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On some arithmetic properties of finite groups

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Abstract. Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$. A group G is called *σ-primary* if G is a finite σ_i-group for some $\sigma_i \in \sigma$. We say that a finite group G is: *σ-soluble* if every chief factor of G is σ-primary; *σ-nilpotent* if G is the direct product of some σ-primary groups.

Being based on these concepts, we develop and unify some aspects of the theories of soluble, nilpotent, supersoluble and quasinilpotent groups, the subgroup lattices theory, the theories of generalized quasinormal and generalized subnormal subgroups.

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Introduction

Throughout this paper, G and G^* always denote a finite group and an arbitrary group, respectively. The subgroups A and B of G^* are called *permutable* if $AB = BA$. In this case, we say also that A *permutes* with B. A *complete Sylow set of* G contains exact one Sylow p-subgroup for every prime p dividing the order of G.

One of the organizing ideas of the group theory is the idea to study groups G depending on the presence in G of a subgroup system $\mathcal{L}$ having desired properties. Often, this approach is more effective if $\mathcal{L}$ forms a sublattice of the lattice $\mathcal{L}(G)$ of all subgroups of G (that is, $A \cap B \in \mathcal{L}$ and $\langle A, B\rangle \in \mathcal{L}$ for all $A, B \in \mathcal{L}$), especially when some subgroups $A, B \in \mathcal{L}$ are permutable since in this case $\langle A, B\rangle = AB$. In order to involve subgroups $A \in \mathcal{L}$ in inductive reasoning, it is also helpful the condition of existence in G subgroup chains of the form

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

with suitable embedding properties for A_{i-1} into A_i. In particular, if here A_{i-1} is normal in A_i for all $i = 1, \ldots, t$, then A is called a *subnormal subgroup* of

G. These circumstances make the general problems of finding sublattices of the lattice $\mathcal{L}(G)$ and studying the conditions of permutability and (generalized) subnormality for subgroups quite important and interesting.

Note that the first results in this line research have been obtained almost simultaneously. Wieland proved [62] that the set $\mathcal{L}_{sn}(G)$ of all subnormal subgroups of G is a sublattice of $\mathcal{L}(G)$. Ore proved [48] that the lattice $\mathcal{L}(G)$ is distributive if and only if G is cyclic, and he showed also, in the paper [49], that if a subgroup H of G is *quasinormal* in G (that is, H permutes with all subgroups of G), then H is subnormal in G. Hall proved [31] that G is soluble if and only if it has a Sylow basis (that is, a complete Sylow set $\mathcal{S}$ of G such that every two members of $\mathcal{S}$ are permutable).

These important results have been developed in a large number of publications. Here, we very briefly mention about some of them.

First of all, recall that the first of the two above-mentioned results of Ore initiated the study of groups G with others constraints on $\mathcal{L}(G)$ and on some sublattices of $\mathcal{L}(G)$. For instance, it was proved that a finite nilpotent group has modular subgroup lattices if and only if every two of its subgroups are permutable. The full description of finite groups with modular subgroup lattice was obtained by Iwasawa [39] (see also Chapter 2 in [52]). Groups G with modular, distributive and complemented lattice $\mathcal{L}_{sn}(G)$ were characterized by Zappa [67], Zacher [65] and Curzio [17], respectively. An original generalization of the lattice $\mathcal{L}_{sn}(G)$ was found by Kegel [19] (see Section 2 below).

Greatly strengthening the second of the above-mentioned results of Ore, Kegel proved [18] that every S-permutable subgroup of G (that is, a subgroup of G which permutes with all Sylow subgroups of G) is subnormal in G, and that the set of all S-permutable subgroups of G forms a sublattice of $\mathcal{L}(G)$. It was also proved (Doerk and Hawkes [21, I, 4.28]) that if $\mathcal{S}$ is a Sylow basis of a soluble group G, then the set $\mathcal{P}(\mathcal{S})$ of all such subgroups of G which permute with all members of $\mathcal{S}$ is a sublattice of $\mathcal{L}(G)$. Deskins proved [20] that if A is an S-permutable subgroup of G, then A/A_G is nilpotent. In the paper [36] (see also Section 3 in [33, VI]), Huppert obtained characterizations of groups G in which every complete Sylow set is a Sylow basis of G. In his other paper [35], Huppert proved that if $\mathcal{S}$ is a basis of a soluble group G and every maximal subgroup of every subgroup $H \in \mathcal{S}$ permutes with all others members of $\mathcal{S}$, then G is supersoluble. In the paper of Asaad and Heliel [5], the last result has been extended to an arbitrary finite group G.

In this review, being based on some ideas in [54, 57, 59], we discuss an arithmetic method for generalizations of all above-mentioned results as well as many other known results, which depends only on the choice of the partition σ of the set of all primes.

1 Basic notation and concepts

In this section, we introduce the most important concepts and notation used in this survey. We use $\mathbb{P}$ to denote the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order $|G|$ of G.

$\Pi_{\mathfrak{J}}$-primary and Π-primary groups. In what follows, $\sigma = \{\sigma_i | \ i \in I \subseteq \{0\} \cup \mathbb{N}\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; Π is always supposed to be a subset of the set σ and $\Pi' = \sigma \setminus \Pi$. We also suppose that $0 \in I$ and $2 \in \sigma_0$.

In practice, we often deal with the following two special partitions of $\mathbb{P}$: $\sigma = \{\{2\}, \{3\}, \ldots\}$ and $\sigma = \{\pi, \pi'\}$.

Modifying the notation for $\sigma(n)$ (n is an integer) in [57, 59], we put

$$\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}; \sigma(G) = \sigma(|G|).$$

We say that: n is a Π-*number* if $\sigma(n) \subseteq \Pi$; G is a Π-*group* if $|G|$ is a Π-number.

Let $\mathfrak{J}$ be a class of finite σ_0-groups which is closed under extensions, epimorphic images and subgroups and which also contains all soluble σ_0-groups. We will say that: (i) G^* is $\Pi_{\mathfrak{J}}$-*primary* provided G^* is either an $\mathfrak{J}$-group or a finite σ_i-group for some $\sigma_i \in \Pi$, where $i \neq 0$; (ii) an integer n is Π-*primary* if n is a σ_i-group for some $\sigma_i \in \Pi$.

In the future, we always omit the symbol $\mathfrak{J}$ in all definitions and notations in the case when $\mathfrak{J}$ is the class of all finite σ_0-groups. Thus we say that G is Π-*primary* if G is a σ_i-group for some $i \in I$.

Note that in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, a Π-primary integer n is simply a power of some prime; a Π-primary group G is simply a *primary* group, that is, G is a group of prime power order.

σ-soluble and σ-nilpotent groups. We say that G is: (i) σ-*soluble* if every chief factor of G is σ-primary; (ii) σ-*nilpotent* if G is the direct product of some σ-primary groups.

It is clear that every σ-nilpotent group is also σ-soluble, and G is σ-soluble if and only if it is σ_i-separable for all $i \in I$.

Example 1. G is soluble (respectively nilpotent) if and only if it is σ-soluble (respectively σ-nilpotent), where $\sigma = \{\{2\}, \{3\}, \ldots\}$.

Example 2. G is π-separable if and only if it is σ-soluble, where $\sigma = \{\pi, \pi'\}$.

Example 3. Let $\pi = \{p_1, \ldots, p_t\}$. Then G is π-soluble if and only if it is σ-soluble, where $\sigma = \{\{p_1\}, \ldots, \{p_t\}, \{p_1, \ldots, p_t\}'\}$.

Example 4. G is p-soluble (respectively p-decomposable) if and only if it is σ-soluble (respectively σ-nilpotent), where $\sigma = \{\{p\}, \{p\}'\}$.

We will use $\mathfrak{S}_\sigma$ and $\mathfrak{N}_\sigma$ to denote the classes of all σ-soluble groups and of all σ-nilpotent groups, respectively.

Π-full groups. A set $1 \in \mathcal{H}$ of subgroups of G is said to be a *complete Hall Π-set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$ and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi \cap \sigma(G)$.

We will say that G is: (i) Π-*full* if G possesses a *complete Hall* Π-*set*; (ii) a D_Π-*group* if G is D_{σ_i}-group for all $\sigma_i \in \sigma$; (iii) a Π-*full group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \Pi$.

Example 5. (i) In view of Theorem A in [54] (see Theorem 8 below), every σ-soluble group is a Π-full group of Sylow type for each $\emptyset \neq \Pi \subseteq \sigma$.

(ii) The group $PSL(2, 11)$ and the Mathieu group M_{11} are σ-full groups of Sylow type, where $\sigma = \{\sigma_i \mid i \in I\}$ such that $\sigma_1 = \{5, 11\}$ and σ_i is a one-element set for all $i \neq 1$; the Lyons group Ly is a σ-full groups of Sylow type, where $\sigma = \{\sigma_i \mid i \in I\}$ such that $\sigma_1 = \{11, 67\}$ and σ_i is a one-element set for all $i \neq 1$.

$\Pi_{\mathfrak{J}}$-subnormal subgroups. Generalizing the concept of σ-subnormality in [57, 59] introduce the following

Definition 1. We say that a subgroup A of G^* is $\Pi_{\mathfrak{J}}$-*subnormal* in G^* if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G^*$$

such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{J}}$-primary for all $i = 1, \ldots, t$.

In this definition, as usual, $(A_{i-1})_{A_i}$ denotes the core of A_{i-1} in A_i.

Note that in the case when $\Pi = \sigma = \{\{2\}, \{3\}, \ldots\}$, a subgroup A of G^* is Π-subnormal in G if and only if A is subnormal in G^*. A subgroup A of G is $\mathfrak{F}$-subnormal in G in the sense of Kegel [19] if and only if it is $\sigma_{\mathfrak{J}}$-subnormal in G, where $\sigma = \{\mathbb{P}\}$ and $\mathfrak{I} = \mathfrak{F}$.

Example 6. Let $C_{29} \rtimes C_7$ be a non-abelian group of order 203 and P be a simple $\mathbb{F}_{11}(C_{29} \rtimes C_7)$-module which is faithful for $C_{29} \rtimes C_7$. Let

$$G = (P \rtimes (C_{29} \rtimes C_7)) \times A_5,$$

where A_5 is the alternating group of degree 5. Let $\sigma = \{\sigma_0, \sigma_1, \sigma_2\}$, where $\sigma_0 = \{2, 3, 5\}$, $\sigma_1 = \{7, 29\}$ and $\sigma_2 = \{2, 3, 5, 7, 29\}'$. Let $\mathfrak{J}$ be the class of all finite soluble σ_0-groups, and let $\Pi = \{\sigma_1, \sigma_2\}$. Then a subgroup H of G of order 4 is σ-subnormal in G but it is neither Π-subnormal in G nor $\sigma_{\mathfrak{J}}$-subnormal in G. The subgroup C_7 is $\Pi_{\mathfrak{J}}$-subnormal in G but it is clearly not subnormal in G.

There are a number of motivations for introducing the concept of $\Pi_{\mathfrak{J}}$-subnormality. First note that being based on this concept, it is possible to expand the class of sublattices of the lattice $\mathcal{L}(G)$ which was found by Kegel in [19] (see Theorem 5 below). On the other hand, the need to introduce and explore such subgroups arises in the theory of Π-permutable subgroups (see Section 6 below). Finally, we demonstrate in this survey, that except of the results in [19], this concept allows to generalize a lot of other known results.

2 $\Pi_{\mathfrak{J}}$-subnormal subgroups

General properties of $\Pi_{\mathfrak{J}}$-subnormal subgroups. Let $\mathfrak{F}$ be a class of finite groups. We use respectively $(G^*)^{\mathfrak{F}}$ and $O^{\sigma_i}(G^*)$ to denote the intersection of all normal subgroups N of G with $G^*/N \in \mathfrak{F}$ and with the property that G^*/N is a finite σ_i-group.

We say that G^* is $\Pi_{\mathfrak{J}}$-*perfect* if $G^* = (G^*)^{\mathfrak{J}}$ and $O^{\sigma_i}(G^*) = G^*$ for all $\sigma_i \in \Pi$ such that $i \neq 0$.

The results of this section collect the most important properties of $\Pi_{\mathfrak{J}}$-subnormal subgroups.

Proposition 1. *(Skiba [56]) Let A, K and N be subgroups of G^*. Suppose that A is $\Pi_{\mathfrak{J}}$-subnormal in G^* and N is normal in G^*.*

(1) $A \cap K$ is $\Pi_{\mathfrak{J}}$-subnormal in K.

(2) If K is a $\Pi_{\mathfrak{J}}$-subnormal subgroup of A, then K is $\Pi_{\mathfrak{F}}$-subnormal in G^*.

(3) If $N \leq K$ and K/N is $\Pi_{\mathfrak{J}}$-subnormal in G^*/N, then K is $\Pi_{\mathfrak{J}}$-subnormal in G^*.

(4) AN/N is $\Pi_{\mathfrak{J}}$-subnormal in G^*/N.

(5) If $K \leq A$ and A is $\Pi_{\mathfrak{J}}$-primary, then K is Π-subnormal in G^*.

(6) If A is $\Pi_{\mathfrak{J}}$-perfect, then A is subnormal in G^*.

Recall that $O^{\Pi}(G)$ denotes the subgroup of G generated by all its Π'-subgroups [59]. A subgroup H of G is called [54, 59]: a *Hall* Π-*subgroup* of G if $|H|$ is a Π-number and $|G : H|$ is a Π'-number; a σ-*Hall* subgroup of G if H is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$.

Proposition 2. *(Skiba [56]) Let $\Pi_1 \subseteq \Pi$ and A be a Π-subnormal subgroup of G.*

(1) If $H \neq 1$ is a Hall Π_1-subgroup of G and A is not a Π'_1-group, then $A \cap H \neq 1$ is a Hall Π_1-subgroup of A.

(2) If A is a Π_1-Hall subgroup of G, then A is normal in G.

(3) If $|G : A|$ is a Π_1-number, then $O^{\Pi_1}(A) = O^{\Pi_1}(G)$.

(4) If N is a Π_1-subgroup of G, then $N \leq N_G(O^{\Pi_1}(A))$.

(5) If $|G : A|$ is a Π'-number, then A is subnormal in G.

We will say that G is $\sigma_{\mathfrak{J}}$-*soluble* if every chief factor of G is $\sigma_{\mathfrak{F}}$-primary.

Proposition 3. *(Skiba [56]) Let A be a $\Pi_{\mathfrak{J}}$-subnormal subgroup of G and let $\mathfrak{F}_0$ be the class of all $\sigma_{\mathfrak{J}}$-soluble Π-groups.*

(1) Let R be the product of some minimal normal subgroups of G and R is not $\Pi_{\mathfrak{J}}$-primary. Suppose also that either $|G : A|$ is a Π'-number or $G = AR$, R is non-abelian and all composition factors of R are isomorphic. Then $R \leq N_G(A)$.

(2) If R is a minimal normal subgroup of G, then $N \leq N_G(A^{\mathfrak{F}_0})$.

(3) If a minimal normal subgroup R of G belongs to $\mathfrak{J}$, then $R \leq N_G(A^{\mathfrak{J}})$.

Proposition 4. *(Skiba [56]) Let A be a $\Pi_{\mathfrak{J}}$-subnormal subgroup of G. Suppose that A is $\Pi_{\mathfrak{F}}$-primary. Then:*

(1) If $A \in \mathfrak{J}$, then $A \leq O_{\sigma_0}(G) \cap G_{\mathfrak{J}}$.

(2) If A is a σ_i-group, then $A \leq O_{\sigma_i}(G)$.

In this proposition $G_{\mathfrak{J}}$ denotes the $\mathfrak{J}$-*radical* of G, that is, the product of all normal subgroups of G belonging $\mathfrak{J}$.

Characterizations of $\Pi_{\mathfrak{J}}$-subnormality. Note that various special cases of Propositions 1–4 have already been used to solve many questions (see, for example, [28, 56, 57, 59]. Let's consider some other applications of these results.

We want start with the following characterization of $\Pi_{\mathfrak{J}}$-subnormality.

Theorem 1. *(Skiba [56]) A subgroup H of G is $\Pi_{\mathfrak{J}}$-subnormal in G if and only if H is $\Pi_{\mathfrak{F}}$-subnormal in $\langle H, x\rangle$ for all $x \in G$.*

Theorem 1 gives the positive answer to Question 4.10 in [59]. From this theorem we get the following well-known result.

Corollary 1. *(Wielandt) A subgroup H of G is subnormal in G if and only if H is subnormal in $\langle H, x\rangle$ for all $x \in G$.*

Nevertheless, the following question is still open now.

Question 1. *Suppose that a subgroup A of G is σ-subnormal in $\langle A, A^x\rangle$ for all $x \in G$. Is it true then that A is σ-subnormal in G?*

One of the key properties of σ-subnormal subgroups is the following (see Proposition 2(1)): *If A is σ-subnormal in G, then $A \cap H$ is a Hall Π-subgroup of A for every Hall Π-subgroup H of G.*

Moreover, the following fact holds.

Proposition 5. *[54] If G is σ-soluble, then a subgroup A of G is σ-subnormal in G if and only if $A \cap H$ is a Hall σ_i-subgroup of A for every Hall σ_i-subgroup H of G and every $i \in I$.*

In view of these observations it seems natural the following question.

Question 2. *Is it true that a subgroup A of the σ-full group G is σ-subnormal in G if and only if $A \cap H$ is a Hall σ_i-subgroup of A for every Hall σ_i-subgroup H of G and every $\sigma_i \in \sigma(G)$?*

Note that in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, the answers to these Questions 1 and 2 are positive (see Wielandt [63] or [21, A, 14.10] and [41], respectively).

The lattice of the $\Pi_{\mathfrak{I}}$-subnormal subgroups. We use $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ to denote the set of all $\Pi_{\mathfrak{I}}$-subnormal subgroups of G.

One of the most known results in the theory of subnormal subgroups is the following fact.

Theorem 2. *(Wielandt) The set $\mathcal{L}_{sn}(G)$ is a sublattice of the lattice $\mathcal{L}(G)$ of all subgroups of G.*

An original generalization of the lattice $\mathcal{L}_{sn}(G)$ was found by Kegel [19]. Let $\mathfrak{F}$ be a class of finite groups and $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ a subgroup chain in G. Then A is called *$\mathfrak{F}$-subnormal in G in the sense of Kegel* [19] or *K-$\mathfrak{F}$-subnormal* in G [13, 6.1.4] if either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ belongs to $\mathfrak{F}$ for all $i = 1, \ldots, t$.

Kegel proved [19] the following fact.

Theorem 3. *(Kegel [19]) If the class $\mathfrak{F}$ is closed under extensions, epimorphic images and subgroups, then the set $\mathcal{L}_{\mathfrak{F}sn}$ of all K-$\mathfrak{F}$-subnormal subgroups of G is a sublattice of the lattice $\mathcal{L}(G)$.*

For every set π of primes, we may choose the class $\mathfrak{F}$ of all π-groups. In this way we obtain infinitely many functors $\mathcal{L}_{\mathfrak{F}sn}$ assigning to every group G a sublattice of $\mathcal{L}(G)$ containing $\mathcal{L}_{sn}(G)$. In the future, this result has been generalized on the basis of the formation theory methods (see Chapter 6 in [13]).

On the base of Propositions 1–4, the following general fact is proved, which gives an another idea to generalize the Wielandt result.

Theorem 4. *(Skiba [56]) $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ is a sublattice of the lattice $\mathcal{L}(G)$.*

Note that in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, we get from this result Theorem 2. Theorem 3 is a corollary of Theorem 4 in the case when $\sigma = \{\mathbb{P}\}$ and $\mathfrak{F} = \mathfrak{I}$. One more special case of Theorem 4 was proved in [59] on the basis of the formation theory.

Corollary 2. *(Skiba [59]) The set of all σ-subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G.*

Zappa [67] characterized conditions under which the lattice $\mathcal{L}_{sn}(G)$ is modular. With respect to the lattice $\mathcal{L}_{\Pi}(G)$ we have the following result.

Theorem 5. *(Skiba [56]) The lattice $\mathcal{L}_{\Pi}(G)$ is modular if and only if the following two conditions hold:*

(i) If $T \leq S$ are Π-subnormal subgroups of G, where T is normal in S and either S/T is Π-primary or $|S/T| = p^3$ (p a prime), then $\mathcal{L}(S/T)$ is modular.

(ii) $\langle A, B\rangle^{\mathfrak{N}_\sigma} \leq A \cap B$ for each $A, B \in \mathcal{L}_{\sigma}(G)$ such that A and B cover $A \cap B$

and $A \cap B$ is not normal both in A and B.

Considering Theorem 5 in the two cases, when $\Pi = \emptyset$ and when $\Pi = \sigma = \{\{2\}, \{3\}, \ldots\}$, we get from this theorem the following

Corollary 3. *(See Zappa [67] or Theorem 9.2.3 in [52]) The following statements are equivalent.*

(I) $\mathcal{L}_{sn}(G)$ is modular.

(ii) If $T \leq S$ are subnormal subgroups of G, where T is normal in S and S/T is a p-group, p a prime, then $\mathcal{L}(S/T)$ is modular.

(III) If $T \leq S$ are subnormal subgroups of G, where T is normal in S and $|S/T| = p^3$ (p a prime), then $\mathcal{L}(S/T)$ is modular.

Question 3. *Characterize groups G with modular lattice $\mathcal{L}_{\Pi_\daleth}(G)$.*

3 σ-nilpotent and σ-quasinilpotent groups

General properties of σ-nilpotent groups. A chief factor H/K of G is said to be *σ-central* (in G) if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary, otherwise it is called *σ-eccentric*. The symbol $Z_\sigma(G)$ denotes the *σ-hypercentre of G*, that is, the product of all normal subgroups N of G such that every chief factor of G below N is σ-central.

It is well-known that the nilpotent groups can be characterized as the groups in which each subgroup, or each Sylow subgroup, or each maximal subgroup is subnormal. The following result demonstrates that there is a quite similar relation between σ-nilpotency and σ-subnormality.

Proposition 6. *[57] Any two of the following conditions are equivalent:*

(i) G is σ-nilpotent.

(ii) Every chief factor of G is σ-central in G.

(iii) G has a complete Hall σ-set $\mathcal{H}$ such that every member of $\mathcal{H}$ is σ-subnormal in G.

(iv) Every subgroup of G is σ-subnormal in G.

(v) Every maximal subgroup of G is σ-subnormal in G.

Corollary 4. *The class $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if E is a normal subgroup of G and $E/\Phi(G) \cap E$ is σ-nilpotent, then E is σ-nilpotent.*

σ-quasinilpotent groups. Recall G is said to be *quasinilpotent* if for every its chief factor H/K and every $x \in G$, x induces an inner automorphism on H/K [37, X, 13.2]. Note that since for every central chief factor H/K of G, an element of G induces the trivial automorphism on H/K, G is quasinilpotent if for every its *eccentric* chief factor H/K and for every $x \in G$, x induces an inner automorphism on H/K. This elementary observation allows us to consider the following σ-analogue of quasinilpotency.

Definition 2. We say that G is *σ-quasinilpotent* (sf. with [30]) if for every σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner.

Example 7. (i) G is quasinilpotent if and only if it is σ-quasinilpotent, where $\sigma = \{\{2\}, \{3\}, \ldots\}$.

(ii) Let $G = (A_5 \wr A_5) \times (A_7 \times A_{11})$ and $\sigma = \{\{2, 3, 5\}, \{2, 3, 5\}'\}$. Then G is σ-quasinilpotent but G is neither σ-nilpotent nor quasinilpotent.

We say that G is *σ-semisimple* if either $G = 1$ or $G = A_1 \times \cdots \times A_t$ is the direct product of non-abelian simple non-σ-primary groups $A_1, \ldots, A_t$.

Note that if $\sigma = \{\{2, 3, 5\}, \{2, 3, 5\}'\}$ and $G = A_7 \times A_{11}$, then G is σ-semisimple and σ-perfect.

The following result shows that the σ-quasinilpotent groups have properties similar to the properties of the quasinilpotent groups.

Theorem 6. *[58] Given group G, the following are equivalent:*
(i) G is σ-quasinilpotent.
(ii) $G/Z_\sigma(G)$ is σ-semisimple.
(iii) $G/F_\sigma(G)$ is σ-semisimple and $G = F_\sigma(G)C_G(F_\sigma(G))$.

From Proposition 6 and Theorem 6 we get

Corollary 5. *Let G be σ-quasinilpotent.*
(i) If G is σ-perfect, then $Z_\sigma(G) = Z(G)$.
(ii) If H is a normal σ-soluble subgroup of G, then $H \leq Z_\sigma(G)$.

Corollary 6. *If a σ-quasinilpotent group $G \neq 1$ is σ-soluble, then $G = O_{\sigma_{i_1}}(G) \times \cdots \times O_{\sigma_{i_t}}(G)$, where $\{\sigma_{i_1}, \ldots, \sigma_{i_t}\} = \sigma(G)$.*

Corollary 7. *Let $\pi = \cup_{\sigma_i \in \Pi} \sigma_i$. If a σ-quasinilpotent group $G \neq 1$ is π-separable, then $G = O_\pi(G) \times O_{\pi'}(G)$.*

Corollary 8. *If a quasinilpotent group G is π-separable, then $G = O_\pi(G) \times O_{\pi'}(G)$.*

The σ-Fitting subgroup and the generalized σ-Fitting subgroup. Recall that if $1 \in \mathfrak{F}$ is a class of finite groups, then the symbols $G^{\mathfrak{F}}$ and $G_{\mathfrak{F}}$ denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$ and the product of all normal $\mathfrak{F}$-subgroups of G, respectively. The subgroup $G^{\mathfrak{F}}$ is called the *$\mathfrak{F}$-residual* of G; $G_{\mathfrak{F}}$ is called the *$\mathfrak{F}$-radical* of G.

The product of all normal respectively σ-soluble, σ-nilpotent, σ-quasinilpotent subgroups of G is said to be respectively the *σ-radical*, the *σ-Fitting subgroup*, the *generalized σ-Fitting subgroup* of G and we denote it respectively by $R_\sigma(G)$, $F_\sigma(G)$, $F^*_\sigma(G)$.

We use $E_\sigma(G)$ to denote the σ-nilpotent residual of $F^*_\sigma(G)$, and we say that $E_\sigma(G)$ is the *σ-layer of* G.

The following result collect the basic properties of the subgroups $F_\sigma(G)$ and $F^*_\sigma(G)$ and describes the main relations between them.

Theorem 7. *(Skiba [55]) Let G be a σ-full group. Let $F_\sigma = F_\sigma(G)$, $F^*_\sigma = F^*_\sigma(G)$ and $E_\sigma = E_\sigma(G)$.*

(i) F^*_σ is σ-quasinilpotent and $F_\sigma = Z_\sigma(F^*_\sigma)$. Hence F^*_σ/F_σ is σ-semisimple and F^*_σ/F_σ is the product of all minimal normal subgroups of G/F_σ contained in $F_\sigma C_G(F_\sigma)/F_\sigma$. Also, $F_\sigma/Z_\sigma(G) = F_\sigma(G/Z_\sigma(G))$ and $F^*_\sigma/Z_\sigma(G) = F^*_\sigma(G/Z_\sigma(G))$.

(ii) $F^*_\sigma = E_\sigma F_\sigma = C_{F^*_\sigma}(F)F_\sigma$ and $F_\sigma = C_{F^*_\sigma}(E_\sigma)$. Also, E_σ is a σ-perfect characteristic subgroup of F^*_σ and $E_\sigma/Z(E_\sigma)$ is σ-semisimple. Hence $E_\sigma(G) = E_\sigma(E_\sigma(G))$.

(iii) A σ-subnormal subgroup H of G is contained in F^* (respectively in F_σ) if and only if it is σ-quasinilpotent (respectively σ-nilpotent). Moreover, if H is a σ-quasinilpotent σ-perfect σ-subnormal subgroup of G, then $H \leq E_\sigma$.

(iv) $C_G(F^*_\sigma) \leq Z(F^*_\sigma)$.

(v) $F_\sigma(G/\Phi(G)) = F_\sigma/\Phi(G)$ and $F^*_\sigma(G/\Phi^*(G)) = F^*_\sigma/\Phi^*(G)$.

Corollary 9. *For every σ-subnormal subgroup V of G we have $F_\sigma(G)\cap V = F_\sigma(V)$ and $F^*_\sigma(G)\cap V = F^*_\sigma(V)$.*

It is clear that if $R \leq E \leq G$, where R is a non-abelian minimal normal subgroup of G and E is normal in G, then R is the product of some minimal normal subgroups of E [21, A, 4.13]. Hence we get from Theorem 7(ii) the following

Corollary 10. *(See [37, X, 13.13]) If G is σ-full, then $F^*_\sigma(G)/F_\sigma(G)$ is the group generated by all minimal normal subgroups of $C_G(F_\sigma(G))F_\sigma(G)/F_\sigma(G)$.*

From Theorem 7(v) we get

Corollary 11. *(Skiba [57]) If G is σ-soluble, then $C_G(F_\sigma(G)) \leq F_\sigma(G)$.*

Corollary 12. *If G is π-separable, then*

$$C_G(O_\pi(G) \times O_{\pi'}(G)) \leq O_\pi(G) \times O_{\pi'}(G).$$

Some other applications of Theorem 7. Theorem 7 not only covers a large number of known results, but it also allows to establish a link between some of these results. Note, for example, that the following known results are special cases of Corollary 10.

Corollary 13. *(See [22, Ch. 6, 1.3]) If G is soluble, then $C_G(F(G)) \leq F(G)$.*

Corollary 14. *(See [22, Ch. 6, 3.2]) If G is π-separable, then the following inclusion holds:*

$$C_{G/O_{\pi'}(G)}(O_\pi(G/O_{\pi'}(G)) \le O_\pi(G/O_{\pi'}(G).$$

Corollary 15. *(Monakhov and Shpyrko [47]) Let G be a π-soluble group.*
(1) $C_G(O_\pi(G) \times O_{\pi'}(G)) \le F(O_\pi(G)) \times O_{\pi'}(G)$.
(2) If $O_{\pi'}(G) = 1$, then $C_G(F(G)) \le F(G)$.

In the case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, we get from Theorem 7 and its Corollaries 9 and 10 the following known results.

Corollary 16. *(See [37, X, 13.13]) $F^*(G)/F(G)$ is the group generated by all minimal normal subgroups of $C_G(F(G))F(G)/F(G)$.*

Corollary 17. *(See [37, X, 13.10]) $F^*(G)$ is quasinilpotent and every subnormal quasinilpotent subgroup of G is contained in $F^*(G)$.*

Corollary 18. *(See [21, A, 8.8]) $F(G)$ is generated by all subnormal nilpotent subgroups of G.*

Corollary 19. *(See [37, X, 13.15]) $F(G) = C_{F^*(G)}(E(G))$.*

4 σ-soluble groups

We say that the integers n and m are *σ-coprime* if $\sigma(n) \cap \sigma(m) = \emptyset$. If A, B and R are subgroups of G and for some $x \in R$ we have $AB^x = B^xA$, then A is said to *R-permute* with B [24]. If every two members of a complete Hall σ-set $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ of G are permutable, then we say that $\{H_1, \ldots, H_t\}$ is a *σ-basis* of G.

By the classical Hall theorem, G is soluble if and only if it has a Sylow basis. The direct analogue of this result for σ-soluble groups is not true in general. Indeed, let $\sigma = \{\{2, 3\}, \{2, 3\}'\}$. Then the alternating group A_5 of degree 5 has a σ-basis and it is not σ-soluble. Nevertheless, the following generalizations of the Hall result are true.

Theorem 8. *[54] Let $R = R_\sigma(G)$ be the σ-radical of G. Then any two of the following conditions are equivalent:*

(i) G is σ-soluble.

(ii) For any Π, G has a Hall Π-subgroup and every σ-Hall subgroup of G R-permutes with every Sylow subgroup of G.

(iii) G has a σ-basis $\{H_1, \ldots, H_t\}$ such that for each $i \ne j$ every Sylow subgroup of H_i R-permutes with every Sylow subgroup of H_j.

Theorem 9. *[54] Let $R = R_\sigma(G)$ be the σ-radical of G. Then G is σ-soluble if and only if for any Π the following hold: G has a Hall Π-subgroup E, every*

Π-subgroup of G is contained in some conjugate of E and E R-permutes with every Sylow subgroup of G.

As a first step in the proof of Theorem 8, the following useful fact was proved in [54].

Proposition 7. *[54] Suppose that $G = A_1A_2 = A_2A_3 = A_1A_3$, where A_1, A_2 and A_3 are σ-soluble subgroups of G. If the three indices $|G : N_G(A_1^{\mathfrak{N}})|$, $|G : N_G(A_2^{\mathfrak{N}})|$, $|G : N_G(A_3^{\mathfrak{N}})|$ are pairwise σ-coprime, then G is σ-soluble.*

Corollary 20. *Suppose that $G = A_1A_2 = A_2A_3 = A_1A_3$, where A_1, A_2 and A_3 are soluble subgroups of G. If the three indices $|G : N_G(A_1)|$, $|G : N_G(A_2)|$, $|G : N_G(A_3)|$ are pairwise coprime, then G is soluble.*

Corollary 21. *(Wielandt) If G has three soluble subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself soluble.*

From Theorem 8 we get the following characterizations of π-separable groups.

Corollary 22. *Let R be the product of all normal π-separable subgroups of G. Then G is π-separable if and only if $G = AB$, where A and B are a Hall π-subgroup and a Hall π'-subgroup of G, respectively, and every Sylow subgroup of A R-permutes with every Sylow subgroup of B.*

Corollary 23. *Let R be the product of all normal π-separable subgroups of G. Then G is π-separable if and only if $G = AB$, where A and B are a Hall π-subgroup and a Hall π'-subgroup of G, respectively, and every Sylow subgroup of G R-permutes with A and with B.*

We say that G is *σ-biprimary* if $|\sigma(G)| = 2$.

Theorem 10. *[54] Let G be σ-full and $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ a complete Hall σ-set of G. Then any two of the following conditions are equivalent:*

(i) G is σ-soluble.

(ii) Every σ-biprimary subgroup of G is σ-soluble and for every chief factor H/K of G and every $A \in \mathcal{H}$ the number $|G : N_G((A \cap H)K)|$ is σ-primary.

(iii) Every σ-biprimary subgroup of G is σ-soluble and for any $k \in \{1, \ldots, t\}$ there is a normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ of G such that the number $|G : N_G((H_k \cap G_i)G_{i-1})|$ is σ-primary for all $i = 1, \ldots, n$.

In view of the Kegel-Wielandt theorem on solubility of products of nilpotent groups we get from Theorem 10 the following facts.

Corollary 24. *Suppose that G has a complete Hall σ-set $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ whose members are nilpotent. Then any two of the following conditions are equivalent:*

(i) G is soluble.

(ii) For every chief factor H/K of G and every $A \in \mathcal{H}$ the number $|G : N_G((A \cap H)K)|$ is σ-primary.

(iii) For any $k \in \{1, \dots, t\}$ there is a normal series $1 = G_0 \leq G_1 \leq \dots \leq G_n = G$ of G such that the number $|G : N_G((H_k \cap G_i)G_{i-1})|$ is σ-primary for all $i = 1, \dots, n$.

From Corollary 24 we get the following known result.

Corollary 25. *(See Zhang [68] or Guo [23]) If for every Sylow subgroup P of G the number $|G : N_G(P)|$ is a prime power, then G is soluble.*

5 σ-supersoluble groups

Analysis of some problems leads to the need for finding σ-analogues of supersoluble groups.

Definition 3. We say that G is *σ-supersoluble* if every σ-eccentric chief factor of G is cyclic.

We will use $\mathfrak{U}$ and $\mathfrak{U}_\sigma$ to denote the classes of all supersoluble and all σ-supersoluble groups, respectively.

It is clear that every σ-supersoluble group is σ-soluble and every σ-nilpotent group is σ-supersoluble. Moreover, G is supersoluble iff G is σ-supersoluble, where $\sigma = \{\{2\}, \{3\}, \dots\}$.

Example 8. Let $G = A_5 \times B$, where A_5 is the alternating group of degree 5 and $B = C_{29} \rtimes C_7$ a non-abelian group of order 203. Let $\sigma = \{\{7\}, \{29\}, \{2, 3, 5\}, \{2, 3, 5, 7, 29\}'\}$. Then $G^{\mathfrak{N}_\sigma} = C_{29}$, so G is a σ-supersoluble group but it is neither soluble nor σ-nilpotent.

A class $1 \in \mathfrak{F}$ of groups is called: a *formation* if for every group G, every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$; a *Fitting class* if for every group G, every normal subgroup of $G_{\mathfrak{F}}$ belongs to $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be: *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *hereditary* if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$.

We say that G is *strongly* σ-supersoluble if every chief factor of G below $G^{\mathfrak{N}_\sigma}$ is cyclic.

The classes $\mathfrak{S}_\sigma$ and $\mathfrak{N}_\sigma$ are hereditary saturated Fitting formations [57]. With respect to the class $\mathfrak{U}_\sigma$ we have

Theorem 11. *(Guo, Skiba [25]) (i) The class of all (strongly) σ-supersoluble groups is a hereditary formation.*

(ii) The formation $\mathfrak{U}_\sigma$ is saturated if and only if $\pi(p-1) \subseteq \sigma_i$ for all $p \in \sigma_i$ such that $|\sigma_i| > 1$.

(iii) The formation $\mathfrak{U}_\sigma$ is a Fitting class if and only if $|\sigma_i| > 1$ for each $\sigma_i \in \sigma$ such that $\sigma_i \neq \{2\}$ and $\pi(p-1) \subseteq \sigma_i$ for all $p \in \sigma_i$. Hence $\mathfrak{U}_\sigma$ is a Fitting

class if and only if $\mathfrak{U}_\sigma = \mathfrak{N}_\sigma$.

The applications of σ-supersoluble groups are based on Theorem 11 and on the following properties of σ-supersoluble and strongly σ-supersoluble groups.

Theorem 12. *(Guo, Skiba [25]) G is σ-supersoluble if and only if the following assertions hold:*

(1) $G^{\mathfrak{N}_\sigma}$ is nilpotent.

(2) G' is σ-nilpotent.

(3) $G^{\mathfrak{U}} \cap G^{\mathfrak{N}_\sigma} \leq \Phi(G) \cap Z_{\mathfrak{U}_\sigma}(G)$.

Theorem 13. *(Guo, Skiba [25]) G is strongly σ-supersoluble if and only if $[G^{\mathfrak{U}}, G^{\mathfrak{N}_\sigma}] = 1$ and $G^{\mathfrak{U}} \cap G^{\mathfrak{N}_\sigma} \leq \Phi(G) \cap Z_{\mathfrak{U}}(G)$.*

Applications of σ-supersoluble groups in the theory generalized CLT-groups. Recall that G is called a *CLT-group* iff it satisfies the following converse to Lagrange's theorem: *for every divisor d of $|G|$, G has a subgroup of order d.* The CLT-group and some interesting subclasses of the class of all CLT-groups have been studied by many authors (see, for example, Chapters 1, 3, 4 and 5 in [61] and the recent papers [7, 42, 43, 44, 45]).

Definition 4. (i) We say that G is a *CLT_σ-group* if G has a complete Hall σ-set $\{1, H_1, \ldots, H_t\}$ such that for all subgroups $A_i \leq H_i$, G has a subgroup of order $|A_1| \cdots |A_t|$.

(ii) We say that G is a *generalized CLT_σ-group* if for all $p_i \in \sigma_i \in \sigma(G)$, G has a subgroup of order $p_1^{n_1} \cdots p_t^{n_t}$ for each $(n_1, \ldots, n_t)$ such that $p_1^{n_1} \cdots p_t^{n_t}$ divides $|G|$.

Remark 1. (i) G is a CLT_σ-group if and only if G has a complete Hall σ-set $\{1, H_1, \ldots, H_t\}$ such that for every i and for every subgroup A_i of H_i, G has a subgroup E_i of order $|A_i||G|_{\sigma_i'}$. Indeed, the intersection $E = E_1 \cap \cdots \cap E_t$ has order $|E| = |A_1| \ldots |A_t|$ since $(|G : E_i|, |G : E_j|) = 1$ for all $i \neq j$.

(ii) G is a generalized CLT_σ-group if and only if for all $p \in \sigma_i \in \sigma(G)$, G has a subgroup of order $p^n|G|_{\sigma_i'}$ for each n such that p^n divides $|G|$ (see (i)).

Note that G is a CLT-group iff G is a CLT_σ-group, where $\sigma = \{\{2\}, \{3\}, \ldots\}$. The group G in Example 8 is a CLT_σ-group but it is not a CLT-group.

Example 9. (i) In view of Proposition 6, $G \neq 1$ is σ-nilpotent if and only if $G = H_1 \times \cdots \times H_t$, where $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ is a complete Hall σ-set of G. Therefore every σ-nilpotent group is a CLT_σ-group, and the σ-nilpotent groups can be characterized as the groups having a complete Hall σ-set $\{1, H_1, \ldots, H_t\}$ such that G has a normal subgroup of order $|A_1| \cdots |A_t|$ for each set $\{A_1, \ldots, A_t\}$ such that A_i is a normal subgroup of H_i.

(ii) Now let G be a σ-soluble group and A a cyclic group of order $|G|$. We shaw that $B = G \times A$ is a generalized CLT_σ-group. Let $p \in \sigma_i \in \sigma(B)$ and p^n

divides $|B| = |G|^2$. Let P be a Sylow p-subgroup of G and $|P| = p^m$. Since G is a σ-soluble, it has a σ_i-complement E such that $EP = PE$ by Theorem 8. Let E_1 be the σ_i-complement of A. First assume that $n \leq m$, and let P_1 be the subgroup of A of order p^n. Then $E \times E_1P_1$ is a subgroup of G of order $p^n|G|_{\sigma_i'}$. Finally, if $n > m$ and P_2 is the subgroup of A of order p^{n-m}, then $PE \times E_1P_2$ is a subgroup of G of order $p^n|G|_{\sigma_i'}$. Therefore B is a generalized CLT_σ-group by Remark 1.

Theorem 14. *(Guo, Skiba [25]) Every CLT_σ-group is σ-soluble.*

Note that the group $G = A_4 \times P$, where A_4 is the alternating group of degree 4 and P a group of order 5, is soluble but it is not a generalized CLT_σ-group, where $\sigma = \{\{3, 5\}, \{3, 5\}'\}$.

The symmetric group S_4 of degree 4 is not supersoluble but it clearly is a CLT-group. Nevertheless, if G is of odd order and every homomorphic image of G is a CLT-group, then G is supersoluble (Humphreys [32]). On the other hand, the Ore-Zappa theorem states (see [48, 66] or Theorem 4.1 in [61, Ch. 1]) that G is supersoluble also in the case when every subgroup of G is a CLT-group. With respect to CLT_σ-groups we get the following fact.

Theorem 15. *(Guo, Skiba [25]) Let $D = G^{\mathfrak{N}_\sigma}$. Suppose that G has a complete Hall σ-set $\{1, H_1, \ldots, H_t\}$ such that H_i is supersoluble whenever $H_i \cap D \neq 1$. Then G is σ-supersoluble if and only if every section of G is a CLT_σ-group.*

Now we consider one special class of strongly σ-supersoluble CLT_σ-groups.

We will say that a subgroup A of G is *H_σ-subnormally embedded* (respectively *H_σ-normally embedded*) in G if A is a σ-Hall subgroup of some σ-subnormal (respectively normal) subgroup of G.

In the special case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, the definition of H_σ-normally embedded subgroups is equivalent to the concept of Hall normally embedded subgroups in [44] and the definition of H_σ-subnormally embedded subgroups is equivalent to the concept of Hall subnormally embedded subgroups in [45].

Example 9(i) is one of the motivations for the following our observation.

Theorem 16. *(Guo, Skiba [25]) Let $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ be a complete Hall σ-set of G, $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Any two of the following conditions are equivalent:*

(i) For all $p, q \in \sigma_i \in \sigma(G)$, where $p \in \pi$, and for every natural number n such that q^n divides $|G|$, G has an H_σ-subnormally embedded subgroup of order $q^n|G|_{\sigma_i'}$.

(ii) For each set $\{A_1, \ldots, A_t\}$, where A_i is a subgroup (respectively a normal subgroup) of H_i for all $i = 1, \ldots, t$, G has an H_σ-subnormally embedded (respectively H_σ-normally embedded) subgroup of order $|A_1| \cdots |A_t|$.

(iii) D is cyclic of square-free order and $|\sigma_i \cap \pi(G)| = 1$ for each $\sigma_i \in \sigma(D)$.

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, Theorem 16 covers Theorem 11 in [44], Theorem 2.7 in [45] and Theorems 3.1 and 3.2 in [42].

6 $\mathcal{L}$-permutable subgroups

Π-permutable subgroups. Let $\mathcal{L}$ be a non-empty set of subgroups of G and E a subgroup of G. Then a subgroup A of G is called *$\mathcal{L}$-permutable* if $AH = HA$ for all $H \in \mathcal{L}$; *$\mathcal{L}^E$-permutable* if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in E$.

Note that in the case when $\mathcal{L}$ is the set of all Sylow subgroups of G, $\mathcal{L}^G$-permutable subgroups are also called *S-quasinormal* or *S-permutable* in G. If $\mathcal{L}$ is the set of all Sylow p-subgroups of G for all $p \in \pi$, $\mathcal{L}^G$-permutable subgroups are also called *π-quasinormal* or *π-permutable* in G [18].

Here we discuss the following generalization of π-permutability.

Definition 5. We say that a subgroup A of G is *Π-permutable* or *Π-quasinormal* in G if G possesses a complete Hall Π-set $\mathcal{H}$ such that A is $\mathcal{H}^G$-permutable.

Example 10. (i) In view of Proposition 6, every subgroup of a σ-nilpotent group is Π-permutable in G for each $\emptyset \neq \Pi \subseteq \sigma$.

(ii) Let G and σ be as in Example 6, and let $\Pi = \{\sigma_0, \sigma_2\}$. Then G is not σ-nilpotent but its subgroup C_7A_5 is Π-permutable in G. This subgroup is not Π_1-permutable in G, where $\Pi_1 = \{\sigma_0, \sigma_1\}$.

Theorem 17. *(Guo, Skiba [29]) Let H be a Π-subgroup of G and $D = G^{N_\sigma}$ the σ-nilpotent residual of G.*

(i) If G is Π-full and possesses a complete Hall Π-set $\mathcal{H}$ such that H is $\mathcal{H}^D$-permutable, then H is σ-subnormal in G and the normal closure H^G of H in G is a Π-group.

(ii) If H is Π-permutable in G and, in the case when $\Pi \neq \sigma(G)$, G possesses a complete Hall Π'-set $\mathcal{K}$ such that H is $\mathcal{K}$-permutable, then H^G/H_G is σ-nilpotent and the normalizer $N_G(H)$ of H is also Π-permutable. Moreover, $N_G(H)$ is $\mathcal{H}^G$-permutable for each complete Hall Π-set $\mathcal{H}$ of G such that H is $\mathcal{H}^G$-permutable.

(iii) If G is a Π'-full group of Sylow type and H is Π'-permutable in G, then H^G possesses a σ-nilpotent Hall Π'-subgroup.

Consider some corollaries of Theorem 17.

Theorem 17(i) immediately implies

Corollary 26. *(Kegel [19]) If H a π-subgroup H of G is π-permutable in G, then H is subnormal in G.*

Now, consider some special cases of Theorem 17(ii). First note that in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Theorem 17(ii) the following results.

Corollary 27. *Let H be a π-subgroup of G. If H is π-permutable in G and, also, H permutes with some Sylow p-subgroup of G for each prime $p \in \pi'$, then the normalizer $N_G(H)$ of H is π-permutable in G.*

In particular, in the case when $\pi = \mathbb{P}$, we have

Corollary 28. *(Schmid [51]) If a subgroup H of G is S-permutable in G, then the normalizer $N_G(H)$ of H is also S-permutable.*

Corollary 29. *Let H be a π-subgroup of G. If H is π-permutable in G and, also, H permutes with some Sylow p-subgroup of G for each prime $p \in \pi'$, then H/H_G is nilpotent.*

Corollary 30. *(Deskins [20]) If a subgroup H of G is S-permutable in G, then H/H_G is nilpotent.*

Recall that G is said to be a *π-decomposable* if $G = O_\pi(G) \times O_{\pi'}(G)$, that is, G is the direct product of its Hall π-subgroup and Hall π'-subgroup.

In the case when $\sigma = \{\pi, \pi'\}$ we get from Theorem 17(ii) the following

Corollary 31. *Suppose that G is π-separable. If a subgroup H of G permutes with all Hall π-subgroups of G and with Hall π'-subgroups of G, then H^G/H_G is π-decomposable.*

In particular, we have

Corollary 32. *Suppose that G is p-soluble. If a subgroup H of G permutes with all Sylow p-subgroups of G and with all p-complements of G, then H^G/H_G is p-decomposable.*

Finally, in the case when $\Pi = \sigma$, we get from Theorem 17(ii) the following

Corollary 33. *(Skiba [59]) Suppose that G is a σ-full group and let H be a subgroup of G. If H is σ-permutable in G, then H^G/H_G is σ-nilpotent.*

From Theorem 17(iii) we get

Corollary 34. *Let H be a π-subgroup of G. If H permutes with every Sylow p-subgroup of G for $p \in \pi'$, then H^G possesses a nilpotent π-complement.*

A subgroup H of G is called an *S-semipermutable* in G if H permutes with all Sylow subgroups P of G such that $(|H|, |P|) = 1$. If H is S-semipermutable in G and $\pi = \pi(H)$, then H is π'-permutable in G. Hence from Corollary 34 we get the following known result.

Corollary 35. *(Isaacs [38]) If a π-subgroup H of G is S-semipermutable in G, then H^G possesses a nilpotent π-complement.*

Note that in the group $G = C_7 \rtimes \mathrm{Aut}(C_7)$ a subgroup of order 3 is π'-permutable in G, where $\pi = \{2, 3\}$, but it is not S-semipermutable.

Groups in which Π-permutability is a transitive relation. A group G is called a *PST-group* if S-permutability is a transitive relation on G, that is, every S-permutable subgroup of a S-permutable subgroup of G is S-permutable in G. In view of the Kegel Corollary 26, the class of all PST-groups coincides with the class of all groups, in which every subnormal subgroup is S-permutable. The description of PST-group was first obtained by Agrawal [1], for the soluble case, and by Robinson in [50], for the general case. In the further publications, authors (see, for example, the recent papers [3, 4, 6, 8, 9, 10, 14, 15, 16, 64] and Chapter 2 in [12]) have found out and described many other interesting characterizations of soluble PST-groups.

The results in [1, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 50, 64] are the motivations for the following question.

Question 4. *Let G be Π-full. What is the structure of G provided that every Π-subnormal subgroup of G is Π-permutable in G?*

We do not know the answer to this question in the case of an arbitrary Π-full group G. Nevertheless, the answer to this question in the case when $\Pi = \sigma$ and G is σ-soluble is known.

Theorem 18. *(Skiba [59]) Let G be σ-soluble. Then every σ-subnormal subgroup of G is σ-permutable in G if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is an abelian σ-Hall subgroup of odd order of G such that every element of M induces a power automorphism of D.*

Corollary 36. *(Agrawal [1]) Let G be a soluble group. Then every σ-subnormal subgroup of G is σ-permutable in G if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is an abelian σ-Hall subgroup of odd order of G such that every element of M induces a power automorphism of D.*

Two characterizations of σ-permutability. Now we give two characterizations of the σ-permutable subgroups. The first of them uses the idea of description of the quasinormal subgroups which dates back to Theorem 5.1.1 in the book of Schmidt [52].

Theorem 19. *(Skiba [58]) Let G be a σ-full group of Sylow type. Then a subgroup A of G is σ-permutable in G if and only if A is σ-subnormal in G and, for each $i \in I$, the equality*

$$E \cap \langle A, H \rangle = \langle A, E \cap H \rangle$$

holds for every Hall σ_i-subgroup H of G and every subgroup E of G containing A.

Theorem 19 remains to be new also in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$.

Corollary 37. *A subgroup A of G is S-permutable in G if and only if A is subnormal in G and the equality*

$$E \cap \langle A, P \rangle = \langle A, E \cap P \rangle$$

holds for every Sylow subgroup P of G and every subgroup E of G containing A.

Theorem 1 is a motivation for the following

Question 5. *Let G be a Π-full group of Sylow type. Suppose that a subgroup A of G is Π-permutable in $\langle A, x\rangle$ for all $x \in G$. Is it true then that A is Π-subnormal in G?*

Partially, the answer to this question is obtained in [59].

Theorem 20. *(Skiba [59]) Let G be a σ-full group of Sylow type. Then a subgroup A of G is σ-permutable in G if and only if A is σ-subnormal in G and A is σ-permutable in $\langle A, x\rangle$ for all $x \in G$.*

Corollary 38. *(See Ballester-Bolinches and Esteban-Romero [11] or Theorem 1.2.13 in [12]) A subgroup A of G is S-permutable in G if and only if A is S-permutable in $\langle A, x\rangle$ for all $x \in G$.*

The lattice of the Π-permutable subgroups. We will use $\mathcal{L}_{\Pi per}(G)$ to denote the set of all Π-subnormal Π-permutable subgroups of G.

Kegel proved [18] that the set of all S-permutable subgroup of G forms a sublattice of the lattice $\mathcal{L}(G)$ of all subgroups of G. This result was generalized in the paper [59], where it was proved that the set $\mathcal{L}_{\sigma per}(G)$ of all σ-permutable subgroups of a σ-full group G is also a sublattice of the lattice $\mathcal{L}(G)$. But in fact, the method of the proof of the last result allows us to prove the following general fact.

Theorem 21. *Suppose that G is a Π-full group of Sylow type. Then the set $\mathcal{L}_{\Pi per}(G)$ forms a sublattice of the lattice $\mathcal{L}(G)$.*

It is not difficult to show that the lattice $\mathcal{L}_{\Pi per}(G)$, in general, is not modular.

Question 6. *Characterize groups G with modular lattice $\mathcal{L}_{\Pi per}(G)$.*

Question 7. *Characterize groups G with distributive lattice $\mathcal{L}_{\Pi per}(G)$.*

Groups in which every n-maximal subgroup is σ-permutable. If $M_n < M_{n-1} < \ldots < M_1 < M_0 = G$, where M_i is a maximal subgroup of M_{i-1}, $i = 1, 2, \ldots, n$, then M_n is said to be an *n-maximal subgroup of G*. Recall also that the *rank* $r(G)$ of a soluble group G is the maximal integer k such that G has a chief factor of order p^k for some prime p (see [33, p. 685]).

Our next observation is the following fact.

Theorem 22. *(Guo, Skiba [28]) Suppose that G is soluble and each n-maximal subgroup of G is σ-permutable in G. Suppose that G has a complete Hall σ-set $\mathcal{H}$ such that for every member H of $\mathcal{H}$ we have $r(H) \leq n-1$. Then $r(G) \leq n-1$.*

Corollary 39. *Suppose that G is soluble and each n-maximal subgroup of G is S-permutable in G. Then $r(G) \leq n-1$.*

Corollary 40. *(Mann [46]) Suppose that G is soluble and each n-maximal subgroup of G is quasinormal. Then $r(G) \leq n-1$.*

It is not difficult to show that if G is not σ-nilpotent and each 3-maximal or each 2-maximal subgroup of G is σ-subnormal in G, then G is soluble. Hence we get also from Theorem 22 the following facts.

Corollary 41. *Suppose that each 3-maximal subgroup of G is σ-permutable in G. If G has a complete Hall σ-set $\mathcal{H}$ such that for every member H of $\mathcal{H}$ we have $r(H) \leq 2$, then G is soluble and $r(G) \leq 2$.*

Corollary 42. *(Agrawal [2]) If every 2-maximal subgroup of G is S-permutable in G, then G is supersoluble.*

Corollary 43. *(Huppert [34]) If every 3-maximal subgroup of G is normal in G, then G is soluble of rank $r(G)$ at most two.*

$\mathcal{H}$-permutable subgroups. We say that a subgroup A of G is *$\mathcal{H}$-permutable* if G possesses a complete Hall σ-set $\mathcal{H}$ such that A is $\mathcal{H}$-permutable.

Before continuing, we need to modify the concepts of complete Hall Π-sets and Π-full groups.

Definition 6. Let $\mathcal{H}$ be a complete Hall Π-set of G and let $\mathfrak{F}$ be a class of groups. Then we say that $\mathcal{H}$ is a *complete Hall $\Pi_{\mathfrak{F}}$-set* of G if every member of $\mathcal{H}$ belongs to $\mathfrak{F}$.

Recall that in the case when $\Pi = \sigma$ and $\mathfrak{F} = \mathfrak{N}$ is the class of all nilpotent groups, a complete Hall $\Pi_{\mathfrak{F}}$-set of G is called a *Wielandt σ-set* [27] of G; if $\Pi = \sigma$ and $\mathfrak{F} = \mathfrak{U}$ is the class of all supersoluble groups, then a complete Hall $\Pi_{\mathfrak{F}}$-set of G is called a *generalized Wielandt σ-set* of G.

In view of the Blessenohl-Slepova Theorem [53, IV, 16.2], the class of all σ-soluble groups having a complete Hall $\Pi_{\mathfrak{F}}$-set is a saturated formation for any $\Pi \subseteq \sigma$ and for any saturated formation $\mathfrak{F}$. On the other hand, in view of the Vorob'ev-Zagurski Theorem [60], the class of all soluble groups having a complete Hall $\Pi_{\mathfrak{F}}$-set is a local Fitting class, for any local Fitting class $\mathfrak{F}$ and any $\Pi \subseteq \sigma$.

The mentioned in Introduction results in [5, 35, 36] and many other related results make natural to ask:

(I) *Suppose that G has a complete Hall $\sigma_{\mathfrak{F}}$-set $\mathcal{H}$ of such that every maximal subgroup of any subgroup in $\mathcal{H}$ permutes with all other members of $\mathcal{H}$. What we can say then about the structure of G? In particular, is it true then that G is supersoluble in the case when $\mathfrak{F} = \mathfrak{U}$ is the class of all supersoluble groups?*

(II) *Suppose that G has a complete Hall σ-set. What we can say then about the structure of G provided every complete Hall σ-set of G forms a σ-basis in G?*

Note that in the paper [5], on the base of the classification of all simple non-abelian groups, it was proved that if G has a complete Sylow set $\mathcal{S}$ such that if every maximal subgroup of any subgroup in $\mathcal{S}$ permutes with all other members of $\mathcal{S}$, then G is supersoluble.

In the paper [26], not using such classification, the authors proved the following two results concerning Question (I).

Theorem 23. *(Guo, Skiba [26]) Suppose that G has a generalized Wielandt σ-set $\mathcal{H}$ such that every maximal subgroup of any non-cyclic subgroup in $\mathcal{H}$ permutes with all other members of $\mathcal{H}$. Then G is supersoluble.*

On the base of Theorem 23, it is proved also the following

Theorem 24. *(Guo, Skiba [26]) Suppose that G has a complete Hall σ-set $\mathcal{H}$ such that every group in $\mathcal{H}$ is a soluble PST-group and every maximal subgroup of any subgroup in $\mathcal{H}$ permutes with all other members of $\mathcal{H}$. Then $G = D \rtimes M$ is a supersoluble group, where the σ-nilpotent residual $D = G^{\mathfrak{N}_\sigma}$ of G is a nilpotent Hall subgroup of G whose maximal subgroups are normal in G and there are no primes p such that p divides $|D|$ and $(p-1, |G|) = 1$. In particular, $|D|$ is odd.*

Recall that G is a soluble PST-group if and only if $G = D \rtimes M$ is supersoluble, where $D = G^{\mathfrak{N}}$ is an abelian Hall subgroup of G of odd order and every subgroup of D is normal in G [12, 2.1.2 and 2.1.8]. Since every nilpotent group is a PST-group, we get from Theorem 24 the following result.

Corollary 44. *G has a complete Hall σ-set $\mathcal{H}$ such that every group in $\mathcal{H}$ is nilpotent and every maximal subgroup of any subgroup in $\mathcal{H}$ permutes with all other members of $\mathcal{H}$ if and only if $G = D \rtimes M$ is a supersoluble group, where the nilpotent residual $D = G^{\mathfrak{N}}$ of G is a nilpotent Hall subgroup of G whose maximal subgroups are normal in G, and there are no primes p such that p divides $|D|$ and $(p-1, |G|) = 1$. In particular, $|D|$ is odd.*

In the paper [27], the following special case of Theorem 23 was proved.

Corollary 45. *(Guo and Skiba [27]) If G has a complete Wielandt σ-set $\mathcal{H}$ such that every subgroup in $\mathcal{H}$ permutes with all maximal subgroups of any member of $\mathcal{H}$, then G is supersoluble.*

The following result was obtained by Asaad and Heliel in [5] by applying the classification of all non-abelian simple groups. The proof of Theorem 24 only requires the classification of simple groups with abelian Sylow 2-subgroups.

Corollary 46. *(Asaad and Heliel [5]) If G has a complete set of Sylow subgroups $\mathcal{S}$ such that every Sylow subgroup in $\mathcal{S}$ permutes with all maximal subgroups of any member of $\mathcal{S}$, then G is supersoluble.*

Now let $p > q > r$ be primes such that qr divides $p-1$. Let P be a group of order p and $QR \leq Aut(P)$, where Q and R are groups with order q and r, respectively. Let $G = P \rtimes (QR)$. Then, in view of the above-mentioned Huppert result in [35], G is not the group such that every complete set of Sylow subgroups forms a Sylow basis of G. But it is easy to see that every complete Hall set of type σ, where $\sigma = \{7\} \cup \{3, 2\} \cup (\mathbb{P} \setminus \{2, 3, 7\})$, is a σ-basis of G. This elementary example is a motivation for the next result, which gives the answer to Question (II) in the universe of all soluble groups.

Theorem 25. *(Guo, Skiba [26]) The class $\mathfrak{H}_\sigma$ is a hereditary formation and it is saturated if and only if $|\sigma| \leq 2$. Moreover, $G \in \mathfrak{H}_\sigma$ if and only if G is soluble and the automorphism group induced by G on every its chief factor of order divisible by p is either a π_i-group, where $p \notin \pi_i \in \sigma$, or a $\pi_i \cup \pi_j$-group, where $\pi_i, \pi_j \in \sigma$ and $p \in \pi_i$.*

In this theorem $\mathfrak{H}_\sigma$ denotes the class of all soluble groups G such that every complete Hall σ-set of G forms a σ-basis of G.

The lattice of the $\mathcal{H}$-permutable subgroups. We use $\mathcal{P}(\mathcal{H})$ to denote the set of all $\mathcal{H}$-permutable subgroups of G.

Theorem 26. *(Skiba [56]) If $\mathcal{H}$ is a σ-basis of G, then $\mathcal{P}(\mathcal{H})$ is a sublattice of $\mathcal{L}(G)$.*

Corollary 47. *(Doerk, Hawkes [21, I, 4.28]) If Σ is a Hall system of a soluble group G, then $\mathcal{P}(\Sigma)$ is a sublattice of $\mathcal{L}(G)$.*

In general, the lattice $\mathcal{P}(\Sigma)$, where Σ is a Hall system of a soluble group G, is not modular. Nevertheless, the following fact is true.

Theorem 27. *(Kimber [40]) Let Σ be a Hall system of a soluble group G. Then $\mathcal{P}(\Sigma)$ is modular if and only if every two subgroups of $\mathcal{P}(\Sigma)$ are permutable.*

Question 8. *Let $\mathcal{H}$ be a σ-basis of G. Characterize groups G with modular lattice $\mathcal{P}(\mathcal{H})$.*

References

[1] R.K. Agrawal: *Finite groups whose subnormal subgroups permute with all Sylow subgroups*, Proc. Amer. Math. Soc., **47**, 77–83, 1975.

[2] R.K. Agrawal: *Generalized center and hypercenter of a finite group*, Proc. Amer. Math. Soc., **54**, 13–21, 1976.

[3] K. Al-Sharo, J.C. Beidleman, H. Heineken, et al.: *Some characterizations of finite groups in which semipermutability is a transitive relation*, Forum Math., **22**, 855–862, 2010.

[4] M. Asaad: *Finite groups in which normality or quasinormality is transitive*, Arch. Math., **83**, n. 4, 289–296, 2004.

[5] M. Asaad, A.A. Heliel: *On permutable subgroups of finite groups*, Arch. Math., **80**, 113–118, 2003.

[6] A. Ballester-Bolinches, J. Cossey: *Totally permutable products of finite groups satisfying SC or PST*, Monatsh. Math., **145**, 89–93, 2005.

[7] A. Ballester-Bolinches, J.C. Beidlemann, R. Esteban-Romero: *On some classes of supersoluble groups*, J. Algebra, **312**, 445–454, 2007.

[8] A. Ballester-Bolinches, J.C. Beidleman, H. Heineken: *A local approach to certain classes of finite groups*, Comm. Algebra, **31**, 5931–5942, 2003.

[9] A. Ballester-Bolinches, J.C. Beidleman, H. Heineken: *Groups in which Sylow subgroups and subnormal subgroups permute*, Illinois J. Math., **47**, 63–69, 2003.

[10] A. Ballester-Bolinches, R. Esteban-Romero, *Sylow permutable subnormal subgroups*, J. Algebra, **251**, 727–738, 2002.

[11] A. Ballester-Bolinches, R. Esteban-Romero: *On finite soluble groups in which Sylow permutability is a transitive relation*, Acta Math. Hungar., **101**, 193–202, 2003.

[12] A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad: Products of Finite Groups, Walter de Gruyter, Berlin-New York, 2010.

[13] A. Ballester-Bolinches, L.M. Ezquerro: Classes of Finite Groups, Springer-Verlag, Dordrecht, 2006.

[14] J.C. Beidleman, B. Brewster, D.J.S. Robinson: *Criteria for permutability to be transitive in finite groups*, J. Algebra, **222**, 400–412, 1999.

[15] J.C. Beidleman, M.F. Ragland: *Subnormal, permutable, and embedded subgroups in finite groups*, Central Eur. J. Math., **9**, n. 4, 915–921, 2011.

[16] R.A. Brice, J. Cossey: *The Wielandt subgroup of a finite soluble groups*, J. London Math. Soc., **40**, 244–256, 1989.

[17] M. Curzio: *Sui sottogruppi di composizione dei gruppi finiti*, Reserche Mat., **7**, 265–280, 1958.

[18] O.H. Kegel: *Sylow-Gruppen and Subnormalteilerendlicher Gruppen*, Math. Z., **78**, 205–221, 1962.

[19] O.H. Kegel: *Untergruppenverbande endlicher Gruppen, die den subnormalteilerverband each enthalten*, Arch. Math., **30**, n. 3, 225–228, 1978.

[20] W.E. Deskins, *On quasinormal subgroups of finite groups*, Math. Z., **82**, 125–132, 1963.

[21] K. Doerk, T. Hawkes: Finite Soluble Groups, Walter de Gruyter, Berlin-New York, 1992.

[22] D. Gorenstein: Finite Groups , Harper & Row Publishers, New York-Evanston-London, 1968.

[23] W. Guo: *Finite groups with given indices of normalizers of Sylow subgroups*, Siberian Math. J., **37**, 207–214, 1996.

[24] W. Guo, K.P. Shum, A.N. Skiba: *X-semipermutable subgroups of finite groups*, J. Algebra, **315**, 31–41, 2007.

[25] W. Guo, A.N. Skiba: *On σ-supersoluble groups and one generalization of CLT-groups*, submitted.

[26] W. Guo, A.N. Skiba: *Finite groups with permutable Hall subgroups*, submitted.

[27] W. Guo, A.N. Skiba: *Finite groups with permutable complete Wielandt sets of subgroups*, J. Group Theory, **18**, 191–200, 2015.

[28] W. Guo, A.N. Skiba: *Finite groups whose n-maximal subgroups are σ-subnormal*, Preprint, 2015.

[29] W. Guo, A.N. Skiba: *On Π-permutable subgroups of finite groups*, submited.

[30] W. Guo, A. N. Skiba: *On some classes of finite quasi-$\mathfrak{F}$-groups*, J. Group Theory, **12**, 407–417, 2009.

[31] P. Hall: *A characteristic property of soluble groups*, J. London Math. Soc., **12**, 205–221, 1937.

[32] J.F. Humphreys: *On goups satifying the converse Lagrang's theorem*, Proc. Camb. Phil. Soc., **75**, 25–32, 1974.

[33] B. Huppert: Endliche Gruppen I, Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[34] B. Huppert: *Normalteiler und maximale Untergruppen endlicher Gruppen*, Math. Z., **60**, 409–434, 1954.

[35] B. Huppert: *Zur Sylowstruktur auflösbarer Gruppen*, Arch. Math., **12**, 161–169, 1961.

[36] B. Huppert: *Zur Sylowstruktur Auflosbarer Gruppen, II*, Arch. Math., **15**, 251–257, 1964.

[37] B. Huppert, N. Blackburn: Finite Groups III, Springer-Verlag, Berlin-New-York, 1982.

[38] I.M. Isaacs: *Semipermutable π-subgroups*, Arch. Math., **102**, 1–6, 2014.

[39] K. Iwasawa: *Über die endlichen Gruppen und die Verbände ihrer Untergruppen*, J. Fac. Sci. Imp. Univ. Tokyo. Sect., **1.4**, 171–199, 1941.

[40] T. Kimber: *Modularity in the lattice of Σ-permutable subgroups*, Arch. Math., **83**, 193–203, 2004.

[41] P.B. Kleidman: *A proof of the Kegel–Wielandt conjecture on subnormal subgroups*, Ann. Math., **133**, 369–428, 1977.

[42] J. Lio, S. Li: *$CLT-$groups with Hall S-quasinormally embedded subgroups*, Ukrain. Math. Journal, **66**, 1281–1287, 2014.

[43] J. Liu, S. Li, J. He: *CLT-groups with normal or subnormal subgroups*, J. Algebra, **362**, 99–106, 2012.

[44] S. Li, J. He, G. Nong, L. Zhou: *On Hall normally embedded subgroups of finite groups*, Comm. Algebra, **37**, 3360–3367, 2009.

[45] S. Li, J. Liu: *On Hall subnormally embedded and generalized nilpotent groups*, J. Algebra, **388**, 1–9, 2013.

[46] A. Mann: *Fnite groups whose n-maximal subgroups are subnormal*, Trans Amer. Math. Soc., **132**, 395–409, 1968.

[47] V.S. Monakhov, O.A. Shpyrko: *The nilpotent π-length of maximal Subgroups in finite π-soluble groups*, Moscow University Mathematics Bulletin, **64**, n. 6, 229–234, 2009.

[48] O. Ore: *Structures and group theory*, Duke Math, **4**, 247–269, 1938.

[49] O. Ore: *Contributions in the theory of groups of finite order*, Duke Math, **5**, 431–460, 1939.

[50] D.J.S. Robinson: *The structure of finite groups in which permutability is a transitive relation*, J. Austral. Math. Soc., **70**, 143–159, 2001.

[51] P. Schmid: *Subgroups permutable with all Sylow subgroups*, J. Algebra, **207**, 285–293, 1998.

[52] R. Schmidt: Subgroup lattices of groups, Walter de Gruyter, Berlin-New York, 1994.

[53] L.A. Shemetkov: Formations of finite groups, Nauka, Main Editorial Board for Physical and Mathematical Literature, Moscow, 1978.

[54] A.N. Skiba: *A generalization of a Hall theorem*, J. Algebra and its Application, **15**, n. 4, 21–36, 2015.

[55] A.N. Skiba: *On some properties of finite σ-soluble and σ-nilpotent groups*, Preprint.

[56] A.N. Skiba: *On the lattice of all* Π*-subnormal subgroups of finite groups*, Preprint.

[57] A.N. Skiba: *On σ-properties of finite groups I*, Problems of Physics, Mathematics and Technics, **4**, n. 21, 89–96, 2014.

[58] A.N. Skiba: *On σ-properties of finite groups II*, Problems of Physics, Mathematics and Technics, **3**, n. 24, 67–81, 2015.

[59] A.N. Skiba: *On σ-subnormal and σ-permutable subgroups of finite groups*, J. Algebra, **436**, 1–16, 2015.

[60] N.T. Vorob'ev, V.N. Zagurski: *Fitting classes with the given properties of Hall subgroups*, Math. notes, **78**, n. 2, 234–240, 2005.

[61] M. Weinstein ed.: Between Nilpotent and Solvable, Polygonal Publishing House, Passaic N.J., 1982.

[62] H. Wielandt: *Eine Verallgemenerung der invarianten Untergruppen*, Math. Z., **45**, 200–244, 1939.

[63] H. Wielandt: *Kriterion für Subnormalität in endlichen Gruppen*, Math. Z., **138**, 199–203, 1974.

[64] X. Yi, A.N. Skiba: *Some new characterizations of PST-groups*, J. Algebra, **399**, 39–54, 2014.

[65] G. Zacher: *Sui gruppi finiti per cui il reticolo dei sottogruppi di composizione e destributivo*, Rend. Sem. Mat. Univ. Padova, **27**, 75–79, 1957.

[66] G. Zappa: *Remark on a recent paper of O. Ore*, Duke. Math. J., **6**, 511–512, 1940.

[67] G. Zappa: *Sui gruppi finiti per cui il reticolo dei sottogruppi di composizione e modulare*, Boll. Un. Mat. Ital., **11**, n. 3, 315–318, 1956.

[68] J. Zhang: *Sylow numbers of finite groups*, J. Algebra, **176**, 111–123, 1995.