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Abstract differential calculus

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Abstract. We consider differentiable maps in the framework of Abstract Differential Geometry and we prove a number of calculus-type results pertaining to the chain rule, restrictions of differentiable maps to subspaces, and the differentiability of maps to and from cartesian products of spaces. We elaborate the latter in the case of functional structure sheaves to obtain a situation similar to that of smooth manifolds. Since the ε, δ-approach does not make sense here, our machinery is the existence of a number of limits in the category of differential triads.

Keywords: differential triad, morphism of differential triads, product, projective system, projective limit, direct sum

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Introduction

Differential calculus is the basic tool for the development of differential geometry, which, in turn, provides a very powerful machinery to deal with problems in many fields of mathematics and numerous applications.

However, the smooth manifold structure is a very strong assumption, not valid on an arbitrary topological space. Besides, the smooth structure breaks down when simple operations are applied on manifolds, such as the consideration of subsets or the forming of quotients. More generally, in the category of manifolds a number of limits, such as (infinite) products, projective or inductive limits, pull-backs and push-outs, do not exist.

There have been several attempts to develop the machinery of CDG on spaces that lack the manifold structure (i.e., "differential" and "diffeological" spaces, as in [3], [15], [23], [24], [25], [26]; see also [16] and the references therein) by introducing new classes of "smooth" functions, enlarging and substituting the ordinary smooth structure sheaf.

In the late 1980's (see [6]) A. Mallios, stimulated by a paper of S. Selesnick [22], innovated by replacing the functional structure sheaves with an abstract

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algebra sheaf $\mathcal{A}$, admitting a differential $\partial : \mathcal{A} \to \Omega$ (in the algebraic sense) taking values in an $\mathcal{A}$-module Ω. The central idea in the work of A. Mallios is that the classical differential geometry of a manifold is deduced from the algebraic properties of its structure sheaf $\mathcal{C}^{\infty}_{X}$, the "smooth" structure of the underlying manifold and the functional character of $\mathcal{C}^{\infty}_{X}$ being of secondary nature. He called the triplets $(\mathcal{A}, \partial, \Omega)$ *differential triads*. Differential triads generalize smooth manifolds (and differential spaces) and also include arbitrary topological spaces with very general, non-functional, structural sheaves. Using such structure sheaves, in this new framework of *Abstract Differential Geometry* (ADG), A. Mallios introduced many basic notions of CDG (such as Riemannian structures, connections, curvature, etc.) and proved numerous results analogous to the classical ones; see [8], [7], [9]. A. Malllios' machinery consists only of the algebraic properties of sheaves and of sheaf cohomology. He focused his study on "*vector sheaves*", the abstract (sheaf-theoretic) analogue of vector bundles. In the same vein, E. Vassiliou studied "*principal sheaves*", the analogue of principal bundles (cf. [28, 29, 30] and [31]). Possible applications of the above abstract setting to theoretical physics have been proposed and discussed in [10] and [14].

On the other hand, the present author (see [19]) introduced, in a similar algebraic way, a notion of "differentiable maps" between spaces with abstract differential structure, organizing thus differential triads into a category, denoted hereafter by $\mathcal{DT}$, which proved to be much richer in "limits" than the category of smooth manifolds.

Although in ADG "we do not use any notion of calculus (smoothness) in the classical sense" ([10, General Preface]), and in spite of E. Galois' claim that "les calculs sont impraticables", in the present paper we prove that the existence of certain limits in $\mathcal{DT}$ (most of which do not exist in the category of smooth manifolds) entails the development of a sort of "abstract differential calculus", thus extending and enforcing the machinery of ADG.

For instance, in contrast with the situation met in smooth manifolds, we prove that every subset of a space with a differential triad inherits the differential structure, and every differentiable map restricted to an arbitrary subset of the original space remains differentiable (Section 2).

In Section 3, we prove first that $\mathcal{DT}$ has (infinite) products (Theorem 3); then we consider maps taking values in a cartesian product of spaces with differential structure, and we obtain that the differentiability of such a "vector map" is equivalent to the differentiability of its "coordinates" (Theorem 4).

Direct sums do not exist in $\mathcal{DT}$, as it is also the case for smooth manifolds and topological spaces. However, we do construct direct sums in the subcategory $\mathcal{DT}_X$ of differential triads over a fixed base space X (Theorem 5). Mimicking the construction of direct sums, we define over $X \times Y$ a differential triad, which we call *the fibre product of differential triads.* Then, under an appropriate condition similar to that of the classical smooth case, we prove that the differentiability of a map on $X \times Y$ with respect to the fibre product is equivalent to the differentiability of the partial maps (Theorem 6).

In the last section, restricting ourselves to *functional* structure sheaves, we prove that the product of differential triads can be considered instead of the fibre product, as it happens in the classical case of smooth manifolds, where one uses the same differential structure (: differential atlas) to study maps either *from* or *to* $X \times Y$ (Proposition 4).

1 The category of differential triads

For the reader's convenience we recall the definitions of differential triads and of their morphisms, and we give a brief account of their constituting a category. Throughout the paper $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$.

Definition 1 ([9]). Let X be a topological space. A *differential triad* over X is a triplet $\delta = (\mathcal{A}, \partial, \Omega)$, where $\mathcal{A}$ is a sheaf of unital, commutative, associative $\mathbb{K}$-algebras over X, Ω is an $\mathcal{A}$-module and $\partial : \mathcal{A} \to \Omega$ is a *Leibniz morphism*, i.e., a $\mathbb{K}$-linear sheaf morphism, satisfying the *Leibniz condition*:

$$\partial(ab) = a\partial(b) + b\partial(a), \qquad (a, b) \in \mathcal{A} \times_X \mathcal{A}.$$

Examples 1. (1) Every *smooth manifold* X is provided with a real differential triad $(\mathcal{C}^\infty_X, d_X, \Omega^1_X)$, induced by the smooth structure: $\mathcal{C}^\infty_X$ is the sheaf of germs of smooth real valued functions on X, Ω^1_X is the sheaf of germs of smooth 1-forms on X, and d_X is the sheafification of the ordinary differential. We call $(\mathcal{C}^\infty_X, d_X, \Omega^1_X)$ the *smooth differential triad* of the manifold X. Thus differential triads generalize and extend the notion of smooth manifolds.

(2) The notion of *differential spaces* and the subsequent differential-geometric concepts on them have been introduced by R. Sikorski ([23], [24]). Their sheaf-theoretic generalization is due to M. A. Mostow ([15]). The latter approach provides an example of a differential triad whose structure sheaf is still a functional algebra sheaf. On the other hand, *Spallek's ∞-standard differential spaces* [16] have structure sheaves induced by the quotient of $\mathcal{C}^\infty(\mathbb{R}^n)$ by some closed ideal $\mathfrak{a}$, thus constituting an example of a differential triad whose structure sheaf is not a functional sheaf in the usual sense. A special case of an algebra which is a quotient of a functional algebra by a certain ideal is Rosinger's *nowhere dense differential algebra of generalized functions.* The resulting differential triad is studied in [12]; see also [13].

(3) Consider now an *algebraized space* $(X, \mathcal{A})$, namely, a topological space X along with a sheaf of (abstract) unital, commutative and associative $\mathbb{K}$-algebras

over X. For every open $U \subseteq X$, the algebra $\mathcal{A}(U)$ of continuous sections of $\mathcal{A}$ over U admits its Kähler differential obtained by Kähler's universal construction ([1, Ch. III, $\mathcal{S}$ 10.10-10.11]). The target spaces of these differentials form a presheaf generating an $\mathcal{A}$-module Ω, while the family of the Kähler differentials constitute a presheaf morphism generating a differential $\partial : \mathcal{A} \to \Omega$ (for details, see [9, Vol. II]). Thus a differential triad arises on every algebraized space *making ADG applicable to arbitrary topological spaces.*

In order to introduce the notion of a morphism of differential triads, we notice that if $\delta_X := (\mathcal{A}_X, \partial_X, \Omega_X)$ is a differential triad over X and $f : X \to Y$ is a continuous map, then *the pull-back of δ_Y by f*

$$f^*(\delta_Y) \equiv (f^*(\mathcal{A}_Y), f^*(\partial_Y), f^*(\Omega_Y))$$

is a differential triad over X.

Definition 2. Let $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$ and $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y)$ be differential triads over the topological spaces X and Y, respectively. A *morphism of differential triads* $\widehat{f} : \delta_X \to \delta_Y$ is a triplet $\widehat{f} = (f, f_{\mathcal{A}}, f_{\Omega})$, where

(i) $f : X \to Y$ is a continuous map;

(ii) $f_{\mathcal{A}} : f^*(\mathcal{A}_Y) \to \mathcal{A}_X$ is a unit preserving morphism of sheaves of algebras;

(iii) $f_{\Omega} : f^*(\Omega_Y) \to \Omega_X$ is an $f_{\mathcal{A}}$-morphism, namely,

$$f_{\Omega}(x, aw) = f_{\mathcal{A}}(x, a) f_{\Omega}(x, w), \quad \forall\ ((x, a), (x, w)) \in f^*(\mathcal{A}_Y) \times_X f^*(\Omega_Y);$$

(iv) The diagram

$$\begin{tikzcd}
f^*(\mathcal{A}_Y) \arrow[r, "f_{\mathcal{A}}"] \arrow[d, "f^*(\partial_Y)"'] & \mathcal{A}_X \arrow[d, "\partial_X"] \\
f^*(\Omega_Y) \arrow[r, "f_{\Omega}"'] & \Omega_X
\end{tikzcd}$$

Diagram 1

is commutative.

Extending the standard terminology, we will say that a continuous mapping $f : X \to Y$ is *differentiable* (*with respect to δ_X and δ_Y*), if it is completed into a morphism $\widehat{f} = (f, f_{\mathcal{A}}, f_{\Omega}) : \delta_X \to \delta_Y$.

Note that if $f : X \to Y$ is differentiable, the morphism $\widehat{f} : \delta_X \to \delta_Y$ *is not uniquely determined.* The set of morphisms over a fixed $f : X \to Y$ is denoted by $Mor_f(\delta_X, \delta_Y)$.

If $\mathcal{S}$ and $\mathcal{T}$ are sheaves over the spaces X and Y, respectively, every continuous map $f : X \to Y$ defines a bijection (in fact, a natural isomorphism of functors)

$$\Phi : Mor(f^*(\mathcal{T}), \mathcal{S}) \longrightarrow Mor(\mathcal{T}, f_*(\mathcal{S})) \tag{1.1}$$

(see [2, $\mathcal{S}$ 4] or [27, Theor. 7.13]). Thus a morphism of differential triads

$$\widehat{f} = (f, f_{\mathcal{A}}, f_{\Omega}) : (\mathcal{A}_X, \partial_X, \Omega_X) \longrightarrow (\mathcal{A}_Y, \partial_Y, \Omega_Y)$$

corresponds bijectively to a triplet $(f,\ \Phi(f_{\mathcal{A}}),\ \Phi(f_{\Omega}))$, where

$$\Phi(f_{\mathcal{A}}) : \mathcal{A}_Y \longrightarrow f_*(\mathcal{A}_X)$$

is a unit preserving morphism of sheaves of algebras and

$$\Phi(f_{\Omega}) : \Omega_Y \longrightarrow f_*(\Omega_X)$$

is a $\Phi(f_{\mathcal{A}})$-morphism, making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{A}_Y \arrow[r, "\Phi(f_{\mathcal{A}})"] \arrow[d, "\partial_Y"'] & f_*(\mathcal{A}_X) \arrow[d, "f_*(\partial_X)"] \\
\Omega_Y \arrow[r, "\Phi(f_{\Omega})"'] & f_*(\Omega_X)
\end{tikzcd}$$

Diagram 2

We call $(f,\ \Phi(f_{\mathcal{A}}),\ \Phi(f_{\Omega}))$ *the push-out of* $\widehat{f}$. Observe that *the push-out of* δ_X *by* f

$$f_*(\delta_X) := (f_*(\mathcal{A}_X), f_*(\partial_X), f_*(\Omega_X))$$

is a differential triad over Y.

Every smooth map $f : X \to Y$ between smooth manifolds X, Y induces a morphism $(f, f_{\mathcal{A}}, f_{\Omega})$ between the smooth differential triads $(\mathcal{C}^\infty_X, d_X, \Omega_X)$, $(\mathcal{C}^\infty_Y, d_Y, \Omega_Y)$, whose push-out is given by the equalities

$$\begin{aligned} \Phi(f_{\mathcal{A}})(\alpha) &:= \alpha \circ f\ , \quad \alpha \in \mathcal{C}^\infty_Y(V)\,, \\ \Phi(f_{\Omega})(\omega) &:= \omega \circ df\ , \quad \omega \in \Omega_Y(V), \end{aligned} \tag{1.2}$$

for every open $V \subseteq Y$. In this case, the commutativity of Diagram 2 is a result of the chain rule.

Remarks 1. (1) Although the push-out version of a morphism of differential triads is a straightforward abstraction of the situation met in the category of manifolds (see (1.2)) and it has been used as the definition of morphism ([19]), the pull-back version given in Definition 2 is more convenient for our purpose.

(2) If smooth differential triads are considered over smooth manifolds, the theory of topological algebras ensures that a differentiable map between the base spaces (in the abstract algebraic sense of Definition 2) is in fact a smooth map in the ordinary sense. Thus, if ADG is applied on situations where CDG is also applicable, both theories give the same results ([5]).

Differential triads and their morphisms form a category, with the following *composition law*: If $\delta_X, \delta_Y, \delta_Z$ are differential triads over the topological spaces X, Y, Z, respectively, and $\widehat{f} = (f, f_{\mathcal{A}}, f_{\Omega}) : \delta_X \to \delta_Y$, $\widehat{g} = (g, g_{\mathcal{A}}, g_{\Omega}) : \delta_Y \to \delta_Z$ are morphisms, then the composite

$$\widehat{g \circ f} = (g \circ f, (g \circ f)_{\mathcal{A}}, (g \circ f)_{\Omega}) : \delta_X \longrightarrow \delta_Z \tag{1.3}$$

is given by

$$\begin{aligned} (g \circ f)_{\mathcal{A}} &:= f_{\mathcal{A}} \circ f^*(g_{\mathcal{A}}) \,, \\ (g \circ f)_{\Omega} &:= f_{\Omega} \circ f^*(g_{\Omega}) \end{aligned} \tag{1.4}$$

(see [19] for the push-out version of $\widehat{g \circ f}$). The category of differential triads, their morphisms and the composition law defined by (1.3) and (1.4) will be denoted by $\mathcal{DT}$. Remark 1(2) implies that *the category of smooth manifolds is a full subcategory of* $\mathcal{DT}$. Note that the identity id_δ of a differential triad $\delta = (\mathcal{A}, \partial, \Omega)$ over X is the triplet $(\mathrm{id}_X, \mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\Omega})$. The subcategory of $\mathcal{DT}$ consisting of all differential triads over a fixed topological space X and of all morphisms over the identity map id_X, will be denoted by $\mathcal{DT}_X$.

Seen from the calculus point of view, the preceding can be rephrased as follows:

Proposition 1. *Let δ_I be a differential triad over the topological space $I = X, Y, Z$ and let $f : X \to Y$ and $g : Y \to Z$ be differentiable maps. Then $g \circ f$ is differentiable. Besides, if $\widehat{f} = (f, f_{\mathcal{A}}, f_{\Omega})$ and $\widehat{g} = (g, g_{\mathcal{A}}, g_{\Omega})$ are morphisms, then a morphism over $g \circ f$ is the triplet $(g \circ f, (g \circ f)_{\mathcal{A}}, (g \circ f)_{\Omega})$, which satisfies the "chain rule" (1.4).*

2 Differential subspaces

If X is a manifold and A is an arbitrary subset of X, in general A does not inherit the manifold structure. In the present section we prove that this shortcoming in dealing with subsets of the spaces under consideration is removed in the category $\mathcal{DT}$. First we give the following.

Definition 3. Let X be a topological space with a differential triad δ. A *differential subspace* of X is an arbitrary subset $A \subseteq X$ provided with the pull-back triad $i^*(\delta)$, where $i : A \to X$ is the canonical injection.

We note that the pull-back of δ by i coincides with the restriction of $(\mathcal{A}, \partial, \Omega)$ to A, namely

$$i^*(\delta) = (\mathcal{A}|_A,\, \partial|_A,\, \Omega|_A), \tag{2.1}$$

where, of course, $\partial|_A$ denotes the restriction of ∂ to $\mathcal{A}|_A$.

Proposition 2 ([20]). *Let X be a topological space with a differential triad $\delta = (\mathcal{A}, \partial, \Omega)$ and let $A \subseteq X$ be provided with the pull-back triad $i^*(\delta)$, where $i : A \to X$ is the canonical injection. Then i is differentiable. More precisely, the triplet*

$$\widehat{i} := (i, \mathrm{id}_{i^*(\mathcal{A})}, \mathrm{id}_{i^*(\Omega)}) : i^*(\delta) \longrightarrow \delta \tag{2.2}$$

is a morphism over i.

On the other hand, the pair $(i^*(\delta), \widehat{i})$ has the following universal property: If δ_A is a differential triad over A making i differentiable and

$$\widehat{j} = (i, j_{\mathcal{A}}, j_{\Omega}) : \delta_A \longrightarrow \delta$$

is a morphism over i, then there exists a unique morphism $\widehat{h} : \delta_A \to i^*(\delta)$ over id_A, with $\widehat{i} \circ \widehat{h} = \widehat{j}$.

Apart from the subsets inheriting the differential structure of the space, differentiable maps restricted to subsets remain differentiable. That is, we have:

Corollary 1. *Let X, Y be endowed with the differential triads δ_X and δ_Y, respectively, let $\widehat{f} : \delta_X \to \delta_Y$ be a morphism over the map $f : X \to Y$ and let $A \subseteq X$ be endowed with the pull-back $i^*(\delta_X)$ of δ_X by the canonical injection $i : A \to X$. Then $f|_A : A \to Y$ is differentiable with respect to $i^*(\delta_X)$ and δ_Y, while a morphism over $f|_A$ is the composite*

$$\widehat{f|_A} = \widehat{f} \circ \widehat{i}. \tag{2.3}$$

A classical property of submanifolds is also valid for differential subspaces.

Proposition 3. *Let δ_X, δ_Y be differential triads over the spaces X, Y, respectively, and let $A \subseteq X$ with $i : A \to X$ the canonical injection. Furthermore, assume that $f : Y \to X$ is a continuous map, such that $f(Y) \subseteq A$. Then $f : Y \to X$ is differentiable with respect to δ_Y and δ_X if and only if $f : Y \to A$ is differentiable with respect to δ_Y and $i^*(\delta_X)$.*

Proof. Since $(i \circ f)^*(\delta_X) = f^*(i^*(\delta_X))$, every morphism $\widehat{i \circ f} : \delta_Y \to \delta_X$ over $i \circ f$ corresponds bijectively to a morphism $\widehat{f} : \delta_Y \to i^*(\delta_X)$ over f. $\boxed{QED}$

3 Products

Our aim is now to relate the differentiability of a map $f : Y \to X$, taking values in a product $X = \prod_{i \in I} X_i$, with that of its coordinates $f_i : Y \to X_i$. To this end, for given differential triads δ_i on the factors X_i, $i \in I$, we need to define a suitable differential triad on the product. It turns out that this triad is the product of δ_i's in $\mathcal{DT}$.

First we define finite products. Details are found in [19]. Let $\delta_i = (\mathcal{A}_i, \partial_i, \Omega_i)$ be a differential triad over X_i, $i = 1, 2$. The *product of* δ_1 *and* δ_2 is a differential triad $\delta = (\mathcal{A}, \partial, \Omega)$ over $X = X_1 \times X_2$. The sheaf of algebras $\mathcal{A}$ is generated by the presheaf

$$U \times V \longmapsto \mathcal{A}_1(U) \otimes \mathcal{A}_2(V), \qquad U \in \tau_{X_1},\ V \in \tau_{X_2}, \tag{3.1}$$

with restrictions

$$\mu^{U \times V}_{U' \times V'} := \rho^U_{U'} \otimes \lambda^V_{V'} : \mathcal{A}_1(U) \otimes \mathcal{A}_2(V) \longrightarrow \mathcal{A}_1(U') \otimes \mathcal{A}_2(V'),$$

where $(\rho^U_{U'})$ and $(\lambda^V_{V'})$ are the restrictions of the presheaves of sections of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. The stalks of $\mathcal{A}$ satisfy

$$\mathcal{A}_{(x,y)} = \mathcal{A}_{1x} \otimes \mathcal{A}_{2y}. \tag{3.2}$$

The $\mathcal{A}$-module Ω is generated by the presheaf

$$U \times V \longmapsto (\mathcal{A}_1(U) \otimes \Omega_2(V)) \times (\Omega_1(U) \otimes \mathcal{A}_2(V)), \qquad U \in \tau_X,\ V \in \tau_Y, \tag{3.3}$$

with restrictions

$$\nu^{U \times V}_{U' \times V'} := (\rho^U_{U'} \otimes \ell^V_{V'}) \times (r^U_{U'} \otimes \lambda^V_{V'}),$$

where $(r^U_{U'})$ and $(\ell^V_{V'})$ are the restrictions of the presheaves of sections of Ω_1 and Ω_2, respectively. Regarding the stalks of Ω, we have

$$\Omega_{(x,y)} = (\mathcal{A}_{1x} \otimes \Omega_{2y}) \times (\Omega_{1x} \otimes \mathcal{A}_{2y}). \tag{3.4}$$

All the tensor products appearing here (and in what follows) are considered with respect to $\mathbb{K}$. Besides, $\partial : \mathcal{A} \to \Omega$ is the sheafification of the presheaf morphism $(\partial_{U \times V})$, where

$$\partial_{U \times V}(\alpha \otimes \beta) := (\alpha \otimes \partial_{2V}(\beta),\ \partial_{1U}(\alpha) \otimes \beta), \tag{3.5}$$

for every decomposable element $\alpha \otimes \beta \in \mathcal{A}_1(U) \otimes \mathcal{A}_2(V)$.

The product of δ_1 and δ_2 *makes the canonical projections* $p_1 : X_1 \times X_2 \to X_1$ *and* $p_2 : X_1 \times X_2 \to X_2$ *differentiable*: they are completed into the morphisms $\widehat{p}_1 = (p_1, p_{1\mathcal{A}}, p_{1\Omega})$ and $\widehat{p}_2 = (p_2, p_{2\mathcal{A}}, p_{2\Omega})$, where

$$p_{1\mathcal{A}} : p_1^*(\mathcal{A}_1) \longrightarrow \mathcal{A}_1 \otimes \mathcal{A}_2 : ((x,y),a) \longmapsto a \otimes 1_y, \tag{3.6}$$

$$p_{1\omega} : p_1^*(\Omega_1) \longrightarrow (\mathcal{A}_1 \otimes \Omega_2) \times (\Omega_1 \otimes \mathcal{A}_2) : ((x,y),\omega) \longmapsto (0, \omega \otimes 1_y) \tag{3.7}$$

and

$$p_{2\mathcal{A}} : p_2^*(\mathcal{A}_2) \longrightarrow \mathcal{A}_1 \otimes \mathcal{A}_2 : ((x,y),b) \longmapsto 1_x \otimes b, \tag{3.8}$$

$$p_{2\omega} : p_2^*(\Omega_2) \longrightarrow (\mathcal{A}_1 \otimes \Omega_2) \times (\Omega_1 \otimes \mathcal{A}_2) : ((x,y),\omega) \longmapsto (1_x \otimes \omega, 0). \tag{3.9}$$

We have the following

Theorem 1 ([19]). *Let $\delta_i = (\mathcal{A}_i, \partial_i, \Omega_i)$ be differential triads over the topological spaces X_i, $i = 1,2$. Then the product $\delta = (\mathcal{A}, \partial, \Omega)$ of δ_1 and δ_2 along with the morphisms $\widehat{p}_1 : \delta \to \delta_1$ and $\widehat{p}_2 : \delta \to \delta_2$ satisfy the universal property of the product in $\mathcal{DT}$.*

Since every pair of objects in $\mathcal{DT}$ has a product, every finite set of objects also has a product. That is, we have:

Corollary 2. *The category $\mathcal{DT}$ has finite products.*

On the other hand, it is also known ([21]) that certain projective systems in $\mathcal{DT}$ have limits:

Theorem 2. *Let $(\delta_i, \widehat{p}_{ij})$ be a projective system of differential triads over the spaces X_i, $i \in I$, where I is directed to the right. Then:*

(i) (X_i, p_{ij}) is a projective system of topological spaces; denote by (X, p_i) its projective limit.

(ii) The system $(\delta_i, \widehat{p}_{ij})$ has a projective limit $(\delta, \widehat{p}_i)$, where δ is a differential triad over X and every $\widehat{p}_i$ is a morphism over the respective canonical projection $p_i : X \to X_i$.

The existence of *infinite* products in $\mathcal{DT}$ is a result of the existence of finite products and that of projective limits: Let $\{X_i\}_{i \in I}$ be a family of topological spaces and let δ_i be a differential triad over X_i. We denote by A the set of all finite subsets of I, directed to the right by the relation

$$\beta \leq \alpha \;\Leftrightarrow\; \beta \subseteq \alpha.$$

For every $\alpha \in A$ we consider the finite product of δ_i, $i \in \alpha$,

$$(\delta_\alpha,\, (\widehat{q}_i^{\,\alpha})_{i \in \alpha})$$

over the space $X_\alpha := \prod_{i\in\alpha} X_i$. If $\{\beta_1,\beta_2\}$ is a partition of α, then δ_α is the product of δ_{β_1} and δ_{β_2}. We denote by $\widehat{q}^{\alpha}_{\beta_1}$ and $\widehat{q}^{\alpha}_{\beta_2}$ the respective projections. Then

$$\widehat{q}^{\beta_k}_i \circ \widehat{q}^{\alpha}_{\beta_k} = \widehat{q}^{\alpha}_i\,, \qquad i\in\beta_k,\ k=1,2.$$

Moreover, if $\gamma\subset\beta\subset\alpha\in A$, then

$$\widehat{q}^{\beta}_\gamma \circ \widehat{q}^{\alpha}_\beta = \widehat{q}^{\alpha}_\gamma.$$

Thus we obtain a projective system $(\delta_\alpha;\ \widehat{q}^{\alpha}_\beta)_{\beta\subseteq\alpha\in A}$ indexed by the directed set A, which has a limit in $\mathcal{DT}$

$$(\delta_X, (\widehat{q}_\alpha)_{\alpha\in A}),$$

over the space $X = \varprojlim X_\alpha = \prod_i X_i$. Since, for each $i\in I$, we have $\{i\}\in A$, the family $\{\widehat{q}_\alpha\}_{\alpha\in A}$ contains the subfamily $\{\,\widehat{q}_{\{i\}}\}_{\{i\}\in A}$. By combining together the universal properties of (finite) products and projective limits, it easily follows that

$$(\delta_X, (\widehat{p}_i := \widehat{q}_{\{i\}})_{i\in I})$$

satisfies the universal property of the product of the family $(\delta_i)_{i\in I}$ in $\mathcal{DT}$. Consequently, we have shown the following

Theorem 3. *Every family of differential triads has a product in $\mathcal{DT}$.*

Rephrasing the universal property of the product, we obtain the following calculus-type result:

Theorem 4. *Assume that X_i, $i\in I$, and Y are topological spaces and δ_i, δ_Y are differential triads over X_i, Y, respectively. Let $(X,(p_i)_{i\in I})$ denote the product of the topological spaces X_i, let $(\delta,(\widehat{p}_i)_{i\in I})$ be the product of $(\delta_i)_{i\in I}$ in $\mathcal{DT}$. Then a continuous map $f: Y\to X$ is differentiable with respect to δ_Y and δ, if and only if each coordinate $f_i := p_i\circ f$ is differentiable with respect to δ_Y and δ_i, $i\in I$.*

Proof. If f is differentiable, then there exists a morphism $\widehat{f}:\delta_Y\to\delta$. By the composition law, $\widehat{p}_i\circ\widehat{f}:\delta_Y\to\delta_i$ is a morphism over $f_i = p_i\circ f$, that is, each f_i is differentiable.

Conversely, if each f_i is differentiable and $\widehat{f_i}$ is the respective morphism, then by the universal property of the product there is a unique morphism

$$\widehat{g} = (g, g_{\mathcal{A}}, g_\Omega) : \delta_Y \longrightarrow \delta$$

such that $\widehat{p}_i\circ\widehat{g} = \widehat{f_i}$. This implies that $p_i\circ g = f_i$, and the universal property of the product of topological spaces assures that $g = f$. Thus $(f, g_{\mathcal{A}}, g_\Omega)$ is a morphism over f and f is differentiable. $\boxed{QED}$

In the preceding theorem, if f is differentiable, there exists a morphism $\widehat{f} : \delta_Y \to \delta_X$ over f. Then each f_i is differentiable, because it is completed into the morphism

$$\widehat{f_i} := \widehat{p_i} \circ \widehat{f}, \tag{3.10}$$

given by the composition law. As we noted before, the morphism $\widehat{f_i}$ over f_i is not uniquely determined, thus there may be other morphisms over f_i. Conversely, if f_i are differentiable and $\widehat{f_i}$ are morphisms over f_i, $i \in I$, then by the universal property of the product, there is a unique morphism $\widehat{f} : \delta_Y \to \delta_X$ satisfying (3.10). Again, there may be other morphisms over f, too.

Thus, there may exist many pairs $(\widehat{f}, (\widehat{f_i})_{i\in I})$ satisfying (3.10), but in each pair $\widehat{f}$ and $(\widehat{f_i})_{i\in I}$ determine each other. In other words, we have

Corollary 3. *Under the assumptions of the preceding theorem, there exists a bijection*

$$Mor_f(\delta_Y, \delta) \cong \prod_i Mor_{f_i}(\delta_Y, \delta_i)$$

induced by the composition law.

4 Direct sums over X

In the category of manifolds, one treats differentiable maps *to* and *from* a product $X \times Y$ of manifolds, by considering on $X \times Y$ the *same differential structure* (: the same atlas), in both cases. In the present abstract setting, the differential structure (: differential triad) which is suitable to treat maps *to* a product is the product of differential triads; but this structure is not appropriate for treating maps *from* a cartesian product of spaces with differential structure. In the latter case, a kind of dual structure (: direct product) is needed. However, direct sums do not exist in $\mathcal{DT}$, because their existence would imply the existence of direct sums in the category of topological spaces: Indeed, let X and Y be topological spaces endowed with differential triads δ_X and δ_Y, respectively, and assume that they have a direct sum $(\delta; \widehat{i}_X, \widehat{i}_Y)$, where δ is a differential triad over some topological space Z and $\widehat{i}_X : \delta_X \to \delta$, $\widehat{i}_Y : \delta_Y \to \delta$ are morphisms in $\mathcal{DT}$, over some continuous maps $i_X : X \to Z$ and $i_Y : Y \to Z$. If δ_K is a differential triad over a topological space K and $\widehat{f}_X : \delta_X \to \delta$, $\widehat{f}_Y : \delta_Y \to \delta$ are morphisms over $f_X : X \to K$, $f_Y : Y \to K$, by the universal property of the direct sum, there would exist a unique morphism $\widehat{f} : \delta \to \delta_K$, such that $\widehat{f} \circ \widehat{i}_X = \widehat{f}_X$ and $\widehat{f} \circ \widehat{i}_Y = \widehat{f}_Y$. The latter equalities imply $f \circ i_X = f_X$ and $f \circ i_Y = f_Y$, a situation not attained, in general, in the category of topological spaces.

Since this difficulty disappears when we restrict ourselves to differential triads over a *fixed* base space X, we consider the same problem in $\mathcal{DT}_X$. Let $\delta_i = (\mathcal{A}_i, \partial_i, \Omega_i) \in \mathcal{DT}_X$, $i \in I$. We define the *fibre product of* $(\delta_i)_{i\in I}$ to be the triplet

$$\delta = (\mathcal{A}, \partial, \Omega) := (\textstyle\prod_X \mathcal{A}_i,\ \prod_X \partial_i,\ \prod_X \Omega_i), \tag{4.1}$$

where $\prod_X \mathcal{A}_i$ and $\prod_X \Omega_i$ are the fibre products of the families $(\mathcal{A}_i)_{i\in I}$ and $(\Omega_i)_{i\in I}$; namely, they are generated by the presheaves

$$(\textstyle\prod_X \mathcal{A}_i)(U) = \prod_{i\in I} \mathcal{A}_i(U),$$
$$(\textstyle\prod_X \Omega_i)(U) = \prod_{i\in I} \Omega_i(U), \qquad U \in \tau_X$$

which define the total spaces

$$\textstyle\prod_X \mathcal{A}_i = \bigcup_{x\in X} \prod_{i\in I} \mathcal{A}_{ix}$$
$$\textstyle\prod_X \Omega_i = \bigcup_{x\in X} \prod_{i\in I} \Omega_{ix}$$

and $\partial = \prod_X \partial_i$ is the sheaf morphism induced by the presheaf morphism $(\partial_U)_{U\in\tau_X}$, where

$$\partial_U : \textstyle\prod_X \mathcal{A}_i(U) \longrightarrow \prod_X \Omega_i(U) : (\alpha_i) \longmapsto (\partial_{iU}(\alpha_i)).$$

It is straightforward that (4.1) is a differential triad.

Theorem 5. *The category $\mathcal{DT}_X$ has direct products.*

Proof. Let $\delta_i = (\mathcal{A}_i, \partial_i, \Omega_i)$, $i \in I$, be a family of differential triads in $\mathcal{DT}_X$ and let $\delta = (\mathcal{A}, \partial, \Omega)$ denote their fibre product. Then, for every $i \in I$, the canonical projections

$$u_{i\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}_i \qquad \text{and} \qquad u_{i\Omega} : \Omega \longrightarrow \Omega_i$$

are sheaf morphisms inducing the morphism of differential triads

$$\widehat{u}_i = (id_X, u_{i\mathcal{A}}, u_{i\Omega}) : \delta_i \to \delta. \tag{4.2}$$

We claim that $(\delta, (\widehat{u}_i))$ has the universal property of the direct product in $\mathcal{DT}_X$. Indeed, let $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$ be a differential triad over X and let

$$\widehat{f}_i = (\mathrm{id}_X, f_{i\mathcal{A}}, f_{i\Omega}) : \delta_i \longrightarrow \delta_X$$

be a family of morphisms. We set

$$f_{\mathcal{A}} := (f_{i\mathcal{A}}) : \mathcal{A}_X \longrightarrow \mathcal{A} : a \longmapsto (f_{i\mathcal{A}}(a)),$$
$$f_\Omega := (f_{i\Omega}) : \Omega_X \longrightarrow \Omega : \omega \longmapsto (f_{i\Omega}(\omega)).$$

Then $\widehat{f} = (\mathrm{id}_X, f_{\mathcal{A}}, f_\Omega) : \delta \to \delta_X$ is the unique morphism in $\mathcal{DT}_X$ which satisfies

$$\widehat{f} \circ \widehat{u}_i = \widehat{f}_i,$$

for every $i \in I$. $\boxed{QED}$

5 Partial maps with respect to the fibre product

The way we construct the direct product in $\mathcal{DT}_X$ can be extended to provide a differential triad over $X \times Y$, which is suitable to handle maps *from* the cartesian product $X \times Y$: Thus, we consider X and Y provided with differential triads $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$ and $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y)$, respectively, and we define on $X \times Y$ the *fibre product*

$$(\mathcal{A}, \partial, \Omega) := (\mathcal{A}_X \times \mathcal{A}_Y, \partial_X \times \partial_Y, \Omega_X \times \Omega_Y). \tag{5.1}$$

The sheaves $\mathcal{A}$ and Ω are the sheafifications of the presheaves

$$\big(\mathcal{A}_X(U) \times \mathcal{A}_Y(V),\, \rho^U_{XU'} \times \rho^V_{YV'}\big)$$
$$\big(\Omega_X(U) \times \Omega_Y(V),\, \lambda^U_{XU'} \times \lambda^V_{YV'}\big)$$

and ∂ is the sheafification of the presheaf morphism

$$\Big(\partial_{XU} \times \partial_{YV} : \mathcal{A}_X(U) \times \mathcal{A}_Y(V) \longrightarrow \Omega_X(U) \times \Omega_Y(V)\Big).$$

The triplet (5.1) obviously is a differential triad.

Lemma 1. *Let $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$ and $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y)$ be differential triads over X and Y, respectively, and let $X \times Y$ be provided with the fibre product (5.1). Then, for every $x_o \in X$ and $y_o \in Y$, the canonical injections*

$$i_{y_o} : X \longrightarrow X \times Y : x \longmapsto (x, y_o),$$
$$i_{x_o} : Y \longrightarrow X \times Y : y \longmapsto (x_o, y)$$

are differentiable.

Proof. We prove the differentiability of i_{y_o}: We first notice that

$$i^*_{y_o}(\mathcal{A}) = \{(x, a, b) \in X \times \mathcal{A}_X \times \mathcal{A}_Y : a \in \mathcal{A}_{X,x} \text{ and } b \in \mathcal{A}_{Y,y_o}\};$$

namely,

$$(i^*_{y_o}(\mathcal{A}))_x = \{x\} \times \mathcal{A}_{X,x} \times \mathcal{A}_{Y,y_o}.$$

We set

$$i_{y_o\mathcal{A}} : i^*_{y_o}(\mathcal{A}) \longrightarrow \mathcal{A}_X : (x, a, b) \longmapsto a.$$

Since the pull-back has the relative topology induced by the product topology of $X \times \mathcal{A}_X \times \mathcal{A}_Y$, $i_{y_o\mathcal{A}}$ is continuous; clearly it is a unit preserving morphism of sheaves of algebras. Analogously, we have that

$$i^*_{y_o}(\Omega) = \{(x, \omega, \phi) \in X \times \Omega_X \times \Omega_Y : \omega \in \Omega_{X,x} \text{ and } \phi \in \Omega_{Y,y_o}\},$$

thus we now set

$$i_{y_o\Omega} : i_{y_o}^*(\Omega) \longrightarrow \Omega_X : (x, \omega, \phi) \longmapsto \omega.$$

Then $i_{y_o\Omega}$ is an $i_{y_o\mathcal{A}}$-morphism and the analog of Diagram 1 is commutative, since, for every $(x, a, b) \in i_{y_o}^*(\mathcal{A}_X \times \mathcal{A}_Y)$, we have

$$\begin{aligned} \partial_X \circ i_{y_o\mathcal{A}}(x, a, b) &= \partial_X(a) = i_{y_o\Omega}(x, \partial_X(a), \partial_Y(b)) \\ &= i_{y_o\Omega} \circ i_{y_o}^*(\partial_X \times \partial_Y)(x, a, b). \end{aligned}$$

Hence, the triplet $(i_{y_o}, i_{y_o\mathcal{A}}, i_{y_o\Omega})$ satisfies the conditions of Definition 2, making i_{y_o} differentiable. Similar reasoning proves that the maps

$$\begin{aligned} i_{x_o\mathcal{A}} : i_{x_o}^*(\mathcal{A}) &\longrightarrow \mathcal{A}_Y : (y, a, b) \longmapsto b\,, \\ i_{x_o\Omega} : i_{x_o}^*(\Omega) &\longrightarrow \Omega_Y : (y, \omega, \phi) \longmapsto \phi \end{aligned}$$

complete i_{x_o} into a morphism $\widehat{i}_{x_o} = (i_{x_o}, i_{x_o\mathcal{A}}, i_{x_o\Omega})$ in $\mathcal{DT}$. $\boxed{QED}$

We consider now a continuous map $f : X \times Y \to Z$, where X, Y, Z have differential triads δ_X, δ_Y, δ_Z, respectively, and $X \times Y$ is provided with the fibre product (5.1). If f is differentiable, a straightforward consequence of the preceding lemma is the differentiability of all its partial maps. We will study now the converse situation.

For every $y_o \in Y$, the differentiability of the partial map $f_{y_o} : X \to Z$ with respect to δ_X and δ_Z implies the existence of a morphism $\widehat{f_{y_o}} : \delta_X \to \delta_Z$; that is, of a pair of sheaf morphisms

$$\begin{aligned} (f_{y_o})_{\mathcal{A}} : f_{y_o}^*(\mathcal{A}_Z) &\to \mathcal{A}_X, \\ (f_{y_o})_{\Omega} : f_{y_o}^*(\Omega_Z) &\to \Omega_X. \end{aligned}$$

Since

$$f_{y_o}^*(\mathcal{A}_Z) = \{(x, c) \in X \times \mathcal{A}_Z : c \in \mathcal{A}_{Z, f(x, y_o)}\}$$

and

$$f^*(\mathcal{A}_Z) = \{(x, y, c) \in X \times Y \times \mathcal{A}_Z : c \in \mathcal{A}_{Z, f(x,y)}\},$$

the differentiability of all f_y's induces the well defined, fibre preserving map

$$f_{Y\mathcal{A}} : f^*(\mathcal{A}_Z) \longrightarrow \mathcal{A}_X : (x, y, c) \longmapsto (f_y)_{\mathcal{A}}(x, c); \tag{5.2}$$

it also induces

$$f_{Y\Omega} : f^*(\Omega_Z) \longrightarrow \mathcal{A}_X : (x, y, w) \longmapsto (f_y)_{\Omega}(x, w). \tag{5.3}$$

Similarly, the differentiability of all f_x's, $x \in X$, results in the existence of two well defined, fibre preserving maps

$$f_{X\mathcal{A}} : f^*(\mathcal{A}_Z) \longrightarrow \mathcal{A}_Y : (x, y, c) \longmapsto (f_x)_{\mathcal{A}}(y, c), \tag{5.4}$$

$$f_{X\Omega} : f^*(\Omega_Z) \longrightarrow \Omega_Y : (x, y, w) \longmapsto (f_x)_{\Omega}(y, w). \tag{5.5}$$

It is clear that the maps (5.2)-(5.5) are sheaf morphisms, if and only if they are continuous. Thus we have the following:

Theorem 6. *Let $\delta_I = (\mathcal{A}_I, \partial_I, \Omega_I)$ be differential triads over the spaces $I = X, Y, Z$. Besides, let $X \times Y$ be endowed with the fibre product $\delta = (\mathcal{A}, \partial, \Omega)$ of δ_X and δ_Y and let $f : X \times Y \to Z$ be a continuous map.*

If f is differentiable with respect to δ and δ_Z, then, for every $x \in X$, the partial map f_x is differentiable with respect to δ_Y and δ_Z. Similarly, for every $y \in Y$, f_y is differentiable with respect to δ_X and δ_Z.

Conversely, if, for every $y \in Y$ and $x \in X$, the partial maps $f_y : X \to Z$ and $f_x : Y \to Z$ are differentiable, so that the maps (5.2)-(5.5) are continuous, then f is differentiable with respect to δ and δ_Z.

In both cases, the induced morphisms over f, f_x and f_y satisfy the relations

$$\begin{aligned} f_{\mathcal{A}}(x, y, a) &:= ((f_y)_{\mathcal{A}}(x, a), (f_x)_{\mathcal{A}}(y, a)) \\ &= (f_{Y\mathcal{A}}(x, y, a), f_{X\mathcal{A}}(x, y, a)) \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} f_{\Omega}(x, y, \omega) &:= ((f_y)_{\Omega}(x, \omega), (f_x)_{\Omega}(y, \omega)) \\ &= (f_{Y\Omega}(x, y, \omega), f_{X\Omega}(x, y, \omega)). \end{aligned} \tag{5.7}$$

Proof. Let f be differentiable and $\widehat{f} = (f, f_{\mathcal{A}}, f_{\Omega})$ a morphism over f. The differentiability of $f_y = f \circ i_y$ and $f_x = f \circ i_x$ is a result of Lemma 1. Moreover, if $(x, y, a) \in f^*(\mathcal{A}_Z)$ and $(x, y, \omega) \in f^*(\Omega_Z)$, then

$$\begin{aligned} f_{\mathcal{A}}(x, y, a) &= (a_X, a_Y) \in \mathcal{A}_{X,x} \times \mathcal{A}_{Y,y}, \\ f_{\Omega}(x, y, w) &= (\omega_X, \omega_Y) \in \Omega_{X,x} \times \Omega_{Y,y}. \end{aligned}$$

Using the composition law (1.4) and the identification

$$(f \circ i_y)^*(\mathcal{A}_Z) \ni (x, a) \equiv (x, (x, y, a)) \in i_y^*(f^*(\mathcal{A}_Z)),$$

we obtain

$$\begin{aligned} (f_y)_{\mathcal{A}}(x, a) &= i_{y\mathcal{A}} \circ i_y^*(f_{\mathcal{A}})(x, (x, y, a)) \\ &= i_{y\mathcal{A}}(x, f_{\mathcal{A}}(x, y, a)) \\ &= i_{y\mathcal{A}}(x, a_X, a_Y) = a_X, \end{aligned} \tag{5.8}$$

and, similarly,

$$(f_y)_\Omega(x,\omega) = i_{y\Omega}(x,\omega_X,\omega_Y) = \omega_X. \tag{5.9}$$

Obviously, we also have

$$(f_x)_\mathcal{A}(y,\alpha) = \alpha_Y, \qquad \text{and} \qquad (f_x)_\Omega(y,\omega) = \omega_Y, \tag{5.10}$$

that is, (5.6) and (5.7) hold.

Conversely, if all partial maps are differentiable so that (5.2)-(5.5) are continuous, then the triplet $(f, f_\mathcal{A}, f_\Omega)$, given by (5.6) and (5.7), is a morphism in $\mathcal{DT}$. Indeed, the continuity of $f_{Y\mathcal{A}}$ and $f_{X\mathcal{A}}$ makes them unit preserving morphisms of sheaves of algebras, thus $f_\mathcal{A}$ has the same property. On the other hand, since $f_{Y\Omega}$ (resp. $f_{X\Omega}$) is an $f_{Y\mathcal{A}}$-morphism (resp. $f_{X\mathcal{A}}$-morphism), f_Ω is an $f_\mathcal{A}$-morphism. The commutativity of Diagram 1 is known stalk-wisely. $\boxed{QED}$

6 Partial maps on functional sheaves

Summarizing our discussion in Sections 3 and 5, we conclude that, given the differential triads δ_X and δ_Y over the spaces X and Y, we treat maps *to* $X \times Y$ by considering the product of δ_X and δ_Y, while maps *from* $X \times Y$ are treated by considering the fibre product $\delta_X \times \delta_Y$. Of course, these two constructions, in general, do not coincide. However, in the full subcategory of manifolds, one treats maps to and from a product $X \times Y$ using the same differential structure (: atlas) on $X \times Y$. This is due to a particular property shared by the differential triads with functional algebra sheaves, as we explain below. In this respect, we recall that a *functional algebra sheaf* over X is a sheaf of algebras which is a subsheaf of the sheaf $\mathcal{C}_X$ of germs of continuous $\mathbb{K}$-valued functions on X.

Lemma 2. *Let $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$, $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y)$ be differential triads over the spaces X, Y and let $X \times Y$ be provided with the product $\delta = (\mathcal{A}, \partial, \Omega)$ of δ_X and δ_Y. If the sheaf $\mathcal{A}_Y$ is functional, then, for every $y \in Y$, the canonical injection $i_y : X \to X \times Y$ is differentiable.*

Proof. First we notice that

$$i_y^*(\mathcal{A}_X \otimes \mathcal{A}_Y) \equiv (\mathcal{A}_X \otimes \mathcal{A}_Y)|_{X\times\{y\}} = \bigcup_{x\in X} \mathcal{A}_{X,x} \otimes \mathcal{A}_{Y,y} = \mathcal{A}_X \otimes \mathcal{A}_{Y,y}.$$

The preceding sheaf is the sheafification of the presheaf

$$\left(\mathcal{A}_X(U) \otimes \mathcal{A}_{Y,y}\,,\; r^U_{U'} \otimes \mathrm{id}_{\mathcal{A}_{Y,y}}\right),$$

where $(r^U_{U'})$ are the restrictions of the presheaf of sections of $\mathcal{A}_X$. We consider the $\mathbb{K}$-bilinear map

$$\mathcal{A}_X(U) \times \mathcal{A}_{Y,y} \longrightarrow \mathcal{A}_X(U) : (\alpha, [\beta]_y) \longmapsto \beta(y) \cdot \alpha$$

and the induced $\mathbb{K}$-linear map

$$(i_y)_{\mathcal{A}U} : \mathcal{A}_X(U) \otimes \mathcal{A}_{Y,y} \longrightarrow \mathcal{A}_X(U),$$

which is a unit preserving algebra morphism. Clearly, the family $((i_y)_{\mathcal{A}U})_{U\in\tau_X}$ is a presheaf morphism inducing the sheaf morphism

$$(i_y)_{\mathcal{A}} : \mathcal{A}_X \otimes \mathcal{A}_{Y,y} \longrightarrow \mathcal{A}_X,$$

given by

$$(i_y)_{\mathcal{A}}([\alpha]_x \otimes [\beta]_y) = \beta(y)[\alpha]_x = [\beta(y)\alpha]_x \tag{6.1}$$

on decomposable elements. In a similar way,

$$i_y^*((\mathcal{A}_X \otimes \Omega_Y) \times (\Omega_X \otimes \mathcal{A}_Y)) = (\mathcal{A}_X \otimes \Omega_{Y,y}) \times (\Omega_X \otimes \mathcal{A}_{Y,y}),$$

the latter being the sheafification of the presheaf

$$((\mathcal{A}_X(U) \otimes \Omega_{Y,y}) \times (\Omega_X(U) \otimes \mathcal{A}_{Y,y}),\ (r^U_{U'} \otimes \mathrm{id}_{\Omega_{Y,y}}) \times (\ell^U_{U'} \otimes \mathrm{id}_{\mathcal{A}_{Y,y}})),$$

where $(\ell^U_{U'})$ are the restrictions of the presheaf of sections of Ω_X. Then a $\mathbb{K}$-linear map

$$\Omega_X(U) \otimes \mathcal{A}_{Y,y} \longrightarrow \Omega_X(U) : \omega \otimes [\beta]_y \longmapsto \beta(y) \cdot \omega$$

is induced by the $\mathbb{K}$-bilinear map

$$\Omega_X(U) \times \mathcal{A}_{Y,y} \longrightarrow \Omega_X(U) : (\omega, [\beta]_y) \longmapsto \beta(y) \cdot \omega$$

and extends to an $(i_y)_{\mathcal{A}U}$-map

$$\begin{aligned}(i_y)_{\Omega U} : (\mathcal{A}(U) \otimes \Omega_y) \times (\Omega_X(U) \otimes \mathcal{A}_{Y,y}) \longrightarrow \Omega_X(U) :\\ (\alpha \otimes [\phi]_y, \omega \otimes [\beta]_y) \longmapsto \beta(y) \cdot \omega.\end{aligned}$$

Similarly, the family $((i_y)_{\Omega U})_{U\in\tau_X}$ is a presheaf morphism inducing the $(i_y)_{\mathcal{A}}$-morphism

$$(i_y)_{\Omega} : (\mathcal{A}_X \otimes \Omega_{Y,y}) \times (\Omega_X \otimes \mathcal{A}_{Y,y}) \longrightarrow \Omega_X$$

given by

$$(i_y)_{\Omega}([\alpha]_x \otimes [\phi]_y, [\omega]_x \otimes [\beta]_y) = \beta(y)[\omega]_x = [\beta(y)\omega]_x \tag{6.2}$$

on decomposable elements. Besides, the pair $((i_y)_{\mathcal{A}}, (i_y)_{\Omega})$ makes Diagram 1 commutative. In fact, $i_y^*(\partial) \equiv \partial|_{i_y^*(\mathcal{A})}$ and

$$\begin{aligned}(i_y)_{\Omega} \circ \partial|_{i_y^*(\mathcal{A})}([\alpha]_x \otimes [\beta]_y) &= (i_y)_{\Omega}([\alpha]_x \otimes \partial_Y([\beta]_y), \partial_X([\alpha]_x) \otimes [\beta]_y)\\ &= \beta(y) \cdot \partial_X([\alpha]_x) = \partial_X(\beta(y) \cdot [\alpha]_x)\\ &= \partial_X \circ (i_y)_{\mathcal{A}}([\alpha]_x \otimes [\beta]_y),\end{aligned}$$

for every decomposable element $[\alpha]_x \otimes [\beta]_y \in i_y^*(\mathcal{A})$. We deduce that the triplet $(i_y, (i_y)_{\mathcal{A}}, (i_y)_{\Omega})$ is a morphism over i_y. $\boxed{QED}$

Similar reasoning proves the differentiability of each i_x, if the sheaf $\mathcal{A}_X$ is functional. As a result, we have the following:

Proposition 4. *Consider the differential triads $\delta_I = (\mathcal{A}_I, \partial_I, \Omega_I)$ over the spaces $I = X, Y, Z$ and let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be functional. Besides, let $X \times Y$ be provided with the product $\delta = (\mathcal{A}, \partial, \Omega)$ of δ_X and δ_Y. If $f : X \times Y \to Z$ is differentiable with respect to δ and δ_Z, then, for every $x \in X$ (resp. $y \in Y$), the partial map f_x (resp. f_y) is differentiable with respect to δ_Y (resp. δ_X) and δ_Z.*

In particular, if $\widehat{f} = (f, f_{\mathcal{A}}, f_{\Omega})$ is a morphism over f, then a pair of morphisms $\widehat{f}_x$ and $\widehat{f}_y$ is given by

$$f_{x\mathcal{A}}(y, c) = \sum_i [\alpha_i(x)\beta_i]_y \tag{6.3}$$

$$f_{y\mathcal{A}}(x, c) = \sum_i [\beta_i(y)\alpha_i]_x \tag{6.4}$$

for every $c \in \mathcal{A}_{Z,f(x,y)}$ with

$$f_{\mathcal{A}}((x, y), c) = \sum_i [\alpha_i]_x \otimes [\beta_i]_y \in \mathcal{A}_{X,x} \otimes \mathcal{A}_{Y,y},$$

and by

$$f_{x\Omega}(y, \psi) = \sum_i [\alpha_i(x)\phi_i]_y \tag{6.5}$$

$$f_{y\mathcal{A}}(x, \psi) = \sum_i [\beta_i(y)\omega_i]_x \tag{6.6}$$

for every $\psi \in \Omega_{Z,f(x,y)}$ with

$$f_{\Omega}((x, y), \psi) = \sum_i ([\alpha_i]_x \otimes [\phi_i]_y, [\omega_i]_x \otimes [\beta_i]_y) \in (\mathcal{A}_{X,x} \otimes \Omega_{Y,y}) \times (\Omega_{X,x} \otimes \mathcal{A}_{Y,y}).$$

Proof. The differentiability of the partial maps is a result of Lemma 2. The relations (6.3)–(6.6) are a consequence of the chain rule. $\boxed{QED}$

Final Remark. In the present work, we have dealt only with the differential calculus within the framework of *commutative* algebras. Preliminary results pertaining to sheaves of specific *non-commutative* algebras have been discussed, for instance in [9, 11, 17, 18]. These results are intended to lead to aspects of symplectic geometry but they may also pave the way to the study of sheaves of C*-algebras and a sheaf-theoretic differential calculus in the sense of A.Connes [4]. This point of view is beyond the scope of the present paper.

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