Note di Matematica Vol.III, 199-211 (1983)

DECOMPOSITION-AGGREGATION APPROACH IN STABILITY PROBLEMS FOR LARGE-SCALE SYSTEMS INCLUDING UNSTABLE ISOLATED SUBSYSTEMS(*)

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Abstract. *Sufficient conditions for asymptotic partial stability of large scale systems including unstable isolated subsystems are given. The proofs are based on the construction of a Lyapunov scalar function, obtained as a weighted sum of the Lyapunov functions used to "measure" either the stability properties of the isolated subsystems or the stabilizing effects of the interconnecting terms.*

1. When the decomposition-aggregation approach is adopted to investigate via Direct Lyapunov Method ([2] -[5] ,[7] -[16]) if a large-scale system has a stability property P, it is usually assumed that a stability analysis performed on the isolated subsystems gives the following result

a) all the isolated subsystems have the property P, which is "measurable" by means of Lyapunov functions. Conditions on the interconnection structure are then specified in order to ensure that the overall system has property P.

(*) This work has been performed under the auspices of G.N.F.M. of Italian Council of Research (C.N.R.).

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However, the analysis of stability of the isolated subsystems may also give the following results

b) none of the isolated subsystems has property P or, anyhow, it is impossible to state it, or

c) a part of the isolated subsystems has property P and the remaining ones have not it or, anyhow, for them it is impossible to state it.

The difficulties which arise when b) or c) happens could be, a priori, theoretically bypassed adopting a different decomposition of the overall system. But, in many practical examples the own structure of processes suggests or imposes a well defined decomposition so that, if b) or c) takes place, the mentioned approach could appear unsuitable. In these cases, obviously, it is not excluded that the overall system could have property P, owing to a possible stabilizing effect of the interconnecting terms. As it has been pointed out ([1], [12]), that may happen in many practical problems concerning with interconnected systems (strongly coupled). Therefore a careful analysis of the stabilizing effects of the interconnection must be performed. We point out that if a "measure" of stabilizing effects of the interconnection structure can be given in terms of Lyapunov functions the decomposition-aggregation approach can be usefully applied even in the cases b) and c).

In our paper, in order to illustrate the above considerations,we give three theorems correspondingly to the cases a), b) and c) in which property P is identified with the uniform asymptotic partial stability. Partial stability problems for large-scale systems have been considered in [3]. They arise, for example, when only some of the state variables of a process may be accessible, whereas it is impossible to obtain informations about the remaining ones because, for instance, of unremovable disturbances.

To prove our theorems we use a scalar Lyapunov function for the overall system obtained as a weighted sum of Lyapunov functions which "measure" the partial stability (uniformly asympotic) of isolated subsystems or the partial stabilizing effects of the interconnecting terms. The proofs are obtained by means of a suitable choice of weights [(1)].

2. Let us consider the system of m differential vectorial equations

$$\dot{z}_i = f_i(z_i,t) + g_i(z_1,z_2,\dots,z_m,t) \qquad (i=1,2,\dots,m), \tag{2.1}$$

where $z_i \in \mathbb{R}^{n_i}$, $\mathbb{R}^{n_i}$ denotes the Euclidean n_i-space with the usual norm. Here f_i and g_i are vectorial functions, $f_i : \Omega_i \times J \to \mathbb{R}^{n_i}$, $g_i : \Omega_1 \times \Omega_2 \times \dots \Omega_m \times J \to \mathbb{R}^{n_i}$, with Ω_i open neighborhood of the origin of $\mathbb{R}^{n_i}$ and $J = [\tau, +\infty[$, τ being a real number.

A system of the form (2.1) is called a *composite system*, because it may be viewed as a nonlinear time-varying interconnection of m isolated subsystems

$$\dot{z}_i = f_i(z_i,t) \qquad (i = 1,2,\dots,m). \tag{2.2}$$

To investigate stability properties with respect to a part of state variables let us set $\forall i \in \{1,2,\dots,m\}$

$$z_i = (x_i,y_i), \qquad \Omega_i = \Omega_{ix} \times \Omega_{iy},$$

(1) In the same context the authors have obtained other results which are contained in [6].

where $x_i \in \mathbb{R}^{k_i}$, $y_i \in \mathbb{R}^{\ell_i}$, $\Omega_{ix} \subseteq \mathbb{R}^{k_i}$, $\Omega_{iy} \subseteq \mathbb{R}^{\ell_i}$, $k_i \geq 0, \ell_i \geq 0$, $k_i + \ell_i = n_i$,

and

$$\xi = (x_1, x_2, \dots, x_m), \quad \xi \in \mathbb{R}^k = \prod_{s=1}^{m} \mathbb{R}^{k_s}, \quad k = k_1 + k_2 + \dots + k_m,$$

$$\eta = (y_1, y_2, \dots, y_m), \quad \eta \in \mathbb{R}^{\ell} = \prod_{s=1}^{m} \mathbb{R}^{\ell_s}, \quad \ell = \ell_1 + \ell_2 + \dots + \ell_m,$$

$$z = (z_1, z_2, \dots, z_m) = (\xi, \eta), \quad z \in \mathbb{R}^n, \quad n = n_1 + n_2 + \dots + n_m .$$

We assume that $\forall (z_{oi}, t_o) \in \Omega_i \times J$ for system (2.2) there exists one and only one solution $z_i(z_{oi}, t_o, t) \equiv (x_i(z_{oi}, t_o, t), y_i(z_{oi}, t_o, t))$ satisfying condition $z_i(z_{oi}, t_o, t_o) = z_{oi}$ and defined $\forall t \geq t_o$. The same hypotheses are assumed for system (2.1) and its solution $z(z_o, t_o, t) \equiv (\xi(z_o, t_o, t), \eta(z_o, t_o, t))$ corresponding to initial data $(z_o, t_o) \in (\prod_{s=1}^{m} \Omega_s) \times J$. Furthermore, we suppose $f_i(0,t) \equiv 0$, $g_i(0,t) \equiv 0$ $\forall t \in J$, $\forall i \in \{1,2,\dots,m\}$, so that systems (2.1) and (2.2) have the null solution.

As usual, we shall say that a function $a : [0,r[\to \mathbb{R}^+$ belongs to class K ($a \in K$) if it is continuous, strictly increasing and $a(0) = 0$.

We shall denote, for each $i \in \{1,2,\dots,m\}$ by L_i the class of functions V_i, $V_i : S_{x_i}(\rho_i) \times \Omega_{iy} \times J \to \mathbb{R}^+$, $S_{x_i}(\rho_i) \equiv \{x_i \in \Omega_{ix} : \|x_i\| < \rho_i, \rho_i > 0\}$,

$V_i \in C^1$, for which there exist two functions a_i and b_i, $a_i : [0,\rho_i[\to R^+$, $b_i : [0,\rho_i[\to R^+$, $a_i, b_i \in K$, such that

$$a_i(\|x_i\|) \leq V_i(x_i,y_i,t) \leq b_i(\|x_i\|) \quad \forall (x_i,y_i,t) \in S_{x_i}(\rho_i) x \Omega_{iy} \times J$$

Furthermore we shall use the following notations

$$S = S_{x_1}(\rho_1) \times \Omega_{1y} \times S_{x_2}(\rho_2) \times \Omega_{2y} \times \ldots \times S_{x_m}(\rho_m) \times \Omega_{my} ,$$

$$S_i = \{z : z = (x_1,y_1,x_2,y_2,\ldots,x_m,y_m) \in S,\ x_i = 0\},$$

$$\tilde{S} = \bigcap_{i=1}^{m} S_i .$$

Finally, setting $I^{(1,m)} = \{1,2,\ldots,m\}$, we shall denote, with respect to a given property $(\cdot)$, by $I^{*(1,m)}_{(\cdot)}$ and $I'^{(1,m)}_{(\cdot)}$ the sets of values $i \in I^{(1,m)}$ for which property $(\cdot)$ is respectively verified or not, and by $S^*_{(\cdot)}$ the set $\bigcap_{i^* \in I^{*(1,m)}_{(\cdot)}} S_{i^*}$

3. Now, correspondingly to the cases considered in the Sect. 1, we are able to give the following three theorems.

THEOREM 1. *Suppose that for each* $i \in I^{(1,m)}$ *there exists a function* V_i, $V_i : S_{x_i}(\rho_i) \times \Omega_{iy} \times J \to \mathbb{R}^+$, *such that the following conditions are satisfied*

i) $V_i \in L_i$ *and there exists a function* $c_i : [0, \rho_i[\to \mathbb{R}^+$, $c_i \in K$, *such that*

$$(3.1) \quad \frac{dV_i}{dt}\Big|_{(2.2)} \equiv \frac{\partial V_i}{\partial t} + \nabla V_i \cdot f_i(z_i,t) \leq -c_i(\|x_i\|)$$

$$\forall (x_i,y_i,t) \in S_{x_i}(\rho_i) \times \Omega_{iy} \times J$$

ii) *there exists a number* $\rho < 1$ *such that one at least of the following inequalities holds* $\forall (z,t) \in S \times J$

$$(3.2) \qquad \nabla V_i \cdot g_i \le \rho c_i(\| x_i \|) - \lambda_i(z) \qquad i \in I^{(1,m)},$$

where the functions $\lambda_i : S \to \mathbb{R}^+$ *are such that*

$$\lambda_i(z) = 0 \qquad \forall\, z \in S_i$$

$$\lambda_i(z) > 0 \qquad \forall\, z \in S \setminus S_i$$

iii) *there exist a function* $\mu : S \to \mathbb{R}$, $\mu(z) \ge 0 \quad \forall z \in S^*_{(3.2)} \setminus \tilde{S}$, *and, for each* $i' \in I'^{(1,m)}_{(3.2)}$, *a positive constant* $\alpha_{i'}$ *such that*

$$(3.3) \qquad \Sigma_{i'} \alpha_{i'} \nabla V_{i'} \cdot g_{i'} \le \Sigma_{i'} \alpha_{i'} \rho\, c_{i'} - \mu \qquad \forall (z,t) \in S \times J$$

and

$$\mathcal{T} = \inf_{z \in S \setminus S^*_{(3.2)}} \left\{ \frac{\mu(z)}{\Sigma_{i^*} \lambda_{i^*}(z)} \right\} > -\infty$$

where $\Sigma_{i'} \equiv \sum_{i' \in I'^{(1,m)}_{(3.2)}}$ *and* $\Sigma_{i^*} \equiv \sum_{i^* \in I^{*(1,m)}_{(3.2)}}$.

Then the null solution of system (2.1) is uniformly asymptotically ξ-stable.

Proof. Hypothesis i) implies that each of the isolated subsystems verifies hypothesis i) of the Theorem in [3]. Furthermore, it is easy to verify that also hypothesis ii) of that theorem is satisfied. Indeed, chosing a positive number $\alpha \geq -\mathscr{T}$ and setting $\forall i^* \in I^{*(1,m)}_{(3.2)}$ $\alpha_{i^*} = \alpha$ from (3.2) and (3.3) it follows that $\forall (z,t) \in (S \setminus S^*_{(3.2)}) \times J$

$$\rho \sum_{i \in I^{(1,m)}} \alpha_i c_i(\|x_i\|) - \sum_{i \in I^{(1,m)}} \alpha_i \nabla V_i \cdot g_i(z,t) =$$

$$= \rho \Sigma_{i'} \alpha_{i'} c_{i'} - \Sigma_{i'} \alpha_{i'} \nabla V_{i'} \cdot g_{i'} + \rho \Sigma_{i^*} \alpha_{i^*} c_{i^*} - \Sigma_{i^*} \alpha_{i^*} \nabla V_{i^*} \cdot g_{i^*} \geq$$

$$\geq \mu(z) + \alpha \Sigma_{i^*} (\rho c_{i^*} - \nabla V_{i^*} \cdot g_{i^*}) \geq \mu(z) + \alpha \Sigma_{i^*} \lambda_{i^*} \geq 0.$$

On the other hand, in S, because of properties of functions c_i and V_i, we have

$$\rho \sum_{i \in I^{(1,m)}} \alpha_i c_i - \sum_{i \in I^{(1,m)}} \alpha_i \nabla V_i \cdot g_i = 0 \qquad \forall t \in J$$

whereas in $S^*_{(3.2)} \setminus \tilde{S}$, being μ non negative, (3.3) implies

$$\rho \sum_{i \in I^{(1,m)}} \alpha_i c_i - \sum_{i \in I^{(1,m)}} \alpha_i \nabla V_i g_i \geq 0 \qquad \forall t \in J$$

Therefore the thesis follows from the mentioned theorem.

THEOREM 2. *Suppose that for each* $i \in I^{(1,m)}$ *there exists a function* V_i, $V_i : S_{x_i}(\rho_i) \times \Omega_{iy} \times J \to \mathbb{R}^+$, *such that the following conditions are satisfied*

i) $V_i \in L_i$, $\nabla V_i = 0 \iff x_i = 0$, $\dfrac{\partial V_i(0,y_i,t)}{\partial t} \leq 0$

ii) *there exist two functions* c_i, c_i : $[0,\rho_i[\to \mathbb{R}^+$, $c_i \in K$, *and* β_i, β_i: $S_{x_i}(\rho_i) \times \Omega_{iy} \times J \to \mathbb{R}$, β_i *continuous such that*

$$(3.4) \qquad \beta_i(z_i,t) \leq - \frac{\frac{\partial V_i}{\partial t} + c_i(\| x_i \|)}{\| \nabla V_i \|} \quad \forall (x_i,y_i,t) \in (S_{x_i}(\rho_i) \setminus \{0\}) \times \Omega_{iy} \times J$$

iii) *there exist a number* $\rho < 1$ *such that one at least of the following inequalities holds*

$$(3.5) \qquad \nabla V_i \cdot g_i \leq \rho c_i(\| x_i \|) - \nabla V_i \cdot (f_i - \beta_i \nabla V_i) - \lambda_i \quad i \in I^{(1,m)} \quad \forall (z,t) \in S \times J$$

where the functions $\lambda_i : S \to \mathbb{R}^+$ *are such that*

$$\lambda_i(z) = 0 \qquad \forall z \in S_i$$

$$\lambda_i(z) > 0 \qquad \forall z \in S \setminus S_i$$

iv) *there exist a function* $\mu : S \to \mathbb{R}$, $\mu(z) \geq 0$ $\forall z \in S^*_{(3.5)} \setminus \tilde{S}$, *and, for each* $i' \in I'^{(1,m)}_{(3.5)}$, *a positive constant* $\alpha_{i'}$ *such that*

$$(3.6) \qquad \Sigma_{i'} \alpha_{i'} \nabla V_{i'} \cdot g_{i'} < \rho \Sigma_{i'} \alpha_{i'} c_{i'} - \Sigma_{i'} \alpha_{i'} \nabla V_{i'} \cdot (f_{i'} - \beta_{i'} \nabla V_{i'}) - \mu \quad \forall (z,t) \in S \times J$$

and

$$\mathcal{T} = \inf_{z \in S \setminus S^*_{(3.5)}} \left\{ \frac{\mu(z)}{\Sigma_{i^*} \lambda_{i^*}(z)} \right\} > -\infty ,$$

where $\Sigma_{i'} \equiv \Sigma_{i' \in I'^{(1,m)}_{(3.5)}}$ *and* $\Sigma_{i^*} \equiv \Sigma_{i^* \in I^{*(1,m)}_{(3.5)}}$

Then, the null solution of system (2.1) is uniformly asymptotically ξ-stable.

Proof. Let us consider the systems

$$\dot{z}_i = \phi_i(z_i,t) \qquad (i = 1,2,\dots,m), \tag{3.7}$$

where

$$\phi_i(z_i,t) = \beta_i \nabla V_i \quad .$$

By virtue of hypotheses i) and ii) the functions ϕ_i are continuous and such that $\phi_i(0,t) = 0 \quad \forall t \in J$, and furthermore

$$(z_i,t) \in (S_{x_i}(\rho_i)\setminus\{0\}) \times \Omega_{iy} \times J \Longrightarrow \nabla V_i \cdot \phi_i \leq -\frac{\partial V_i}{\partial t} - c_i(\|x_i\|),$$

whereas if $(z_i,t) \equiv (0,y_i,t)$, $y_i \in \Omega_{iy}$ and $t \in J$ we have

$$\frac{\partial V_i}{\partial t} + \nabla V_i \cdot \phi_i = \frac{\partial V_i}{\partial t} \leq 0.$$

Therefore $\forall (z_i,t) \in S_{x_i}(\rho_i) \times \Omega_{iy} \times J$ we get

$$\left.\frac{dV_i}{dt}\right|_{(3.7)} = \frac{\partial V_i}{\partial t} + \nabla V_i \cdot \phi_i \leq - c_i(\|x_i\|)$$

and, being $V_i \in L_i$, systems (3.7) satisfy hypothesis i) of the Theorem in [3].

Furthermore, chosing a positive number $\alpha \geq -\mathscr{T}$ and setting for each $i^* \in I^{*(1,m)}_{(3.5)}$ $\alpha_{i^*}=\alpha$, from (3.5) and (3.6) it follows that

$$\rho \sum_{i\in I^{(1,m)}} \alpha_i c_i(\|x_i\|) - \sum_{i\in I^{(1,m)}} \alpha_i \nabla V_i \cdot (f_i+g_i-\phi_i) =$$

$$= \rho\Sigma_{i'} \alpha_{i'} c_{i'} - \Sigma_{i'} \alpha_{i'} \nabla V_{i'} \cdot (f_{i'} + g_{i'} - \beta_{i'} \nabla V_{i'}) +$$

$$+ \rho\Sigma_{i*} \alpha_{i*} c_{i*} - \Sigma_{i*} \alpha_{i*} \nabla V_{i*} \cdot (f_{i*} + g_{i*} - \beta_{i*} \nabla V_{i*}) \geq$$

$$\geq \mu(z) + \alpha\Sigma_{i*} \lambda_{i*}(z) .$$

Therefore, reasoning as in the proof of Theorem 1, we have $\forall (z,t) \in$ $\in S \times J$

$$\rho \sum_{i \in I^{(1,m)}} \alpha_i c_i(\|x_i\|) - \sum_{i \in I^{(1,m)}} \alpha_i \nabla V_i \cdot (f_i + g_i - \phi_i) \geq 0 .$$

Then, for system (2.1) written in the form

$$\dot{z}_i = \phi_i(z_i,t) + \tilde{g}_i(z,t) \qquad (i=1,2,\dots,m)$$

with $\tilde{g}_i = f_i + g_i - \phi_i$, all the hypotheses of the Theorem in [3] are verified and its null solution is uniformly asymptotically ξ-stable.

THEOREM 3. *Suppose that for each* $i \in I^{(1,m)}$ *there exists a function* V_i, $V_i : S_{x_i}(\rho_i) \times \Omega_{iy} \times J \to \mathbb{R}^+$, *such that the following conditions are satisfied*

i) $\forall i \in I^{(1,p)}$, $0 \leq p \leq m$, *hypothesis* i) *of Th.1 holds and, if* $p < m$, $\forall i \in I^{(p+1,m)}$ *hypotheses* i) *and* ii) *of Th.2 hold;*

ii) *there exists a number* $\rho < 1$ *such that one at least of sets* $I^{*(1,p)}_{(3.2)}$ *and* $I^{*(p+1,m)}_{(3.5)}$ *is not empty;*

iii) *there exist a function* $\mu : S \to \mathbb{R}$ *such that*

a) $\mu(z) \geq 0 \quad \forall z \in \Theta \setminus \tilde{S}$ *where* $\Theta \equiv \bigcap_{i^* \in I^{*(1,p)}_{(3.2)} \cup I^{*(p+1,m)}_{(3.5)}} S_{i^*}$

b) *for each* $i' \in I'^{(1,p)}_{(3.2)} \cup I'^{(p+1,m)}_{(3.5)}$ *there exists a positive constant* $\alpha_{i'}$ *such that* $\forall (z,t) \in S \times J$

$$\Sigma_{i'} \alpha_{i'} \nabla V_{i'} \cdot g_{i'} \leq$$

$$\leq \rho \Sigma_{i'} \alpha_{i'} c_{i'} - \Sigma'_{i'} \alpha_{i'} \nabla V_{i'} \cdot (f_{i'} - \beta_{i'} \nabla V_{i'}) - \mu$$

c) $$\mathscr{F} \equiv \inf_{z \in S \setminus \Theta} \left\{ \frac{\mu(z)}{\Sigma_{i^*} \lambda_{i^*}(z)} \right\} > -\infty$$

where $\Sigma_{i'} \equiv \Sigma_{i' \in I'^{(1,p)}_{(3.2)} \cup I'^{(p+1,m)}_{(3.5)}}$, $\Sigma'_{i'} \equiv \Sigma_{i' \in I'^{(p+1,m)}_{(3.5)}}$,

$\Sigma_{i^*} \equiv \Sigma_{i^* \in I^{*(1,p)}_{(3.2)} \cup I^{*(p+1,m)}_{(3.5)}}$.

Then the null solution of system (2.1) is uniformly asymptotically ξ-stable.

The proof is similar to those of Theorems 1 and 2 wich are actually particular cases of this theorem.

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Lavoro pervenuto alla Redazione il 21 Maggio 1983
ed accettato per la pubblicazione il 16 Febbraio 1984
su parere favorevole di G. Andreassi e C. Risito