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ON SOME TOPOLOGICAL PROPERTIES OF THE V-PRIME ELEMENTS OF A PARTIALLY ORDERED SET (*)

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Sunto. *D. Drake e W.J. Thron hanno dato in* [1] *una caratterizzazione degli elementi* v-*irriducibili e degli elementi fortemente* v-*irriducibili di un reticolo distributivo* $(L,\leq)$. *Tra l'altro in* [1] *è stato provato che un elemento* $c\in L$ *è irriducibile se e solo se* $(L,\leq)$ *si può identificare tramite un isomorfismo reticolare* f, *con un sottoreticolo* $(L',\subseteq)$ *del reticolo delle parti* $\mathcal{P}(X)$ *di un opportuno insieme* X *in modo tale che* f(c) *è la chiusura in* L' *di un certo elemento* $x\in X$ (*cioè* f(c) *è il più piccolo elemento di* L', *rispetto all'inclusione insiemistica, cui appartiene* x).

Come è ben noto un elemento di un reticolo distributivo è v-*irriducibile se e solo se esso è* v-*primo. Questa proprietà è usata in maniera essenziale in* [1]. *In questo lavoro noi prendiamo lo spunto da tale proprietà per dare una caratterizzazione degli elementi* v-*primi e degli elementi fortemente* v-*primi di un qualsiasi insieme parzialmente ordinato.*

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Precisiamo che qui, in analogia con una caratterizzazione degli elementi v-primi e degli elementi fortemente v-primi di un reticolo, un elemento c *di un insieme ordinato* $(S,\leq)$ *è detto* v-*primo se il sottoinsieme* $D_c := \{s \in S : c \nleq s\}$ *è superiormente diretto; inoltre* c *è detto fortemente* v-*primo se* $D_c = \emptyset$ *oppure* D_c *è dotato di massimo. Allora noi proviamo che un elemento* $c \in S$ *è* v-*primo in* $(S,\leq)$ *se e solo se possiamo identificare* $(S,\leq)$*, tramite una biezione isotona* f*, con una base di chiusi di uno spazio topologico, in modo tale che* f(c) *è la chiusura in* $(f(S), \subseteq)$ *di un elemento di* $\bigcup_{s \in S} f(s)$*; inoltre proviamo che se* c *è un elemento non minimo di* S *allora esso è fortemente* v-*primo in* $(S,\leq)$ *se e solo se per ogni biiezione* f *del tipo su menzionato l'insieme* f(c) *è la chiusura in* $(f(S), \subseteq)$ *di un punto di* $\bigcup_{s \in S} f(s)$.

INTRODUCTION. A characterization of v-irreducible elements and of strongly v-irriducible elements of a distributive lattice $(L,\leq)$ was given by D. Drake and W.J.Thron in [1]. Among other things in [1] it was proven that an element $c \in L$ is v-irreducible iff one can identify $(L,\leq)$, by means of a lattice isomorphism f, with a sublattice $(L',\subseteq)$ of the power set $\mathscr{P}(X)$ of a suitable set X, in such a way that f(c) is the point closure in L' of an element $x \in X$ (i.e. f(c) is the minimum element in L', with respect to the set inclusion, including x).

As is well-known, an element of a distributive lattice is v-irreducible iff it is v-prime. This property is exploited in an essential manner in [1]. Now then in our paper we took this property as a starting point for a characterization of v-prime and of strongly v-prime elements of a partially ordered set (shortly "poset"). Here, on the analogy of a characterization of v-prime elements and of strongly v-prime elements of a lattice, an element c of a poset $(S,\leq)$ is said v-prime iff the subset $D_c = \{s \in S : c \nleq s\}$ is v-directed (i.e.

$D_c = \emptyset$ or for every $x_1, x_2 \in D_c$ there exists $t \in D_c$ such that $x_1 \leq t$ and $x_2 \leq t$); moreover c is said strongly v-prime if $D_c = \emptyset$ or D_c has a maximum element. Then we prove that an element $c \in S$ is v-prime in $(S, \leq)$ iff we can identify $(S, \leq)$, by means of an order isomorphism f, with a base of the closed sets of a topological space in such a way that f(c) is the point closure of an element of $\bigcup_{s \in S} f(s)$; moreover we prove that if c is a non-minimum element of S, then it is strongly v-prime in $(S, \leq)$ iff for all function f of the above type the set f(c) is the closure in $(f(S), \subseteq)$ of an element of $\bigcup_{s \in S} f(s)$.

N. 1. PRELIMINARY CONSIDERATIONS.

We recall that a lattice is said a set lattice (see [1] p. 57) iff its elements are subsets of a suitable set X and the order relation is the set inclusion; in particular if the lattice is a sublattice of the power set $\mathscr{P}(X)$ then it is called a proper set lattice.

More generally we shall say that a set lattice $(L', \subset)$ is a "U-proper set lattice" iff the lattice join is equal to the set union.

We recall also that a proper set representation of a lattice $(L, \leq)$ is an ordered pair $(L', \subseteq), f)$, where $(L', \subseteq)$ is a proper set lattice and f is an isomorphism from $(L, \leq)$ onto $(L', \subseteq)$. If $(L', \subseteq)$ is a U-proper set we shall call $(L', \subseteq), f)$ a "U-proper set representation".

We want to extend the previous definitions to the case of an arbitrary partially ordered set.

In the meantime we observe that in a set lattice $(L', \subseteq)$ the lattice join is equal to the set union iff $L' \cup \{\emptyset\}$ is a base for the closed sets of a topology on $\bigcup_{Y \in L'} Y$ (i.e. the set complements in $\bigcup_{Y \in L'} Y$ of the elements of $L' \cup \{\emptyset\}$ are a base of a topology) on $\bigcup_{Y \in L'} Y$.

On the analogy of this fact we shall say that a poset $(S,\leq)$ is a U-proper set poset iff its elements are subsets of a suitable set X, $\leq$ is the set inclusion and $S\cup\{\emptyset\}$ is a base for the closed sets of a topology on X; thus we shall say that an ordered pair $((S',\subseteq),f)$, where $(S',\subseteq)$ is a U-proper set poset and f is a function, is a U-proper set representation of a poset $(S,\leq)$ iff f is an order isomorphism from S onto S'. Dually we can give the notions of $\cap$-proper set poset and $\cap$-proper set representation.

Now let $\mathscr{C}$ be a subset of $\mathscr{P}(S)$ (the power set of S), $x\in S$ and $\mathscr{C}_x = \{X \in \mathscr{C} : x \in X\}$. Then one can define, for every $x,y\in S$,

(i) $$x \underset{\sim}{<} y(\mathscr{C}) \text{ iff } \mathscr{C}_x \subseteq \mathscr{C}_y .$$

Clearly the defined relation is a preorder relation.

REMARK 1. *It is easy to verify that if* $Y\in\mathscr{C}$ *and* $x\in Y$, *then* Y *is a point closure of* x *in* $\mathscr{C}$ *if and only if, for every* $y\in Y$, $x \underset{\sim}{<} y(\mathscr{C})$.

We recall that a right tail of an ordered set $(S,\leq)$ is every $Y\subseteq S$ such that $\forall x,y\in S$: $(x\in Y$ and $x \leq y) \Rightarrow y \in Y$. In particular the set $r(x) := \{y\in S | x \leq y\}$ (the "principal filter" generated by $x\in S$) is a right tail of $(S,\leq)$.

Obviously, if $\mathscr{C}'$ is the set of the principal filters of a poset $(S,\leq)$ and $\mathscr{C}$ is the set of their set complements in S, then:

(j) $$x \leq y \Leftrightarrow y \underset{\sim}{<} x(\mathscr{C}) \quad x \underset{\sim}{<} y(\mathscr{C}') .$$

N. 2. A CHARACTERIZATION OF V-PRIME AND STRONGLY V-PRIME ELEMENTS

Let $(S,\leq)$ be a poset. Then we can consider the function $g:S\to\mathscr{P}(S)$ mapping $x\in S$ into the set $g(x) := \{y\in S : x \nleq y\} = S-r(x)$.

REMARK 2. *Clearly* g *is an order isomorphism between* $(S,\leq)$ *and* $(g(S),\subseteq)$. *Moreover* $((g(S),\subseteq),g)$ *is a* U-*proper set representation of* $(S,\leq)$. *In fact the right tails of* $(S,\leq)$ *are the open sets of a topology of* S *and the principal filters are a base for them; then, by De Morgan's properties, our assertion holds.*

Now we can give the following

THEOREM 3. *Let* c *be an element of* S *and let* c *be non-minimum in* $(S,\leq)$. *Then* c *is strongly* v-*prime in* $(S,\leq)$ *iff, for ever* U-*proper set representation* $((f(S),\subseteq),f)$ *of* $(S,\leq)$, f(c) *is a point closure in* f(S).

Proof. Let f(c) be a point closure in f(S) for every U-proper set representation $((f(S),\subseteq),f)$ and let us consider the U-proper set répresentation of remark 2; thus the set $\{y \in S : c \nleq y\} = g(c)$ is a point closure in g(S). Hence, as a consequence of remark 1 and (j) in N. 1, g(c) has a maximum element, therefore c is a strongly v-prime element of $(S,\leq)$.

The second part of the proof is an immediate consequence of the following considerations. Let X be a non-empty set and $\mathscr{S} \subseteq \mathscr{P}(X)$, moreover let C be a non-minimum and v-prime element of $(\mathscr{S},\subseteq)$. Then the set $\{Y \in \mathscr{S} : C \nsubseteq Y\}$ has a maximum M. Now, for every $x \in C-M$, C is the minimum element of $\mathscr{S}$ containing x. In fact if $x \in Z \in \mathscr{S}$ and $C \nsubseteq Z$ then $x \in Z \subseteq M$. This is absurd, since $x \notin M$.

Q.E.D.

THEOREM 4. *An element* $c \in S$ *is* v-*prime in* $(S,\leq)$ *iff a* U-*proper set representation* $((f(S),\subseteq)$ *of* $(S,\leq)$ *exists such that* f(c) *is a point* *closure in* f(S).

Proof. Let $((f(S),\subseteq),f)$ be a U-proper set representazion of $(S,\leq)$ such that f(c) is the point closure in f(S) of x; moreover let $y,z \in S$ be such that $c \nleq y$ and $c \nleq z$. Then $f(c) \nsubseteq f(y)$ and $f(c) \nsubseteq f(z)$, thus $x \notin f(y)$ and $x \notin f(z)$; as a consequence since $(f(S) \cup \{\emptyset\}$ is a base of closed sets) an element $t \in S$ exists such tha $f(t) \supseteq f(y) \cup f(z)$ and $x \notin f(t)$, thus $f(c) \nsubseteq f(t)$ and hence $c \nleq t$ but $y \leq t$ and $z \leq t$. This means that the subset $\{s \in S : c \nleq s\}$ is v-directed.

Conversely let c be a v-prime element in $(S,\leq)$ and $((f(S),\subseteq),f)$ a U-proper set representation of $(S,\leq)$. If f(c) is a point closure in f(S) we have nothing to prove. If not, it is sufficient to fix an element $x \notin \bigcup_{s \in S} f(s)$ and adjoin it to every f(s) including f(c)[1].

Q.E.D.

We conclude with the following

THEOREM 5. *If* $(S,\leq)$ *has at least a* v-*prime element, then there is a* U-*proper set representation* $((f(S),\subseteq),f)$ *of* $(S,\leq)$ *such that* f *maps every* v-*prime element of* $(S,\leq)$ *in a point closure.*

Proof. Let P be the set of all v-prime elements of $(S,<)$, $((f(S),\subseteq),f)$ a U-proper set representation of $(S,\leq)$ (cfr. remark 2), and P_1 the set of all the elements of P mapped into a point closure. If $P_1 = P$ we have nothing to prove. Otherwise, for every $p \in P-P_1$, we can fix (by a bijective function) an element $x_p \notin \bigcup_{s \in S} f(s)$. Then we can consider the function f' that maps every $s \in S$ into the set $f(s) \cup X_s$,

(1) In theorem 5 we shall apply this method in a rigorous and more general manner.

where $X_s := \{x_p\}_{p \leq s}$. Clearly f' is an injective and isotone function. Now let U be the set of all upper bounds of $\{s_1, s_2\} \subseteq S$; we shall prove that $\bigcap_{u \in U} (f(u) \cup X_u) = (f(s_1) \cup X_{s_1}) \cup (f(s_2) \cup X_{s_2})$. Indeed $\bigcap_{u \in U} (f(u) \cup X_u) = (\bigcap_{u \in U} f(u)) \cup (\bigcap_{u \in U} X_u)$, moreover $\bigcap_{u \in U} f(u) = f(s_1) \cup f(s_2)$ (since $f(S) \cup \{\emptyset\}$ is a base) and $\bigcap_{u \in U} X_u \supseteq X_{s_1} \cup X_{s_2}$; then it is sufficient to prove that $\bigcap_{u \in U} X_u \subseteq X_{s_1} \cup X_{s_2}$. Now if $x_p \in \bigcap_{u \in U} X_u$, then p is an element of $P - P_1$ such that $p \leq u$ for every $u \in U$. As a consequence, since p is v-prime and U is the set of all the upper bounds of $\{s_1, s_2\}$, then $p \leq s_1$ or $p \leq s_2$, hence $x_p \in X_{s_1} \cup X_{s_2}$. One can easily verify that $((f'(S), \subseteq), f')$ is the requested U-proper set representation.

Q.E.D.

REFERENCE

[1] D. DRAKE - W.J. THRON: "*On the representations of an abstract lattice as the family of closed sets of a topological space*". Trans. of Amer. Math. Soc. 120(1965), 57-71.

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