(T) = 4 \). Since \( T \) is not a complete intersection, a general hyperplane section of \( T \) is a smooth rational curve of degree 4. Since \( h^1(H, \mathcal{I}_{C \cap H, H}(t)) = 0 \), we have \( h^1(T, \mathcal{I}_C(2)) = 2 \) (4)}.
0 for all $t \geq 2$ and $h^0(\mathcal{O}_{C\cap H}(t)) = 4t + 1$, $t = 3, 4$, we get $h^1(H, \mathcal{I}_{C\cap H,H}(3)) \leq d - 13$ and $h^1(\mathcal{I}_{C\cap H,H}(4)) = 0$. We get $h^1(\mathcal{I}_C(3)) \geq h^1(\mathcal{I}_C(4))$ and $h^1(\mathcal{I}_C(2)) \geq h^1(\mathcal{I}_C(3)) + 13 - d \geq d + 4$. Hence $h^0(\mathcal{I}_C(2)) \geq 18 - d$, a contradiction.

Now assume deg($T$) = 5. In this case $T$ is linked to a plane by the complete intersection $T$ and hence $T \cap H$ is linked to a plane by a complete intersection of a quadric and a cubic. Hence $T \cap H$ is arithmetically Cohen-Macaulay with degree 5 and arithmetic genus 2 ([18, Theorem 1.1 (a)], [22], [21, Proposition 3.1]). Thus $h^1(H, \mathcal{I}_{C\cap H,H}(4)) = h^1(T \cap H, \mathcal{I}_{C\cap H,H}(4)) = 0$ and $h^1(H, \mathcal{I}_{C\cap H,H}(3)) \leq 2$. We get $h^1(\mathcal{I}_C(2)) \leq 2d - 11$ and hence $h^0(\mathcal{I}_C(2)) \geq 3$, a contradiction.

(a2) Now assume that there is a line $L \subset \mathbb{P}^4$ with deg($L \cap C$) $\geq 7$. By Lemma 4 we may assume $h^1(\mathcal{I}_C(5)) \geq 2d - 11$. Lemma 7 gives $h^1(\mathcal{I}_C(4)) \geq 2d - 10$ and hence $h^1(\mathcal{I}_C(3)) \geq 2d - 12 \geq 7$. We get $h^0(\mathcal{I}_C(3)) > 5$. We repeat the proof of step (a1) with a loss of 1; for instance, if deg($T$) = 4 (resp. deg($T$) = 5) we get $h^1(\mathcal{I}_C(2)) \geq d + 3$ and $h^0(\mathcal{I}_C(2)) \geq 17 - d$ (resp. $h^1(\mathcal{I}_C(2)) \geq 2d - 12$ and hence $h^0(\mathcal{I}_C(2)) \geq 2$), a contradiction.

(a3) Assume the existence of a conic $D$ with deg($D \cap C$) $\geq 12$, but that there is no line $L \subset \mathbb{P}^4$ with deg($L \cap C$) $\geq 7$. By Lemma 6 we may assume that $D$ is smooth.

(a3.1) Assume for moment $h^1(\mathcal{I}_C(5)) \geq 2d - 12$. Lemma 7 gives $h^1(\mathcal{I}_C(4)) \geq 2d - 11$. The case $t = 4$ of (3) and [10, Lemma 3.9] give $h^1(\mathcal{I}_C(3)) \geq 2d - 13$. Hence $h^0(\mathcal{I}_C(3)) \geq 35 - 14 - d > 5$. As in step (a1) we first get $h^1(\mathcal{I}_C(3)) \geq h^1(\mathcal{I}_C(4))$ and then $h^1(\mathcal{I}_C(2)) \geq h^1(\mathcal{I}_C(3)) - 1$. Thus $h^0(\mathcal{I}_C(2)) \geq 2$, contradicting our assumption.

(a3.2) Now we justify the assumption made in step (a3.1). If $Q$ is a quadric with vertex a line, then we may assume $h^1(\mathcal{I}_C(5)) \geq 2d - 10$ by Remark 1. If $Q$ is a quadric cone with vertex a point $o$ and $o \notin C$, then we may assume $h^1(\mathcal{I}_C(3)) \geq 2d - 12$ by Remark 1. Now assume that $C$ is a contained in a quadric cone $Q$ with vertex a point $o \in C$. It is sufficient to prove that for each irreducible component $\Delta$ of the set of all non-degenerate $Y \subset M_q$ with $Y \subset C$ and $o \in Y$ a general $Y \in \Delta$ has no conic $D$ with deg($D \cap Y$) $\geq 12$ or that if $C \subset \Delta$, then it may be deformed to $Y \in \Delta$ with no offending conic. Bezout gives $D \subset Q$. We need to distinguish the case $o \in D$ and $o \notin D$. First assume $o \in D$. Fix $Z \subset D \cap C$ with deg($Z$) = 12 and $o \in Z_{\text{red}}$. Since $D$ has $\infty^{12}$ zero-dimensional schemes with degree 12 and $Q$ has $\infty^{6}$ conics through $o$, it is sufficient to prove that $h^0(N_{C,Q}(-Z)) < 3d + 1 - 5 - 12$. We have $h^0(N_{C,Q}(-Z)) \leq 3d + 1 - 12 - 7$ by Lemma 1. If $o \notin D$ we use the same proof, just using that $Q$ has $\infty^{6}$ conics.

The case of a smooth $Q$ is similar.

(b) Now assume $h^0(\mathcal{I}_C(2)) \geq 2$. By Lemmas 2 and 3 $C$ is contained in an integral complete intersection of 2 quadrics and we may assume that
$h^1(I_C(5)) \geq 4d-24$. Hence as in step (a) we get $h^1(I_C(3)) \geq 4d-24$, $h^1(I_C(2)) \geq 3d-13$ and hence $h^0(I_C(2)) > 2$, contradicting Lemma 3.

3 Proof of Proposition 1

Remark 2. Fix an integer $d \geq 13$ and $C \in M_d$ contained in a hyperplane $H \subset \mathbb{P}^4$. Since $h^0(H, I_C(5)) = 56$, we have $h^1(I_C(5)) \geq 5(d-11) > 0$.

Proof of Proposition 1: Take $C \in M_d$ contained in a hyperplane $H \subset \mathbb{P}^4$ and contained in some $W \in \mathcal{W}$. Let $S \subset H$ be a degree $\alpha$ hypersurface. Since $\alpha$ is the minimal degree of a surface of $H$ containing $C$ and $C$ is irreducible, $S$ is irreducible. Since $C \subset W \cap H$, we have $\alpha \leq 5$.

(a) Assume $\alpha = 2$. If $S$ is smooth, then up to a change of the ruling of $S$ we may assume $C \in |O_S(1, d-1)|$. Since $d-1 > 5$, $W \supset S$, contradicting the Lefschetz theorem which implies that all surfaces contained in $W$ have degree divisible by 5. If $S$ is a cone, then any smooth curve on it is projectively normal ([11, Ex. V.2.9]), contradicting Remark 2.

(b) Assume $\alpha = 3$. Bezout implies $h^0(H, I_C(3)) = 1$. By the Lefschetz theorem we have $S \nsubseteq W$. Since $C \subseteq S \cap W$, we get $d \leq 15$. The case $d = 15$ is excluded, because the $\omega_{S\cap W} \cong O_{S\cap W}(4)$ and so $S \cap W \neq C$. The case $d = 14$ is excluded, because it would give that the complete intersection $S \cap W$ would link $C$ to a line and hence it is arithmetically normal ([18], [21], [22]), contradicting Remark 2. Now assume $d = 13$. In this case $S \cap W$ links $C$ to a degree 2 locally Cohen-Macaulay curve $D$. If $D$ is a plane curve, then $C$ is arithmetically Cohen-Macaulay, contradicting Remark 2. If $D$ is a disjoint union of 2 lines, then $p_a(D) = -1$, contradicting [21, Proposition 3.1]. Now assume that $D$ is a double structure on a line $L$, but it is not a conic, i.e. that $D$ is not a conic. Since $S \cap W$ links $C \cup L$ to $L$, $C \cup L$, we have $p_a(C \cup L) - p_a(L) = 2(11-1)$ ([21, Proposition 3.1]), i.e. $p_a(C \cup L) = 20$, and hence $\deg(C \cap L) = 21$, contradicting the inequality $d < 21$.

(c) Assume $\alpha = 4$. Since $C \subseteq W \cap S$, we have $d \leq 20$. We exclude the cases $d = 20$ and $d = 19$ as in step (b). Now assume $d = 18$. $S \cap W$ links $C$ to a degree 2 locally Cohen-Macaulay curve $D$. If $D$ is a plane curve, then $C$ is arithmetically Cohen-Macaulay, contradicting Remark 2. Now assume that $D$ is a double structure on a line $L$, but it is not a conic, i.e. that $D$ is not a conic. Since $S \cap W$ links $C \cup L$ to $L$, $C \cup L$, we have $p_a(C \cup L) - p_a(L) = (17-1)5/2$ ([21, Proposition 3.1]) and hence $\deg(C \cap L) > 40$, a contradiction.
References


