A METRIC FOR WEAK CONVERGENCE OF MULTIPLE DISTRIBUTION FUNCTIONS

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SOMMARIO. Si introduce una metrica nello spazio delle funzioni di ripartizione multiple e si mostra che la convergenza in tale metrica equivale alla convergenza debole delle funzioni di ripartizione.

1 - INTRODUCTION.

As was shown in [3], weak convergence in the space $\Delta_r$ of multiple distribution function (d.f.'s) is slightly different from the convergence in the product topology of the space $\Delta \times \Delta \times \ldots \times \Delta$, $\Delta$ being the space of simple d.f.'s. A metric $d$ can be introduced in $\Delta$ such that convergence in $d$ is equivalent to weak convergence of (simple) d.f.'s; in fact, two such metrics exist (see [3],[1],[2]). It is thus natural to ask whether a metric on $\Delta_r$ exists such that convergence in that metric is equivalent to weak convergence in $\Delta_r$ ($r \geq 2$). To answer this question is the purpose of the present note. All the results and their proofs are given in the case $r=2$; for $r>2$ everything can be repeated with the obvious changes.

2. THE METRIC ON $\Delta_2$

Let the functions $\Phi_{ab} : \mathbb{R} \rightarrow [0,1]$ (a,b $\in \mathbb{Q}$ a < b) be defined as in [2], i.e.

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\[ \phi_{ab}(x) = (b-a)^{-1} \int_{x}^{+\infty} 1_{[a,b]}(t) \, dt. \]

By a suitable enumeration, one can define a sequence of functions \( \gamma_r : \mathbb{R}^2 \to [0,1] \) (re\( \mathbb{N} \)) by \( \gamma_r(x,y) = \phi_{ab}(x) \phi_{cd}(y) \) or \( \gamma_r(x,y) = \phi_{ab}(x) \) or again by \( \gamma_r(x,y) = \phi_{cd}(y) \) (re\( \mathbb{N} \), a\( \mathbb{R} \), c\( \mathbb{R} \), d\( \mathbb{R} \); \( a < b, c < d \), \( (x,y) \in \mathbb{R}^2 \)).

We shall shortly show that the mapping \( d_2 : \Delta_2 \times \Delta_2 \to \mathbb{R}^+ \) defined by

\[
(1) \quad d_2(F,G) := \sum_{r=1}^{\infty} 2^{-r} \left( \int_{\mathbb{R}^2} |\gamma_r(x,y)| \, dx \right) \left( \int_{\mathbb{R}^2} |\gamma_r(x,y)| \, dy \right)
\]

is the requested metric on \( \Delta_2 \) (see theorems 2 and 3 below).

The proofs rest on the following formula of integration by parts for double Lebesgue-Stieltjes integrals.

**Lemma 1.** If \( F \in \Delta_2 \), then, for every \( a, b, c, d \in \mathbb{R} \), with \( a < b \) and \( c < d \),

\[
(2) \quad \int_{\mathbb{R}^2} \phi_{ab}(x) \phi_{cd}(y) \, dx, \, y \, F(x,y) = (b-a)^{-1}(d-c)^{-1} \int_{a}^{b} dx \int_{c}^{d} F(x,y) \, dy.
\]

**Proof.** Let \( P \) be the probability measure on \( \mathbb{R}^2 \), \( \mathcal{B}(\mathbb{R}^2) \) corresponding to the d.f. \( F \). Then

\[
(b-a)(d-c) \int_{\mathbb{R}^2} \phi_{ab}(x) \phi_{cd}(y) \, dP(x,y) =
\]

\[
= \int_{\mathbb{R}^2} \, dP(x,y) \int_{x}^{+\infty} 1_{[a,b]}(s) \, ds \int_{y}^{+\infty} 1_{[c,d]}(t) \, dt =
\]

\[
= \int_{a}^{b} dx \int_{c}^{d} \int_{-\infty}^{\infty} P([-\infty, s] \times [-\infty, t]) \, dy.
\]

Q.E.D.
THEOREM 2. The function \( d_2 : \Delta_2 \times \Delta_2 \to \mathbb{R}^+ \) defined by (1) is a metric on \( \Delta_2 \).

Proof. The only property that is not immediately obvious is the implication \( d_2(F,G) = 0 \iff F=G \). This we set to prove.

\[
\begin{align*}
\sum_{r=1}^{\infty} 2^{-r} \left| \int_{\mathbb{R}^2} \gamma_r \, dF - \int_{\mathbb{R}^2} \gamma_r \, dG \right| &= 0 \\
\implies \int_{\mathbb{R}^2} \gamma_r \, dF &= \int_{\mathbb{R}^2} \gamma_r \, dG \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (\forall r \in \mathbb{N}).
\end{align*}
\]

This last equality implies three kinds of relationship according as to whether, for a given \( r \in \mathbb{N} \), \( \gamma_r(x,y) = \phi_{ab}(x) \phi_{cd}(y) \) or \( \gamma_r(x,y) = \phi_{ab}(x) \) or \( \gamma_r(x,y) = \phi_{cd}(y) \). We shall deal separately with the three cases.

(i) \( \gamma_r(x,y) = \phi_{ab}(x) \phi_{cd}(y) \). In this case equality (3) implies 
\[
F(x,y) = G(x,y) \quad \forall (x,y) \in \mathbb{R}^2.
\]
In fact, (3) reads, in this case
\[
\int_{\mathbb{R}^2} \phi_{ab}(x) \phi_{cd}(y) \, d_{x,y} F(x,y) = \int_{\mathbb{R}^2} \phi_{ab}(x) \phi_{cd}(y) \, d_{x,y} G(x,y)
\]

\((\forall a,b,c,d \in \mathbb{Q}, a < b, c < d)\).

Because of (2), this last inequality entails

\[
\int_a^b \int_c^d F(x,y) \, dy \, dx = \int_a^b \int_c^d G(x,y) \, dy \, dx \quad (\forall a,b,c,d \in \mathbb{Q}, a < b, c < d);
\]

(4) implies the assertion \( F(x,y) = G(x,y) \quad \forall (x,y) \in \mathbb{R}^2 \). Indeed, assume, \textit{ab absurdo}, the existence of a point \( (x_*, y_*) \in \mathbb{R}^2 \) such
that \( F(x_0, y_0) \neq G(x_0, y_0) \), for instance \( F(x_0, y_0) < G(x_0, y_0) \).

Set \( \varepsilon = \left\{ G(x_0, y_0) - F(x_0, y_0) \right\} / 2 \). Since \( F \) is right-continuous and non-decreasing in both variables, there exist \( x' \) and \( y' \in \mathbb{R} \), with \( x' > x_0 \) and \( y' > y_0 \), such that

\[
F(x_0, y_0) \leq F(x, y_0) \leq F(x_0, y_0) + \varepsilon / 2 \quad (x \in [x_0, x'])
\]

and

\[
F(x_0, y_0) \leq F(x, y_0) \leq F(x', y_0) \leq F(x', y') \leq F(x_0, y_0) + \varepsilon.
\]

Thus, if \( a, b \in [x_0, x'] \) and \( c, d \in [y_0, y'] \) one has

\[
\int_a^b \int_c^d F(x, y) \, dy \, dx \leq (b-a)(d-c)(F(x_0, y_0) + \varepsilon) =
\]

\[
= (b-a)(d-c)(F(x_0, y_0) + G(x_0, y_0)) / 2 < (b-a)(d-c) \cdot G(x_0, y_0) \leq \int_a^b \int_c^d G(x, y) \, dy \, dx
\]

which contradicts (4).

(ii) If \( \gamma_r(x, y) = \Phi_{ab}(x) \), then (3) impies

\[
\int_{\mathbb{R}^2} \Phi_{ab}(x) \, dF(x, y) = \int_{\mathbb{R}^2} \Phi_{ab}(x) \, dG(x, y) \quad (a, b \leq Q, a < b)
\]

or

\[
\int_{\mathbb{R}^2} \Phi_{ab}(x) \, dF_1(x) = \int_{\mathbb{R}^2} \Phi_{ab}(x) \, dG_1(x) \quad (a, b \in \mathbb{R}, a < b)
\]

where \( F_1 \) and \( G_1 \) are marginals of \( F \) and \( G \) respectively: \( F_1(x) = F(x, +\infty) \).
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\( G_1(x) = G(x, +\infty) \). But then theorem 4 in [2], applied to \( F_1 \) and \( G_1 \) gives \( F_1 = G_1 \).

(iii) If \( \gamma_r(x, y) = \Phi_{cd}(y) \) then the same argument as above yields \( F_2 = G_2 \) where \( F_2 \) and \( G_2 \) are the marginals \( F_2(y) = F(+\infty, y) \) and \( G_2(y) = G(+\infty, y) \).

Then one can conclude from (i) that \( F \) and \( G \) coincide on the set \( \{(x, y) \in \mathbb{R}^2 : x < +\infty, y < +\infty \} \) and from (ii) and (iii) that both marginals of \( F \) and \( G \) are equal. Therefore \( F = G \).

Q.E.D.

THEOREM 3. Let \( F_n, F \in \mathcal{D}_2 \) (\( n \in \mathbb{N} \)); then \( d_2(F_n, F) \to 0 \) if \( \frac{F_n}{F} \rightarrow F \) (weakly, i.e. \( F_n(x, y) \rightarrow F(x, y) \) at every point \((x, y)\) at which \( F \) is continuous).

Proof. \((\Rightarrow)\) Assume \( d_2(F_n, F) \to 0 \). Set

\[ \delta_2(r, n) := \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \gamma_r(x, y) dF_n(x, y) - \int_{\mathbb{R}^2} \gamma_r(x, y) dF(x, y) \right| \quad (r, n \in \mathbb{N}). \]

Then

\[ 0 \leq \delta_2(r, n) \leq 2^r d_2(F_n, F) \tag{5} \]

so that \( \lim_{n \to \infty} \delta_2(r, n) = 0 \) \( \forall r \in \mathbb{N} \). If \( \gamma_r(x, y) = \phi_{ab}(x) \phi_{cd}(y) \), then on account of (2), this means

\[ \lim_{n \to \infty} \int_a^b \int_c^d F_n(x, y) dy dx = \int_a^b \int_c^d F(x, y) dy dx \quad (\forall a, b, c, d \in \mathbb{Q}, a, b, c < d). \]

We shall presently show that (6) implies \( F_n(x, y) \to F(x, y) \) if \((x, y) \in \mathbb{R}^2 \) is a point of continuity for \( F \).
Set \( F^*(x,y) := \lim_{n \to \infty} \sup_n F_n(x,y) \). Then, because of (6)

\[
(b-a)(d-c)F^*(a,c) = \lim_{n \to \infty} \sup_n (b-a)(d-c) F_n(a,c) \leq \\
\leq \lim_{n \to \infty} \sup_n \int_a^b \int_c^d F_n(x,y) \,dy \,dx = \int_a^b \int_c^d F(x,y) \,dy \,dx \leq (b-a)(d-c)F(b,d),
\]

i.e., \( F^*(a,c) \leq F(b,d) \). Let \( b+a \) to obtain \( F^*(a,c) \leq F(a,d) \),

then let \( d+c \) to obtain \( F^*(a,c) \leq F(a,c) \). Let \((x,y) \in \mathbb{R}^2\) be

such that \( x < +\infty \) and \( y < +\infty \) and take \( a > x, c > y, a, c \in \mathbb{Q} \);

then \( F^*(x,y) \leq F^*(a,c) \leq F(a,c) \). Let \( a+x \) and \( c+y \) so that \( F^*(x,y) \leq F(x,y) \).

A similar argument yields \( F_*(x,y) := \liminf_{n \to \infty} F_n(x,y) \geq F(b-0,d-0) \)

where \( b < x, d < y, b, d \in \mathbb{Q} \), so that

\[
F(x-0,y-0) \leq F_*(x,y) \leq F^*(x,y) \leq F(x,y).
\]

Therefore if \((x,y)\) is point of continuity for \( F \), then

\[
\lim_{n \to \infty} F_n(x,y) = F(x,y).
\]

If, on the other hand, at least one, say \( y \), of the coordinates of the continuity point \((x,y)\) equals \(+\infty\) then, considering the terms in \( d_2 \) with \( \gamma_{r}(x,y) = \phi_{ab}(x) \),

one has from (5)

\[
\lim_{n \to \infty} \int_a^b F_n(x,\infty) \,dx = \int_a^b F(x,\infty) \,dx \quad (a,b \in \mathbb{Q}, a < b).
\]

From this last equality it follows as in theorem 5 [2] that

\[
\lim_{n \to \infty} F_n(x,\infty) = F(x,\infty).
\]

An analogous argument holds for a continuity point of the type \((+\infty,y)\).

\((\Leftarrow)\) Conversely, if \( F_n \xrightarrow{w} F \), then \( \delta_2(r,n) \to 0 \) (WreN). Since

\( 0 \leq \delta_2(r,n) \leq 2 \), one has

\[
\lim_{n \to \infty} d_2(F_n, F) = \lim_{r \to \infty} \sum_{r=1}^{\infty} 2^{-r} \delta_2(r,n) = 0
\]
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by the dominated convergence theorem applied to the counting measure on \( \mathbb{N} \).

Q.E.D.

REFERENCES


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