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Congruences modulo 3 for two interesting partitions arising from two theta function identities

Kuwali Das
Department of Mathamatical Sciences, Bodoland University, Kokrajhar-783370, Assam, INDIA kwldas90@gmail.com

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Abstract. We find several interesting congruences modulo 3 for 5-core partitions and two color partitions.

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Introduction

A partition of n is a non-increasing sequence of positive integers, called parts, whose sum is n. A partition of n is called a t-core of n if none of the hook numbers is a multiple of t. If $a_t(n)$ denotes the number of t-cores of n, then the generating function for $a_t(n)$ is [5]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}, \tag{1}$$

where, here and throughout the sequel, for any complex number a and $|q|<1$,

$$(a;q)_\infty := \prod_{n=1}^{\infty}(1-q^n).$$

The following exact formula for $a_5(n)$ in terms of the prime factorization of $n+1$ can be found in [5, Theorem 4]. This theorem follows from an identity of Ramanujan recorded in his famous manuscript on the partition function and tau-function now published with his lost notebook [6, p. 139].

Theorem 1. *Let $n+1 = 5^c p_1^{a_1} \dots p_s^{a_s} q_1^{b_1} \dots q_t^{a_t}$ be the prime factorization of $n+1$ into primes $p_i \equiv 1,4 \pmod 5$, and $q_j \equiv 2,3 \pmod 5$. Then*

$$a_5(n) = 5^c \prod_{i=1}^{s} \frac{p_i^{a_i+1}-1}{p_i-1} \prod_{j=1}^{t} \frac{q_j^{b_j+1}-1}{q_j-1}.$$

Many arithmetical identities and congruences easily follow from the above theorem. For example, for any positive integer n and non-negative integer r, we have

$$a_5(5^\alpha n - 1) = 5^\alpha a_5(n-1) \equiv 0 \pmod{5^\alpha}.$$

Similarly, for a prime $p \equiv 2,3 \pmod 5$ and any non-negative integers n and r, we can easily deduce that

$$a_5(p^\alpha(pn+1)-1) = \frac{p^{\alpha+1}+(-1)^\alpha}{p+1}\, a_5(pn) \equiv 0 \left(\mathrm{mod}\ \frac{p^{\alpha+1}+(-1)^\alpha}{p+1}\right).$$

Next, let $p_k(n)$ denote the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of k. Then the generating function for $p_k(n)$ is given by

$$\sum_{n=0}^{\infty} p_k(n) q^n = \frac{1}{(q;q)_\infty (q^k;q^k)_\infty}.$$

Recently, the following result was proved by Baruah, Ahmed and Dastidar [1].

Theorem 2. *If $k \in \{0,1,2,3,4,5,10,15,20\}$, then for any non-negative integer n,*

$$p_k(25n+\ell) \equiv 0 \pmod 5,$$

where $k+\ell = 24$.

In this paper, we find some interesting congruences modulo 3 for 5-core partitions and $p_5(n)$ by employing Ramanujan's theta functions and their dissections. Since our proofs mainly rely on various properties of Ramanujan's theta functions and dissections of certain q-products, we end this section by defining a t-dissection and Ramanujan's general theta function and some of its special cases. If $P(q)$ denotes a power series in q, then a t-dissection of $P(q)$ is given by

$$P(q) = \sum_{k=0}^{t-1} q^k P_k(q^t),$$

where P_k are power series in q^t. In the remainder of this section, we introduce

Ramanujan's theta functions and some of their elementary properties, which will be used in our subsequent sections. For $|ab| < 1$, Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

In this notation, Jacobi's famous triple product identity [3, p. 35, Entry 19] takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2}$$

Three important special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \frac{(q^2;q^2)_\infty(-q;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{4}$$

and

$$f(-q) := f(-q,-q^2) = (q;q)_\infty, \tag{5}$$

where the product representations in (4) and (5) arise from (2) and the last equality in (5) is Euler's famous pentagonal number theorem. After Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty} \tag{6}$$

and hence,

$$\chi(-q) := (q;q^2)_\infty = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}. \tag{7}$$

Furthermore, the q-product representations of $\varphi(-q)$ and $\psi(-q)$ can be written in the forms

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}, \quad \psi(-q) = \frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty} \tag{8}$$

Lemma 1. [4, Theorem 2.2] *For any prime $p \geq 5$,*

$$(q;q)_\infty = \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2};q^{p^2})_\infty, \quad (9)$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6; \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, for $\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2}$ *and* $k \neq \frac{(\pm p-1)}{6}$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

Some congruences modulo 3 for 5-core partitions

We first introduce an important lemma which will be used later.

Lemma 2. *We have*

$$(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2 \equiv q^2(q^6;q^6)_\infty^2(q^{30};q^{30})_\infty^2 + (q^6;q^6)_\infty^4 + 2q^4(q^{30};q^{30})_\infty^4 \pmod 3. \quad (10)$$

Proof. We note that [3, p. 258],

$$\varphi^2(q) - \varphi^2(q^5) = 4q\chi(q)f(-q^5)f(-q^{20}) \quad (11)$$

and

$$5\varphi^2(q^5) - \varphi^2(q) = 4\chi(q)\chi(-q^5)\psi^2(-q). \quad (12)$$

Multiplying (11) and (12)

$$6\varphi^2(q)\varphi^2(q^5) - \varphi^4(q) - 5\varphi^4(q^5) = 16q\chi^2(q)\chi(q^5)f(-q^5)f(-q^{20})\psi(-q). \quad (13)$$

Employing (5), (6), (7) and (8) in the last equation, we obtain

$$6\varphi^2(q)\varphi^2(q^5) - \varphi^4(q) - 5\varphi^4(q^5) = 16q(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2. \quad (14)$$

Taking congruences modulo 3, we obtain

$$q(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2 \equiv -\varphi(q)\varphi(q^3)+\varphi(q^5)\varphi(q^{15}) \pmod 3. \tag{15}$$

Recalling [3, p.49, Corollary (i)], we have

$$\varphi(q)=\varphi(q^9)+2qf(q^3,q^{15}) \tag{16}$$

Replacing q by q^5 in the above equation, we have,

$$\varphi(q^5)=\varphi(q^{45})+2q^5f(q^{15},q^{75}) \tag{17}$$

Employing the above two equation in (15), we find that

$$\begin{aligned} q(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2 \equiv & -\varphi(q^3)\varphi(q^9)+q\varphi(q^3)f(q^3,q^{15})+\varphi(q^{15})\varphi(q^{45}) \\ & +2q^5\varphi(q^{15})f(q^{15},q^{75}) \pmod 3. \end{aligned} \tag{18}$$

Note that

$$\varphi(q)f(q,q^5)=\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty^3(q^4;q^4)_\infty^3(q^6;q^6)_\infty} \tag{19}$$

Since, $(q;q)_\infty^3 \equiv (q^3;q^3)_\infty$, so (19) can be written as

$$\varphi(q)f(q,q^5) \equiv (q^2;q^2)_\infty^4 \pmod 3. \tag{20}$$

Employing (15) and (20) in (18), we obtain

$$\begin{aligned} & q(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2 \equiv \\ & \quad q^3(q^6;q^6)_\infty^2(q^{30};q^{30})_\infty^2+q(q^6;q^6)_\infty^4+2q^5(q^{30};q^{30})_\infty^4 \pmod 3. \end{aligned} \tag{21}$$

From above congruence we can easily obtain (10). QED

Theorem 3. *We have,*

$$\sum_{n=0}^{\infty} a_5(3n)q^n \equiv (q^3;q^3)_\infty(q^5;q^5)_\infty \pmod 3, \tag{22}$$

$$\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv (q;q)_\infty(q^{15};q^{15})_\infty \pmod 3, \tag{23}$$

and

$$\sum_{n=0}^{\infty} a_5(3n+2)q^n \equiv 2\sum_{n=0}^{\infty} a_5(n)q^n \pmod 3. \tag{24}$$

Proof. Putting $t=5$ in (3.7) and replacing q by q^2, we have

$$\sum_{n=0}^{\infty} a_5(n)q^{2n} = \frac{(q^{10};q^{10})_\infty^5}{(q^2;q^2)_\infty}. \tag{25}$$

Taking congruences modulo 3,

$$\sum_{n=0}^{\infty} a_5(n)q^{2n} \equiv \frac{(q^{30};q^{30})_\infty (q^2;q^2)_\infty^2 (q^{10};q^{10})_\infty^2}{(q^6;q^6)_\infty} \pmod 3. \tag{26}$$

With the help of (10), (26) can be rewritten as

$$\begin{aligned}\sum_{n=0}^{\infty} a_5(n)q^{n} \equiv & \frac{(q^{30};q^{30})_\infty}{(q^6;q^6)_\infty}(q^2(q^6;q^6)_\infty^2(q^{30};q^{30})_\infty^2 & (27)\\ & + (q^6;q^6)_\infty^4 + 2q^4(q^{30};q^{30})_\infty^4) \pmod 3 \\ \equiv & q^2(q^6;q^6)_\infty(q^{30};q^{30})_\infty^3 + (q^6;q^6)_\infty^3(q^{30};q^{30})_\infty & (28)\\ & + 2q^4\frac{(q^{30};q^{30})_\infty^5}{q^6;q^6)_\infty} \pmod 3 \\ \equiv & q^2(q^6;q^6)_\infty(q^{90};q^{90})_\infty + (q^{18};q^{18})_\infty(q^{30};q^{30})_\infty & (29)\\ & + 2q^4\sum_{n=0}^{\infty} a_5(n)q^{6n} \pmod 3. \end{aligned}$$

Comparing the terms involving q^{6n}, q^{6n+2} and q^{6n+4} respectively from both sides of the above congruence, we can easily obtain (22)–(24). QED

Applying the mathematical induction in (24), we can easily obtain the following

Corollary 1. *For any nonnegative integers k and n, we have*

$$a_5\left(3^k n + 3^k - 1\right) \equiv 2^k a_5(n) \pmod 3.$$

Theorem 4. *We have,*

$$a_5(15n+6) \equiv 0 \pmod 3, \tag{30}$$

$$a_5(15n+12) \equiv 0 \pmod 3, \tag{31}$$

and

$$a_5(15n+9) \equiv 2a_5(3n+1) \pmod 3. \tag{32}$$

Proof. From [3, p. 270, Entry 12(v)], we recall that

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left(\frac{A(q^5)}{B(q^5)} - q - q^2 \frac{B(q^5)}{A(q^5)} \right) \tag{33}$$

where

$$A(q) = \frac{f(-q^{10},-q^{15})}{f(-q^5,-q^{20})} \text{ and } B(q) = \frac{f(-q^5,-q^{20})}{f(-q^{10},-q^{15})}.$$

Replacing q by q^3 in (33) and then employing in (22), we obtain

$$\sum_{n=0}^{\infty} a_5(3n)q^n \equiv (q^5;q^5)_\infty (q^{75};q^{75})_\infty \left(\frac{A(q^{15})}{B(q^{15})} - q^3 - q^6 \frac{B(q^{15})}{A(q^{15})} \right) \pmod 3 \tag{34}$$

Extracting the terms involving q^{5n+3} from both sides of the congruence, we obtain,

$$\sum_{n=0}^{\infty} a_5(3(5n+3))q^n \equiv 2(q;q)_\infty (q^{15};q^{15})_\infty \pmod 3 \tag{35}$$

Employing (23) in (35), we can easily obtain (32).

We have seen that in the right hand side of the congruence (34), there is no terms involving q^{5n+2} and q^{5n+4} and hence we can easily obtain (30) and (31) respectively. QED

Theorem 5. *We have,*

$$a_5(15n+4) \equiv 2a_5(3n) \pmod 3, \tag{36}$$

$$a_5(15n+10) \equiv 0 \pmod 3, \tag{37}$$

and

$$a_5(15n+13) \equiv 0 \pmod 3. \tag{38}$$

Proof. Employing (33) in (23), we obtain

$$\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv (q^{15};q^{15})_\infty (q^{25};q^{25})_\infty \left(\frac{A(q^5)}{B(q^5)} - q - q^2 \frac{B(q^5)}{A(q^5)} \right) \pmod 3 \tag{39}$$

Extracting the terms involving q^{5n+1} from both sides of the congruence, we obtain,

$$\sum_{n=0}^{\infty} a_5(3(5n+1)+1)q^n \equiv 2(q^3;q^3)_\infty(q^5;q^5)_\infty \pmod 3 \tag{40}$$

Employing (22) in (40), we can easily obtain (36).

It is clear that in the right hand side of the congruence (39), there is no terms involving q^{5n+3} and q^{5n+4} and hence we can easily obtain (37) and (38) . QED

Theorem 6. *For any prime $p \geq 5$ with $\left(\frac{-15}{p}\right) = -1$ and for any non-negative integers k and n,*

$$a_5\left(3\cdot p^{2k}n + p^{2k} - 1\right) \equiv a_5(3n) \pmod 3. \tag{41}$$

Proof. With the help of (9), (22) can be rewritten as

$$\begin{aligned}\sum_{n=0}^{\infty} a_5(3n)q^n \equiv &\Big[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{3\cdot\frac{3k^2+k}{2}} f(-q^{3\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{3\cdot\frac{3p^2-(6k+1)p}{2}}) + \\ &(-1)^{\frac{\pm p-1}{6}} q^{3\cdot\frac{p^2-1}{24}} f(-q^{3p^2})\Big] \\ &\times \Big[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{5\cdot\frac{3k^2+k}{2}} f(-q^{5\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{5\cdot\frac{3p^2-(6k+1)p}{2}}) \\ &+ (-1)^{\frac{\pm p-1}{6}} q^{5\cdot\frac{p^2-1}{24}} f(-q^{5p^2})\Big] \pmod 3. \end{aligned} \tag{42}$$

Now we consider the congruence

$$3\cdot\frac{(3k^2+k)}{2} + 5\cdot\frac{(3m^2+m)}{2} \equiv\ 8\cdot\frac{(p^2-1)}{24} \pmod p, \tag{43}$$

where $-(p-1)/2 \leq k,\ m \leq (p-1)/2$, with $\left(\frac{-15}{p}\right) = -1$. Since the above congruence is equivalent to

$$(18k+3)^2 + 15\cdot(6m+1)^2 \equiv 0 \pmod p,$$

and $\left(\frac{-15}{p}\right) = -1$, there is only one solution $k = m = \pm p - 1/6$ for (43). That is, there are no other k and m such that $3\frac{(3k^2+k)}{2} + 5 \cdot \frac{(3m^2+m)}{2}$ and $8 \cdot \frac{(p^2-1)}{24}$ are in the same residue class modulo p. Therefore, equating the terms involving $q^{pn+8\cdot\frac{p^2-1}{24}}$ from both sides of (42), we deduce that

$$\sum_{n=0}^{\infty} a_5\left(3\left(pn + 8\cdot\frac{p^2-1}{24}\right)\right)q^n = \sum_{n=0}^{\infty} a_5\left(3pn + p^2 - 1\right)q^n$$
$$\equiv (q^{3p};q^{3p})_\infty (q^{5p};q^{5p})_\infty \pmod 3.$$

Thus,

$$\sum_{n=0}^{\infty} a_5\left(3p^2n + p^2 - 1\right)q^n \equiv (q^3;q^3)_\infty (q^5;q^5)_\infty \pmod 3. \tag{44}$$

From (44) and (22), we arrive at

$$a_5\left(3p^2n + p^2 - 1\right) \equiv a_5(3n) \pmod 3.$$

Now (41) can be established easily by mathematical induction. QED

Corollary 2. *For any prime* $p \geq 5$ *with* $\left(\frac{-15}{p}\right) = -1$ *and for any non-negative integers* k *and* n*,*

$$a_5\left(3\cdot p^{2k+2}n + (3i+p)p^{2k+1} - 1\right) \equiv 0 \pmod 3,$$

where $i = 1,\ 2,\ \ldots,\ p-1$.

Proof. As in the proof of the previous theorem, it can also be shown that

$$\sum_{n=0}^{\infty} a_5\left(3\cdot p^{2k}(pn + 8\frac{p^2-1}{24}) + p^{2k} - 1\right)q^n \equiv (q^{3p};q^{3p})_\infty (q^{5p};q^{5p})_\infty \pmod 3,$$

that is,

$$\sum_{n=0}^{\infty} a_5\left(3\cdot p^{2k+1}n + p^{2k+2} - 1\right)q^n \equiv (q^{3p};q^{3p})_\infty (q^{5p};q^{5p})_\infty \pmod 3.$$

Since there are no terms on the right side of the above congruence in which the powers of q are congruent to $1,\ 2,\ \ldots,\ p-1$ modulo p, it follows, for $i = 1,\ 2,\ \ldots,\ p-1$, that

$$a_5\left(3\cdot p^{2k+1}(pn+i) + p^{2k+2} - 1\right) \equiv 0 \pmod 3,$$

which is clearly equivalent to the proffered congruence. QED

Corollary 3. *For any prime $p \geq 5$ with $\left(\frac{-15}{p}\right) = -1$ and for any non-negative integers k and n,*

$$a_5\left(15 \cdot p^{2k}n + (3r+1)p^{2k} - 1\right) \equiv 0 \pmod 3,$$

where $r = 2,\ 4$.

Proof. It can also be shown that

$$\sum_{n=0}^{\infty} a_5\left(3 \cdot p^{2k}n + p^{2k} - 1\right) q^n \equiv (q^3;q^3)_\infty (q^5;q^5)_\infty \pmod 3.$$

In (33), replacing q by q^3, we can see that $(q^3;q^3)_\infty(q^5;q^5)_\infty$ has no terms q^{5n+r} where $r = 2,\ 4$, it follows that

$$a_5\left(3 \cdot p^{2k}(5n+r) + p^{2k} - 1\right) \equiv 0 \pmod 3,$$

which is clearly equivalent to the proffered congruence. $\boxed{QED}$

Theorem 7. *For any prime $p \geq 5$ with $\left(\frac{-15}{p}\right) = -1$ and for any non-negative integers k and n,*

$$a_5\left(3 \cdot p^{2k}n + 2 \cdot p^{2k} - 1\right) \equiv a_5(3n+1) \pmod 3. \tag{45}$$

Proof. With the help of (9), (23) can be rewritten as

$$\begin{aligned}\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv & \Big[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}})+ \\ & (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2})\Big] \\ & \times \Big[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{15\cdot\frac{3k^2+k}{2}} f(-q^{15\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{15\cdot\frac{3p^2-(6k+1)p}{2}}) \\ & + (-1)^{\frac{\pm p-1}{6}} q^{15\cdot\frac{p^2-1}{24}} f(-q^{15p^2})\Big] \pmod 3.\end{aligned} \tag{46}$$

Consider the congruence

$$\cdot\frac{(3k^2+k)}{2} + 15 \cdot \frac{(3m^2+m)}{2} \equiv\ 16 \cdot \frac{(p^2-1)}{24} \pmod p, \tag{47}$$

where $-(p-1)/2 \leq k,\ m \leq (p-1)/2$, with $\left(\frac{-15}{p}\right) = -1$. Since the above congruence is equivalent to

$$(6k+1)^2 + 15 \cdot (6m+1)^2 \equiv 0 \pmod{p},$$

and $\left(\frac{-15}{p}\right) = -1$, there is only one solution $k = m = \pm p - 1/6$ for (47). That is, there are no other k and m such that $\frac{(3k^2+k)}{2} + 15 \cdot \frac{(3m^2+m)}{2}$ and $16 \cdot \frac{(p^2-1)}{24}$ are in the same residue class modulo p. Therefore, equating the terms involving $q^{pn+16\cdot\frac{p^2-1}{24}}$ from both sides of (46), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} a_5\left(3\left(pn + 16\cdot\frac{p^2-1}{24}\right)+1\right)q^n &= \sum_{n=0}^{\infty} a_5\left(3pn + 2p^2 - 1\right)q^n \\ &\equiv (q^p;q^p)_\infty (q^{15p};q^{15p})_\infty \pmod{3}.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} a_5\left(3p^2 n + 2p^2 - 1\right)q^n \equiv (q;q)_\infty (q^{15};q^{15})_\infty \pmod{3}. \tag{48}$$

From (48) and (23), we arrive at

$$a_5\left(3p^2 n + 2p^2 - 1\right) \equiv a_5(3n+1) \pmod{3}.$$

Now (45) can be established easily by mathematical induction. QED

The following results follow in a similar fashion. So we omit the proof.

Corollary 4. *For any prime $p \geq 5$ with $\left(\frac{-15}{p}\right) = -1$ and for any non-negative integers k and n,*

$$a_5\left(3\cdot p^{2k+2} + (3i+2p)p^{2k+1} - 1\right) \equiv 0 \pmod{3},$$

where $i = 1,\ 2,\ \ldots,\ p-1$.

Corollary 5. *For any prime $p \geq 5$ with $\left(\frac{-15}{p}\right) = -1$ and for any non-negative integers k and n,*

$$a_5\left(15\cdot p^{2k} + (3r+2)p^{2k} - 1\right) \equiv 0 \pmod{3},$$

where $r = 3,\ 4$.

Some congruences modulo 3 for two color partitions $p_5(n)$

Theorem 8. *We have,*

$$\sum_{n=0}^{\infty} p_5(3n)q^n \equiv \frac{(q^3;q^3)_\infty}{(q^5;q^5)_\infty} \pmod 3, \tag{49}$$

$$\sum_{n=0}^{\infty} p_5(3n+1)q^n \equiv (q;q)_\infty (q^5;q^5)_\infty \pmod 3, \tag{50}$$

and

$$\sum_{n=0}^{\infty} p_5(3n+2)q^n \equiv 2\frac{(q^{15};q^{15})_\infty}{(q;q)_\infty} \pmod 3. \tag{51}$$

where

$$\sum_{n=0}^{\infty} p_5(n)q^n := \frac{1}{(q;q)_\infty (q^5;q^5)_\infty}.$$

Proof. We have

$$\sum_{n=0}^{\infty} p_5(n)q^n = \frac{1}{(q;q)_\infty (q^5;q^5)_\infty} = \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty (q^5;q^5)_\infty^6}.$$

Taking congruences modulo 3 in the above equation, we obtain,

$$\begin{aligned}\sum_{n=0}^{\infty} p_5(n)q^n &\equiv \frac{(q^5;q^5)_\infty^5}{(q^{15};q^{15})_\infty^2 (q;q)_\infty} \pmod 3 \\ &\equiv \frac{1}{(q^{15};q^{15})_\infty^2} \sum_{n=0}^{\infty} a_5(n)q^n \end{aligned} \tag{52}$$

Comparing the terms involving q^{3n}, q^{3n+1} and q^{3n+2} respectively from the both sides of the above congruence, we have

$$\sum_{n=0}^{\infty} p_5(3n)q^n \equiv \frac{1}{(q^5;q^5)_\infty^2} \sum_{n=0}^{\infty} a_5(3n)q^n \pmod 3, \tag{53}$$

$$\sum_{n=0}^{\infty} p_5(3n+1)q^n \equiv \frac{1}{(q^5;q^5)_\infty^2} \sum_{n=0}^{\infty} a_5(3n+1)q^n \pmod 3, \tag{54}$$

and

$$\sum_{n=0}^{\infty} p_5(3n+2)q^n \equiv 2\frac{1}{(q^5;q^5)_\infty^2}\sum_{n=0}^{\infty} a_5(3n+2)q^n \pmod 3. \tag{55}$$

Employing (3) in the above congruences, we can easily obtain the (8). QED

Theorem 9. *We have,*

$$p_5(15n+9) \equiv p_5(3n+2) \pmod 3, \tag{56}$$
$$p_5(15n+6) \equiv 0 \pmod 3, \tag{57}$$

and

$$p_5(15n+12)q^n \equiv 0 \pmod 3. \tag{58}$$

Proof. Replacing q by q^3 in(33) and then employing in(49), we obtain

$$\sum_{n=0}^{\infty} p_5(3n)q^n \equiv \frac{(q^{75};q^{75})_\infty}{(q^5;q^5)_\infty}\left(\frac{A(q^{15})}{B(q^{15})} - q^3 - q^6\frac{B(q^{15})}{A(q^{15})}\right) \pmod 3 \tag{59}$$

Comparing the terms involving $q^{5n+3}$from both sides of the above congruence we can easily arrive at (56). There is no terms involving q^{5n+2}, q^{5n+4} in the right hand side of the congruence (59). So, we can easily obtain (57) and (58). QED

Theorem 10. *For any $k \geq 1$, we have*

$$\sum_{n=0}^{\infty} p_5\left(3^{2k-1}n + \frac{3^{2k-1}+1}{4}\right)q^n \equiv 2^{k+1}(q;q)_\infty(q^5;q^5)_\infty \pmod 3. \tag{60}$$

Proof. We prove the result by mathematical induction. For $k=1$, we obtain

$$\sum_{n=0}^{\infty} p_5(3n+1)q^n \equiv (q;q)_\infty(q^5;q^5)_\infty \pmod 3, \tag{61}$$

which is the (50).

Let (60) be true for some positive integer k. Therefore we can write (60) as

$$\sum_{n=0}^{\infty} p_5\left(3^{2k-1}n + \frac{3^{2k-1}+1}{4}\right)q^n \tag{62}$$

$$\equiv 2^{k+1}(q^3;q^3)_\infty(q^{15};q^{15})_\infty \times \frac{1}{(q;q)_\infty^2(q^5;q^5)_\infty^2} \pmod 3 \tag{63}$$

$$\equiv 2^{k+1}(q^3;q^3)_\infty(q^{15};q^{15})_\infty \times \left(\sum_{n=0}^{\infty} p_5(n)q^n\right)^2 \pmod 3.$$

Extracting the terms involving q^{3n+2} from both sides of the above congruence, we obtain

$$\sum_{n=0}^{\infty} p_5\left(3^{2k-1}(3n+2)+\frac{3^{2k-1}+1}{4}\right)q^n \equiv$$
$$2^{k+1}(q;q)_\infty(q^5;q^5)_\infty\times$$
$$\Big[\Big(\sum_{n=0}^{\infty} p_5(3n+1)q^n\Big)^2 + 2\sum_{n=0}^{\infty} p_5(3n)q^n \sum_{n=0}^{\infty} p_5(3n+2)q^n\Big] \pmod 3. \quad (64)$$

Employing (49), (50) and (51) in (64),

$$\sum_{n=0}^{\infty} p_5\left(3^{2k-1}(3n+2)+\frac{3^{2k-1}+1}{4}\right)q^n \quad (65)$$
$$\equiv 2^{k+1}(q;q)_\infty(q^5;q^5)_\infty\Big[(q;q)_\infty^2(q^5;q^5)_\infty^2 + 4\frac{(q^3;q^3)_\infty(q^{15};q^{15})_\infty}{(q^5;q^5)_\infty(q;q)_\infty}\Big] \pmod 3$$
$$\equiv 2^{k+2}(q;q)_\infty(q^5;q^5)_\infty \pmod 3.$$

Hence the result is true for all $k \geq 1$. QED

Applying (33) we can see easily that $(q;q)_\infty(q^5;q^5)_\infty$ has no terms containing q^{5n+r} for $r = 3,\ 4$. From the last result we can conclude the following

Theorem 11. *For any $k \geq 1$, we have*

$$\sum_{n=0}^{\infty} p_5\left(3^{2k-1}n+\frac{(4r+1)3^{2k-1}+1}{4}\right)q^n \equiv 0 \pmod 3, \quad (66)$$

where, $r = 3,\ 4$.

Theorem 12. *For any prime $p \geq 5$ with $\left(\frac{-5}{p}\right) = -1$ and for any non-negative integers k and n,*

$$p_5\left(3\cdot p^{2k}n+\frac{3p^{2k}+1}{4}\right) \equiv p_5(3n+1) \pmod 3. \quad (67)$$

Proof. With the help of (9), (50) can be rewritten as

$$\sum_{n=0}^{\infty} p_5(3n+1)q^n \equiv \Big[\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}})+$$

$$(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2})\Big]$$

$$\times \Big[\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{5\cdot\frac{3k^2+k}{2}} f(-q^{5\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{5\cdot\frac{3p^2-(6k+1)p}{2}})$$

$$+ (-1)^{\frac{\pm p-1}{6}} q^{5\cdot\frac{p^2-1}{24}} f(-q^{5p^2})\Big] \pmod 3. \tag{68}$$

Consider the following congruence

$$\frac{(3k^2+k)}{2} + 5\cdot\frac{(3m^2+m)}{2} \equiv 6\cdot\frac{(p^2-1)}{24} \pmod p, \tag{69}$$

where $-(p-1)/2 \leq k,\ m \leq (p-1)/2$, with $\left(\frac{-5}{p}\right) = -1$. Since the above congruence is equivalent to

$$(6k+1)^2 + 5\cdot(6m+1)^2 \equiv 0 \pmod p,$$

and $\left(\frac{-5}{p}\right) = -1$, there is only one solution $k = m = \pm p - 1/6$ for (47). That is, there are no other k and m such that $\frac{(3k^2+k)}{2} + 5\cdot\frac{(3m^2+m)}{2}$ and $6\cdot\frac{(p^2-1)}{24}$ are in the same residue class modulo p. Therefore, equating the terms involving $q^{pn+\cdot\frac{p^2-1}{4}}$ from both sides of (68), we deduce that

$$\sum_{n=0}^{\infty} p_5\left(3\left(pn+\cdot\frac{p^2-1}{4}\right)+1\right)q^n = \sum_{n=0}^{\infty} p_5\left(3pn+\frac{3p^2+1}{4}\right)q^n$$

$$\equiv (q^p;q^p)_\infty (q^{5p};q^{5p})_\infty \pmod 3.$$

Thus,

$$\sum_{n=0}^{\infty} p_5\left(3p^2n+\frac{3p^2+1}{4}\right)q^n \equiv (q;q)_\infty (q^5;q^5)_\infty \pmod 3. \tag{70}$$

From (70) and (50), we arrive at

$$p_5\left(3p^2n+\frac{3p^2+1}{4}\right) \equiv p_5(3n+1) \pmod 3.$$

Now (67) can be established easily by mathematical induction. *QED*

The following results follow in a similar fashion. So we omit the proof.

Corollary 6. *For any prime $p \geq 5$ with $\left(\frac{-5}{p}\right) = -1$ and for any non-negative integers k and n,*

$$p_5\left(3 \cdot p^{2k+2} + \frac{(12i+3p)p^{2k+1}+1}{4}\right) \equiv 0 \pmod 3,$$

where $i = 1,\ 2,\ \ldots,\ p-1$.

Corollary 7. *For any prime $p \geq 5$ with $\left(\frac{-5}{p}\right) = -1$ and for any non-negative integers k and n,*

$$p_5\left(15 \cdot p^{2k} + \frac{(12r+3)p^{2k}+1}{4}\right) \equiv 0 \pmod 3,$$

where $r = 3,\ 4$.

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