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A METRIC FOR WEAK CONVERGENCE OF MULTIPLE DISTRIBUTION FUNCTIONS

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SOMMARIO. *Si introduce una metrica nello spazio delle funzioni di ripartizione multiple e si mostra che la convergenza in tale metrica equivale alla convergenza debole delle funzioni di ripartizione.*

1 - INTROCUCTION.

As was shown in [3], weak convergence in the space Δ_r of multiple distribution function (d.f.'s) is slightly different from the convergence in the product topology of the space $\Delta\times\Delta\times\ldots\times\Delta$, Δ being the space of simple d.f.'s. A metric d can be introduced in Δ such that convergence in d is equivalent to weak convergence of (simple) d.f.'s; in fact, two such metrics exist (see [3],[1],[2]). It is thus natural to ask whether a metric on Δ_r exists such that convergence in that metric is equivalent to weak convergence in Δ_r $(r \geq 2)$. To answer this question is the purpose of the present note. All the results and their proofs are given in the case r=2; for r>2 everything can be repeated with the obvious changes.

2. THE METRIC ON Δ_2

Let the functions $\phi_{ab} : \bar{\mathbb{R}} \to [0,1]$ $(a,b\in\mathbf{Q}\ a<b)$ be defined as in [2], i.e.

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$$\phi_{ab}(x) := (b-a)^{-1} \int_x^{+\infty} 1_{[a,b]}(t)\,dt \;.$$

By a suitable enumeration, one can define a sequence of functions $\gamma_r : \bar{\mathbb{R}}^2 \to [0,1]\,(r\in\mathbb{N})$ by $\gamma_r(x,y) = \phi_{ab}(x)\,\phi_{cd}(y)$ or $\gamma_r(x,y) = \phi_{ab}(x)$ or again by $\gamma_r(x,y) = \phi_{cd}(y)$ $(r\in\mathbb{N}, a,b,c,d\in\mathbb{Q};\ a<b,\ c<d,\ (x,y)\in\bar{\mathbb{R}}^2)$. We shall shortly show that the mapping $d_2 : \Delta_2 \times \Delta_2 \to \bar{\mathbb{R}}^+$ defined by

$$d_2(F,G) := \sum_{r=1}^{\infty} 2^{-r} \left| \int_{\bar{\mathbb{R}}^2} \gamma_r(x,y)\,dF(x,y) - \int_{\bar{\mathbb{R}}^2} \gamma_r(x,y)\,dG(x,y) \right| \tag{1}$$

is the requested metric on Δ_2 (see theorems 2 and 3 below).

The proofs rest on the following formula of integration by parts for double Lebesgue-Stieltjes integrals.

LEMMA 1. *If* $F \in \Delta_2$ *then, for every* $a,b,c,d\in\mathbb{Q}$, *with* $a<b$ *and* $c<d$, *one has*

$$\int_{\bar{\mathbb{R}}^2} \phi_{ab}(x)\,\phi_{cd}(y)\,d_{x,y}F(x,y) = (b-a)^{-1}(d-c)^{-1} \int_a^b dx \int_c^d F(x,y)\,dy. \tag{2}$$

Proof. Let P be the probability measure on $(\bar{\mathbb{R}}^2, \mathscr{B}(\bar{\mathbb{R}}^2))$ corresponding to the d.f. F. Then

$$(b-a)(d-c) \int_{\bar{\mathbb{R}}^2} \phi_{ab}(x)\,\phi_{cd}(y)\,dP(x,y) =$$

$$= \int_{\bar{\mathbb{R}}^2} dP(x,y) \int_x^{+\infty} 1_{[a,b]}(s)\,ds \int_y^{+\infty} 1_{[c,d]}(t)\,dt =$$

$$= \int_a^b ds \int_c^d dt\; P([-\infty,s] \times [-\infty,t]) = \int_a^b dx \int_c^d F(x,y)\,dy.$$

Q.E.D.

THEOREM 2. *The function* $d_2:\Delta_2\times\Delta_2 \to \mathbb{R}^+$ *defined by* (1) *is a metric on* Δ_2.

Proof. The only property that is not immediately obvious is the implication $d_2(F,G) \Longrightarrow F=G$. This we set to prove.

$$d_2(F,G)=0 \Longrightarrow \sum_{r=1}^{\infty} 2^{-r} \left| \int_{\mathbb{R}^2} \gamma_r \, dF - \int_{\mathbb{R}^2} \gamma_r \, dG \right| = 0 \Longrightarrow$$

(3)
$$\Longrightarrow \int_{\mathbb{R}^2} \gamma_r \, dF = \int_{\mathbb{R}^2} \gamma_r \, dG \qquad (\forall r \in \mathbb{N}).$$

This last equality implies three kinds of relationship according as to whether, for a given $r\in\mathbf{N}$, $\gamma_r(x,y) = \phi_{ab}(x)\,\phi_{cd}(y)$ or $\gamma_r(x,y) = \phi_{ab}(x)$ or $\gamma_r(x,y) = \phi_{cd}(y)$. We shall deal separately with the three cases.

(i) $\gamma_r(x,y) = \phi_{ab}(x)\ \ \phi_{cd}(y)$. In this case equality (3) implies $F(x,y) = G(x,y)\ \forall\ (x,y) \in \mathbb{R}^2$. In fact, (3) reads, in this case

$$\int_{\mathbb{R}^2} \phi_{ab}(x)\, \phi_{cd}(y)\, d_{x,y}F(x,y) = \int_{\mathbb{R}^2} \phi_{ab}(x)\, \phi_{cd}(y) d_{x,y}\, G(x,y)$$

$$(\forall a,b,c,d \in \mathbf{Q},\ a<b,\ c<d).$$

Because of (2), this last inequality entails

(3) $$\int_a^b dx \int_c^d F(x,y)dy = \int_a^b dx \int_c^d G(x,y)dy \qquad (\forall a,b,c,d\in\mathbf{Q},\ a<b,\ c<d);$$

(4) implies the assertion $F(x,y) = G(x,y)\ \forall (x,y)\in\mathbb{R}^2$. Indeed, assume, ab absurdo , the existence of a point $(x_\circ,y_\circ)\in\mathbb{R}^2$ such

that $F(x_\circ,y_\circ) \neq G(x_\circ,y_\circ)$, for instance $F(x_\circ,y_\circ) < G(x_\circ,y_\circ)$. Set $\varepsilon = \{G(x_\circ,y_\circ)-F(x_\circ,y_\circ)\}/2$. Since F is right-continuous and non-decreasing in both variables, there exist x' and $y' \in \mathbb{R}$, with $x' > x_\circ$ and $y' > y_\circ$, such that

$$F(x_\circ,y_\circ) \leq F(x,y_\circ) \leq F(x_\circ,y_\circ) + \varepsilon/2 \qquad (x \in [x_\circ,x'])$$

and

$$F(x_\circ,y_\circ) \leq F(x,y_\circ) \leq F(x',y_\circ) \leq F(x',y') \leq F(x_\circ,y_\circ) + \varepsilon .$$

Thus, if $a,b \in [x_\circ,x']$ and $c,d \in [y_\circ,y']$ one has

$$\int_a^b dx \int_c^d F(x,y)dy \leq (b-a)(d-c)\{F(x_\circ,y_\circ) + \varepsilon\} =$$

$$= (b-a)(d-c)\{F(x_\circ,y_\circ)+G(x_\circ,y_\circ)\}/2 < (b-a)(d-c)\ G(x_\circ,y_\circ) \leq \int_a^b dx \int_c^d G(x,y)dy$$

which contradicts (4).

(ii) If $\gamma_r(x,y) = \phi_{ab}(x)$, then (3) impies

$$\int_{\mathbb{R}^2} \phi_{ab}(x)\ dF(x,y) = \int_{\mathbb{R}^2} \phi_{ab}(x)\,dG(x,y) \qquad (a,b \leq Q,\ a<b)$$

or

$$\int_{\mathbb{R}^2} \phi_{ab}(x)\ dF_1(x) = \int_{\mathbb{R}} \phi_{ab}(x)\ dG_1(x) \qquad (a,b \in Q,\ a< b)$$

where F_1 and G_1 are marginals of F and G respectively: $F_1(x):=F(x,+\infty)$,

$G_1(x) = G(x,+\infty)$. But then theorem 4 in [2], applied to F_1 and G_1 gives $F_1 = G_1$.

(iii) If $\gamma_r(x,y) = \phi_{cd}(y)$ then the same argument as above yields $F_2 = G_2$ where F_2 and G_2 are the marginals $F_2(y):=F(+\infty,y)$ and $G_2(y):=G(+\infty,y)$.

Then one can conclude from (i) that F and G coincide on the set $\{(x,y)\in\bar{\mathbb{R}}^2 : x < +\infty,\ y < +\infty\}$ and from (ii) and (iii) that both marginals of F and G are equal. Therefore $F = G$.

Q.E.D.

THEOREM 3. *Let* F_n, $F\in\Delta_2$ $(n\in N)$; *then* $d_2(F_n,F)\to 0$ *iff* $F_n \xrightarrow{w} F$ (*weakly, i.e.* $F_n(x,y) \to F(x,y)$ *at every point* (x,y) *at which* F *is continuous*).

Proof. ($\Longrightarrow$) Assume $d_2(F_n,F)\to 0$. Set

$$\delta_2(r,n) := \left| \int_{\mathbb{R}^2} \gamma_r(x,y)\,dF_n(x,y) - \int_{\mathbb{R}^2} \gamma_r(x,y)\,dF(x,y) \right| \qquad (r,n\in\mathbb{N}).$$

Then

$$(5) \qquad 0 \le \delta_2(r,n) \le 2^r\, d_2(F_n,F)$$

so that $\lim_{n\to\infty} \delta_2(r,n) = 0$ $\forall r\in\mathbb{N}$. If $\gamma_r(x,y) = \phi_{ab}(x)\ \phi_{cd}(y)$, then on account of (2), this means

$$(6) \qquad \lim_{n\to\infty} \int_a^b dx \int_c^d F_n(x,y)\,dy = \int_a^b dx \int_c^d F(x,y)\,dy \qquad (\forall a,b,c,d\in\mathbb{Q}, a<b, c<d).$$

We shall presently show that (6) implies $F_n(x,y) \to F(x,y)$ if $(x,y)\in\mathbb{R}^2$ is a point of continuity for F.

Set $F^*(x,y):=\limsup_{n\to\infty} F_n(x,y)$. Then, because of (6)

$$(b-a)(d-c)F^*(a,c) = \limsup_{n\to\infty} (b-a)(d-c)\, F_n(a,c) \leq$$

$$\leq \limsup_{n\to\infty} \int_a^b dx \int_c^d F_n(x,y)dy = \int_a^b dx \int_c^d F(x,y)dy \leq (b-a)(d-c)F(b,d),$$

i.e. $F^*(a,c) \leq F(b,d)$. Let $b \downarrow a$ to obtain $F^*(a,c) \leq F(a,d)$, then let $d \downarrow c$ to obtain $F^*(a,c) \leq F(a,c)$. Let $(x,y) \in R^2$ be such that $x < +\infty$ and $y < +\infty$ and take $a > x, c > y, a, c \in Q$; then $F^*(x,y) \leq F^*(a,c) \leq F(a,c)$. Let $a \downarrow x$ and $c \downarrow y$ so that $F^*(x,y) \leq F(x,y)$. A similar argument yields $F_*(x,y) := \liminf_{n\to\infty} F_n(x,y) \geq F(b-0,d-0)$ where $b < x$, $d < y$, $b,d \in Q$, so that

$$F(x-0,y-0) \leq F_*(x,y) \leq F^*(x,y) \leq F(x,y).$$

Therefore if (x,y) is point of continuity for F, then $\lim_{n\to\infty} F_n(x,y) = F(x,y)$. If, on the other hand, at least one, say y, of the coordinates of the continuity point (x,y) equals $+\infty$ then, considering the terms in d_2 with $\gamma_r(x,y) = \phi_{ab}(x)$, one has from (5)

$$\lim_{n\to\infty} \int_a^b F_n(x,+\infty)dx = \int_a^b F(x,+\infty)dx \qquad (a,b \in Q,\ a < b).$$

From this last equality it follows as in theorem 5 [2] that $\lim F_n(x,+\infty) = F(x,+\infty)$. An analogous argument holds for a continuity point of the type $(+\infty,y)$.

($\Leftarrow$) Conversely, if $F_n \xrightarrow{w} F$, then $\delta_2(r,n) \to 0$ ($\forall r \in N$). Since $0 \leq \delta_2(r,n) \leq 2$, one has

$$\lim_{n\to\infty} d_2(F_n,F) = \lim_{n\to\infty} \sum_{r=1}^{\infty} 2^{-r}\ \delta_2(r,n) = 0$$

by the dominated convergence theorem applied to the counting measure on $\mathbb{N}$.

Q.E.D.

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