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SPECTRAL GEOMETRY OF SUBMANIFOLDS

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PREFACE. These notes reflect the lectures given by the first author at the University of Lecce in June-July 1984. The work deals with geometry of Laplacian's spectrum of submanifolds in the complex projective space. The second part contains a new way to study this subject given by the second author in his Ph. Dr. (Granada, 1984). This idea can be extended to other symmetric spaces.

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For convenience to the reader we give the content of this paper as fallovs:

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1. INTRODUCTION. Let Ω be a very regular bounded domain with nice boundary $\partial\Omega$ in R^d (for d=2, one can think of Ω as a vibrating membrane fixed along $\partial\Omega$). The vibrations of Ω are the functions $F:\Omega\times R \longrightarrow R$ $(p,t)\longrightarrow F(p,t)$ with

(1.1) $$\Delta F + \partial^2F/\partial t^2 = 0 \qquad F/(\partial\Omega\times R) = 0$$

being $\Delta = -\sum_{i=1}^{d} \partial^2/\partial x_i^2$ (the Laplacian in R^d).

It is important to study solutions of the type $F(p,t)=f(p).\theta(t)$ with $f:\Omega\longrightarrow R$ and $\theta:R\longrightarrow R$. For these solutions one obtains

(1.2) $$\Delta f/f = -\theta''/\theta = \lambda$$

where λ must be constant and it is very related to the frecuencies of our vibrations since $\theta'' + \lambda\theta = 0$. That is a good reason to be interested in the spectrum of Ω,

(1.3) $$\mathrm{Spec}(\Omega) = \{0=\lambda_0<\lambda_1\ldots\lambda_1<\lambda_2\ldots\lambda_2\ldots\uparrow\infty\}$$

consisting of all λ's (real numbers) such that there exists some $f\neq 0$ with $\Delta f=\lambda f$, $f/\partial\Omega=0$. Each λ written in Spec(Ω) as many times as its multiplicity indicates. Being multiplicity of λ = dim$\{f/\Delta f= f\}$.

Despite its simplicity and its phisical background, Spec(Ω) is almost unknown. For example Spec(Ω) is known when Ω is a ball or a rectangular parallelopiped and few other Ω.

A natural and very old problem is the following: Let Ω_1 and Ω_2 be two plane regions bounded by curves Γ_1 and Γ_2 respectively and consider the eigenvalue problems:

$$\begin{array}{lll} (1.4) & \Delta f = \lambda f \text{ in } \Omega_1 & \Delta g = \mu g \text{ in } \Omega_2 \\ & f = 0 \text{ in } \partial\Omega_1 & g = 0 \text{ in } \partial\Omega_2 \end{array}$$

Suppose that for each $n \in N$ the eigenvalue λ_n for Ω_1 is equal to the eigenvalue μ_n for Ω_2 : Are the regions Ω_1 and Ω_2 congruent in the sense of Euclidean geometry?.

This problem is called the inverse problem because it is well-known that: If ϕ is an isometry of the plane then $\mathrm{Spec}(\phi(\Omega))=\mathrm{Spec}(\Omega)$ for any domain Ω in R^2.

The inverse problem was firstly posed for Bochner and Bers said about it that it can be posed in the following amusing way: Can one hear the shape of a drum?.

Of course this problem is still unsolved but it is clear that the spectrum of Ω contains an ordinate information about the geometry of Ω. $\mathrm{Spec}(\Omega)$ is like a "secret code" of information about the geometry of Ω which we need to discover.

To be less greedy than asking to hear the complete shape of Ω, one can ask only deduce from $\mathrm{Spec}(\Omega)$ some information on the geometry of Ω. For example : One can hear the area of Ω. This is an old problem posed by H.A.Lorentz (1910) which was solved by H.Weyl. In fact he proved

$$(1.5) \qquad \lim_{\lambda \to \infty} N(\lambda)/\lambda = \mathrm{vol}(\Omega)/(2\pi)$$

where $N(\lambda)$ is the number of eigenvalues less than λ.

The length of $\partial\Omega$ is also a spectral invariant and therefore we can prove that: You can hear the shape of a circular drum. In fact if L denotes the length of $\partial\Omega$, the classical isoperimetric inequality states

$$(1.6) \qquad L^2 \geq 4\pi\,\mathrm{vol}(\Omega)$$

with the equality holding if and only if Ω is a circular disk.

This result joint the facts than $\mathrm{vol}(\Omega)$ and L are spectral invariants allow us to prove that : You can hear the shape of a circular drum.

The spectral problem appear in many parts of mathematics and physics: statistical mechanics, diffusion theory, quantum statistics etc. (see [28]).

In this paper we are interested in the spectral problem but for free vibrating membranes, that is, for Riemannian manifolds which are C^∞, compact, connected and vithout boundary. All manifolds in this paper are assumed to have these properties.

2. SOME PROBLEMS RELATED WITH THE SPECTRUM OF A RIEMANNIANN MANIFOLD. Let M be a compact, connected, without boundary Riemannian manifold (along this paper we will omit these properties), (also all geometric objets are assumed to be C^∞). We will denote by g the metric tensor of M. As usual $C^\infty(M)$ (respectively $\Lambda^p(M)$) will denote the real valued functions over M (respectively the exterior p-forms over M). $\Lambda(M)$ will be the Grassmann algebra over M, d (respectively δ) will denote exterior derivative (respectively the coderivative).

The Laplace-Beltrami operator of (M,g) is defined over $\Lambda(M)$ by

$$\Delta : \Lambda(M) \longrightarrow \Lambda(M) \qquad \Delta(\omega) = (d\delta+\delta d)(\omega) \tag{2.1}$$

it is clear that the degree of ω is preserved by Δ.

In particular we are interested in the Laplacian acting over functions. In this case various definitions exist for Δ (of course equivalent), ve recall some of them:

(a) Suppose $N=\dim M$, if $m\in M$ and $f\in C^\infty(M)$, we choose an orthonormal set of geodesics $\{\gamma_i\}$, $i=1,\ldots,n$ parametrized by arc length and passing through m at $t=0$, then

$$(\Delta f)(m) = -\sum_{i=1}^{n} (d^2(f.\gamma_i)/dt^2)(0) \tag{2.2}$$

(b) Last definition is obviously equivalent to the folloving one, first we consider the Hessian of f like the symmetric 2-times covariant tensor over M given by

$$\operatorname{Hess} f(X,Y) = XYf - (\nabla_X Y)f \qquad X,Y\in\chi(M) \tag{2.3}$$

$\chi(M)$ being the Lie algebra of vector fields over M, then

$$\Delta f = -\operatorname{Trace}(\operatorname{Hess} f) = -\sum_{i=1}^{n} \operatorname{Hess} f(E_i,E_i) \tag{2.4}$$

where $\{E_i\}$ $i=1,\ldots,n$ is a local orthonormal frame of vector fields.

For other definitions of Δ see [9].

The canonical measure of (M,g) will be denoted by dv. One defines a pre-Hilbert structure over $C^\infty(M)$ by taking

(2.5) $$\langle f,g\rangle = \int_M f.g\,dv \qquad \text{for all } f,g\in C^\infty(M)$$

the completed of $C^\infty(M)$ with respect to this inner product is $L^2(M)$. An eigenvalue of Δ is a real number λ such that $\Delta f=\lambda f$ for some non-trivial function $f\in C^\infty(M)$. The set of all eigenvalues of Δ will be called the spectrum of (M,g) which will be denoted by $\mathrm{Spec}(M,g)$.

Since Δ is elliptic, $\mathrm{Spec}(M,g)$ is discrete and the multiplicity of each λ, that is, $\dim V_\lambda = \dim \{f\in C^\infty(M)/\Delta f=\lambda f\}$ is finite. In particular, since M is compact, $\dim V_0=1$ (V_0 consists only of constant functions). Therefore

(2.6) $$\mathrm{Spec}(M,g) = \{0=\lambda_0<\lambda_1\ldots\lambda_1<\lambda_2\ldots\lambda_2<\ldots\uparrow\infty\}$$

As it is well known, the decomposition $\sum V_\lambda, \lambda\in\mathrm{Spec}(M,g)$ is orthogonal with respect to (2.5). Moreover $\sum V_\lambda$ is dense in $C^\infty(M)$ (in L^2-sense).

Similarly the spectrum of Δ acting over exterior p-forms will be denoted by $\mathrm{Spec}^p(M,g)$. One has

(2.7) $$\mathrm{Spec}^p(M,g) = \{{}^p\lambda_0\ldots{}^p\lambda_0<{}^p\lambda_1\ldots{}^p\lambda_1<\ldots\uparrow\infty\}$$

From Hodge theory one has:

(i) If n denotes the dimension of M, then $\mathrm{Spec}^p(M,g)=\mathrm{Spec}^{n-p}(M,g)$

(ii) 0 belongs to $\mathrm{Spec}^p(M,g)$ if and only if the pth Betti number of M is non-zero (being this Betti number the multiplicity of 0 in $\mathrm{Spec}^p(M,g)$,

Now one can put the following general problems: Given a Riemannian manifold (M,g), one has $\mathrm{Spec}(M,g)$ which only dependes on the Riemannian structure

(1) What is the behaviour of the elements of $\mathrm{Spec}(M,g)$?

(2) What information can one obtain from $\mathrm{Spec}(M,g)$ about the geometry and the topology of (M,g) and M respectively?

These kind of problems and others related are known under the general name of Spectral geometry.

Now, we will briefly treat about some of this problems:

(A) How to determine $Spec(M,g)$ for a given Riemannian manifold?

This problem is very difficult and we only know a few about it:

- We can compute the spectrum of a Riemannian product from the spectrums of the factors.
- We can also relate the spectrums of two Riemannian manifolds connected by means of a Riemannian submersion with totally geodesic fibers.
- If dimM=l and L denotes the length of (M,g) then one has

$$Spec(M,g) = \{4\pi^2 n^2/L^2 \text{ with } n\in N\} \tag{2.8}$$

each eigenvalue with multiplicity 2 except $\lambda_0=0$. Therefore, when M is 1-dimensional, the spectrum characterizes the Riemannian structure and even λ_1 does.

- The spectrum is completely known for (S^n,g_0), the n-dimensiona1 sphere endowed with its canonica1 Riemannian structure. This is also true for compact symmetric spaces of rank one with their canonica1 structures.

The spectrum is, in some sense, also completely known for symmetric spaces and some homogeneous spaces, at least in theory,[36].

- Consider a flat torus $M=R^n/\Lambda$ where Λ is any lattice of R^n, endowed with g_0/Λ, g_0 being the canonica1 metric on R^n (this metric has no problems in going to the quotient since it is invariant by translations). We put

$$\Lambda^* = \{\xi\in R^n/(\xi,\eta)\in Z, \text{ for all } \eta\in\Lambda\} \tag{2.9}$$

the aual lattice of Λ. Then

$$Spec(R^n/\Lambda, g_0/\Lambda) = \{4\pi^2|\xi|^2/\xi\in\Lambda^*\} \tag{2.10}$$

the multiplicity of $4\pi^2|\xi|^2$ is the number of $\eta\in\Lambda^*$ with $|\eta|^2=|\xi|^2$.

(B) Since it is very difficult to compute $Spec(M,g)$, one must obtain estimates of eigenvalues by means of geometrica1 invariants of (M,g).

The more important results in this direction have been obviously obtained for the first non-trivial eigenvalue. Now we recall some of them.

The first result is due to A.Lichnerowicz [33] and M.Obata [38]

THEOREM 2.1.- Let M be an n-dimensional Riemannian manifold, $(n>1)$. Suppose there exists a constant $k>0$ such that the Ricci curvature S of M satisfies $S \geq kg$ (g being the metric tensor of M). Then:

(a) $\lambda_1 \geq (n/(n-1))k$ (Lichnerowicz)

(b) $\lambda_1 = (n/(n-1))k$ if and only if (M,g) is isometric to (S^n, g_o) (Obata).

This estimate of Lichnerowicz-Obata was extended later by R.C. Reilly ([44]) to manifolds with boundary where he treated the Dirichlet boundary value problem ($\Delta u=F(u)$ with $u=0$ in ∂M) under the aditional assumption that ∂M has non-negative mean curvature.

There exist some classical papers due to Faber-Krahn, Polyá-Szegö, Payne, Weinberger etc. ([8]) which give estimates for the first eigenvalue of a domain Ω in R^n. For instance an estimate due to Faber-Krahn for the first eigenvalue of a vibrating membrane Ω in R^2 is

$$\lambda_1 \geq \pi j^2/\mathrm{vol}(\Omega) \tag{2.11}$$

the equality holding if and only if Ω is a disk. Here j is the first zero of the first Bessel's function.

The estimate of Faber-Krahn can not be extended to Riemannian manifold, in fact one has the following examples:

- Consider the flat torus R^2/Λ where Λ is the lattice of R^2 of lengths $(t,1/t)$. Its area is 1, but for small values of t, $\lambda_1=4\pi^2t^2$ is close to zero.
- Consider two spheres E_1 and E_2 connected by means of a cylinder of radius r and length L. One defines a function of measure zero f such that: f is constant k over E_1, f is constant-k over E_2 and f is linear over the cylinder being constant over its sections. Then one obtains

$$|\nabla f|^2/|f|^2 \leq 4\pi r/(L\,\mathrm{vol}(M)) \quad (\nabla f \text{ being the gradient of } f)$$

but the inequality of Poincaré gives

$$\int_M |\nabla f|^2 dv \geq \lambda_1 \int_M f^2 dv \quad \text{for all } f \text{ with measure zero}$$

tnerefore one obtains the following estimate

$$\lambda_1 \leqslant 4\pi r/(L\mathrm{vol}(M)) \tag{2.12}$$

this estimate, for small values of r, is close to zero while vol(M) is close to $2\mathrm{vol}(S^2,g_o)$.

These examples prove us that in a possible lower bound of λ_1, must participate other geometrica1 invariants apart from the volume and the diameter.

After the work of Lichnerowicz and Obata, J.Cheeger ([12]) obtained a lower bound of λ_1 by a constant which is involved in a certain type of isoperimetric inequality.

In [1] T.Aubin gave a lower bound of λ_1 in terms of the following elements: a lower bound of the volume, an upper bound of the diameter, a lower bound of the sectional curvature, an upper bound of the Ricci curvature and a lower bound of the injectivity radius.

It seems obvious to relax the dependency of the Aubin estimate on the geometric quantities. In this sense, S.T.Yau ([60] showed that one can estimate λ_1 from bellow by a lower bound involving the following elements: a lower bound of the volume, an upper bound of the diameter and a lower bound of the Ricci curvature. Moreover, basing on an upper bound of λ_1 obtained by S.Y.Cheng ([21]), Yau conjectured that one should be able to drop the dependency of the volume in the last estimate. Therefore by combining with Cheng's result this would give the best possible estimate of λ_1 for a general compact Riemannian manifold.

This conjecture was solved by P.Li ([31]). His method depended bassically on a gradient estimate of the first eigenfuctions. In particular, for compact Riemannian manifolds with non-negative Ricci curvature, he obtained $\lambda_1 \geqslant \pi^2/(4d^2)$, d being the diameter of the manifold. Because for these kind of spaces, the upper estimate of Cheng is $\lambda_1 \leqslant n\pi^2/d^2$, n being the dimension of the manifold. One has the following estimation of λ_1 for a compact Riemannian manifold with non-negative Ricci curvature

$$\pi^2/4d^2 \leqslant \lambda_1 \leqslant n\pi^2/d^2 \tag{2.13}$$

Recently, P.Li and S.T.Yau ([32]) extended the above estimates to compact manifolds with boundary:

(a) For Dirichlet boundary valued problem, the estimate also depends on the lower bound of the mean curvature of ∂M.

(b) For Neumann boundary valued problem, they need to assume that the second fundamental form of ∂M is positive semidefinite.

When one considers Δ acting over exterior p-forms, there exist some interesting estimates of ${}^p\lambda_1$, in this way one can see for instance [24],[42],[51].

(c) To find the regularities of the eigenvalues sequence in the infinity (to study the behaviour of the spectrum in the limit).

This problem has been intensely studied and one has some satisfactory answers for it. The central idea can be explained as follows: Given a compact Riemannian manifold. one consider the function

$$Z(t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \qquad \lambda_i \in \mathrm{Spec}(M,g) \tag{2.14}$$

(the same problem can be considered for $\mathrm{Spec}^p(M,g)$).

The function $Z(t)$ is well defined for each $t>0$ (by means of the fundamental solution of heat equation) and its behaviour when $t\to 0^+$ is completely described by the well-known asymptotic expansion formula of Minakshisundaram-Pleijel-Gaffney

$$\sum_i e^{-\lambda_i t} \underset{t\to 0+}{\sim} (4\pi t)^{-n/2}(a_0+a_1 t+a_2 t^2+\dots) \tag{2.15}$$

the coefficients a_i involving only Riemannian invariants of corresponding degrees. For example, one knows a few coefficients:

$$a_0 = \mathrm{vol}(M)$$

$$a_1 = 1/6 \int_M \rho \, dv$$

$$a_2 = 1/360 \int_M (2|R|^2-2|S|^2+5\rho^2)dv$$

where R,S and ρ denotes respectively the Riemannian curvature tensor, the Ricci curvature and the scalar curvature (see for instance [9]). The coefficient a_3 has been computed for T.Sakai [49].

3. THE INVERSE PROBLEM. As we said before, one can think in the

following problem: Can one hear the shape of a manifold? (manifolds are considered like vibrating membranes because they are not boundary). This problem can be formulated with respect to the injectivity of a map whose image looks too formidable today (to see the first problem). In fact one considers the map

$$\text{Spec}:\{\text{Riemannian structures}\}\longrightarrow R^N$$

is it injective?.

We know that the injectivity of this map is false in general. In fact Witt discovered in 1941 two lattices Λ_1 and Λ_2 in R^{16} which are not isometric but with the same number of elements of any given norm. Therefore the two corresponding flat torus have the same spectrum (2.10) and they are not isometric [35].

In 1967, M.Kneser, [29], obtained the same result for two flat torus of dimension 12.

Recently, M.Vigneras, [58], obtained two surfaces of Poincaré (that is, surfaces with constant curvature -1) which have the same spectrum but they are not isometric.

Since these examples have a discrete character, an open problem is to decide whether or not the exists a nontrivial family $(M,g(t))$ of Riemannian structures over M with $\text{Spec}(M,g(t))=\text{Spec}(M,g(o))$ for every t (isospectral deformation). In this sense, V.Guillemin and D.Kazhdan ([25]) showed that under certain conditions, every isospectral deformation of a negatively curved Riemannian manifold is isometric (and so the deformation is trivial).

Por Kaehlerian isospectral deformations, one knows thatthe complex projective space CP" with its canonica1 Kaehlerian structure does not admit non-trivial Kaehlerian isospectral deformations, [9], (here Kaehlerian deformations means that each metric in the deformation must be Kaehlerian with the fixed complex structure). In the same way M.Barros, [3], showed that the complex quadric of complex dimension 3 with its canonica1 Kaehlerian structure (which is obtained from the fact that it is a complex hypersurface of CP^4) does not admit non-trivial Kaehlerian isospectral deformation.

Also, we can think of some special cases of injectivity, for instance, one can study the following problem: Wh mannian manifolds can be characterized by its spectrum?. It is ar that the first candidate to be characterized by its spectrum . st be the sphere with its canonical metric. Many authors obtained partial answers to the following problem: If $Spec^p(M,g)=Spec^p(S^n,g_o)$ for some p, then (M,g) is isometric to (S^n,g_o) ?.

For instance (S^2,g_o) is characterized by its spectrum over functions ([9]). S.Tanno ([56]), by using the spectrum of Δ acting over 1-forms gave an affirmative answer to the above problem for n=2,3, 16,...93.

Other candidate to be characterized by its spectrum is the complex projective space CP^n endoved with the Fubini-Study metric g_o, which, as it is well-known is the corresponding model in Kaehlerian geometry with constant positive holomorphic sectional curvature. Therefore the corresponding problem is : Let (M,J,g) a Kaehlerian manifold and assume that $Spec^p(M,g)=Spec^p(CP^n,g_o)$, can ve say that (M,J,g) is holomorphically isometric to (CP^n,J_o,g_o) ? (here J_o denotes the canonical complex structuré over CP^n).

Probably the most complete ansver to this problem was obtained by B.Y.Chen and L.Vanhecke, ([20]), which gave an affirmative answer to the above problem when p=2 and $n\neq 8$.

An important contribution to this problem was obtained by M.Barros and B.Y.Chen, ([5]): Let Q_2 be the complex quadric of complex dimension 2. In general Q_n is defined extrinsically as a complex hypersurface of CP^{n+1} and after linear submanifolds it is the most important submanifold of CP^{n+1}, (it will play an important role in this paper), then

THEOREM 3.1.- Let (M,J,g) be a Kaehler surface vith $Spec(M,g) = Spec(Q_2,g_o)$. If (M,J,g) satisfies one of the following conditions:

(1) the arithmetic genus $a(M)\geqslant 1$ or

(2) the Hirzebruch index $t(M)\leqslant 0$ or

(3) the Euler characteristic $\chi(M)\geqslant 4$

then (M,J,g) is holomorphically isometric to (Q_2,J_o,g_o).

Proof.- We will denote by S (respectively S_o) the Ricci tensor of M (respectively of Q_2) and by ρ (respectively ρ_o) the scalar curvature of M (respectively of Q_2). Each assuption (1), (2) or (3) implies an integral inequality for M and Q_2 in terms of their corresponding quadratics invariants. For instance (3) implies

$$\int_M \{|R|^2 - 4|S|^2 + \rho^2\} dv \geq \int_{Q_2} \{|R_o|^2 - 4|S_o|^2 + \rho_o^2\} dv_o \tag{3.1}$$

therefore combining this inequality with the equality $a_2 = a_2^o$ where a_2 (respectively a_2^o) denotes the corresponding coefficient in the Minashisundaran-Pleijel-Gaffney asymptotic expansion for M (respectively for Q_2), one obtains

$$\int_M \{2|S|^2 + \rho^2\} dv \leq \int_{Q_2} \{2|S_o|^2 + \rho_o^2\} dv_o \tag{3.2}$$

We recall that in general $|S|^2 \geq \rho^2/4$ and the equality holds if and only if (M,J,g) is Einstein. Then by using this fact joint (3.2) and the fact that Q_2 is Einstein (and so ρ_o is constant) joint the Schwartz inequality, one can prove that (M,J,g) is an Einstein Kaehler surface. Now we use a well-known result due to H.Donelly, ([22]) to prove that (M,g) is a symmetric space which must be simply connected because its Ricci tensor is definite positive, ([30]). Therefore (M, J,g) is a Hermitian symmetric space. If it is ireducible, it must be CP^2 ([27]) but this contradicts every assumption (1),(2) or (3). Thus (M,J,g) is a reducible Hermitian symmetric space and so the Riemannian product of two complex projective lines and holomorphically isometric to (Q_2, J_o, g_o).

Remark.- (1) For general dimension, the problem of obtaining intrinsic characterizations of Q_n by its spectrum looks formidable today. (2) Q_2 is the first space characterized by its spectrum which has not constant curvature (in the correct sense). (3) The above result has important consequences, for instance, (Q_2, J_o, g_o) is completely characterized by its spectrum among all rational surfaces. Also among all topological reducible Kaehler surfaces. In these cases, a(M) =1 and t(M)=0, respectively.

There exists other important contributions to the inverse problem, to see for instance [43] and [57].

4. BASIC FORMULAS IN THEORY OF SUBMANIFOLDS. Now we will recall some fundamental facts in theory of submanifolds (a fundamental reference for that is [13].

Let M^n and $\bar{M}^{n+p}$ be two Riemannian manifolds with dimension n and n+p respectively. We will denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections of M and $\bar{M}$ respectively. Now we assume that $f:M\longrightarrow\bar{M}$ is an isometric immersion, then for any point $x\in M$ one has: The tangent space of M at x, T_xM; the normal space of M via f at x, $(T_x^{\perp}M)_f$ (sometimes we will omit f) and the connection $\nabla^{\perp}$ determined by f in the normal vector bundle $T^{\perp}M$. Also $\chi(M)$ will denote the Lie algebra of vector fields tangent to M and $\chi^{\perp}(M)$ the space of vector fields normal to M via f.

As it is well-known, the most important geometric object associated to f is its second fundamental form σ, which is well defined via the Gauss equation

$$\sigma(X,Y) = \bar{\nabla}_XY-\nabla_XY \tag{4.1}$$

for all $X,Y\in\chi(M)$. That is, $\sigma(X,Y)$ is the normal component of $\bar{\nabla}_XY$. Properties of symmetry and bilinearlity of σ are well-known.

The Weingarten endomorphisms of f are defined by means of Weingarten's formula

$$A_\xi X = \nabla^{\perp}_X\xi-\bar{\nabla}_X\xi \tag{4.2}$$

for all $X,Y\in\chi(M)$ and $\xi\in\chi(M)^{\perp}$. That is $A_\xi X$ is, up the sign, the tangential component of $\bar{\nabla}_X\xi$.

One has the relation

$$g(A_\xi X,Y) = g(\sigma(X,Y),\xi) \quad \text{for all } X,Y\in\chi(M) \text{ and } \xi\in\chi(M)^{\perp} \tag{4.3}$$

g denoting the metric over M and $\bar{M}$.

The mean curvature vector of f is a vector field $H\in\chi(M)^{\perp}$ given by

$$H = (1/n)\sum_{i=1}^{n} \sigma(E_i,E_i) \tag{4.4}$$

where $\{E_i\}$ $i=1,\dots,n$, denotes an orthonormal basis for the tangent space of M at the corresponding point.

One can define the covariant derivative of σ by

(4.5) $$(\nabla\sigma)_X(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z)$$

for all $X,Y,Z\in\chi(M)$.

- If $\sigma=0$, the isometric immersion f is called totally geodesic.
- If $H=0$, the isometric immersion f is called minimal.
- If $\nabla\sigma=0$, the isometric immersion f is called parallel.

We will use the following sign for the Riemannian curvature R of M (the same for others curvatures)

(4.6) $R(X,Y,Z,W) = g(([\nabla_X,\nabla_Y]-\nabla_{[X,Y]})Z,W)$ for all $X,Y,Z,W\in\chi(M)$

The equations of Gauss and Codazzi for the curvature are given by

(4.7) $\bar{R}(X,Y,Z,W) = R(X,Y,Z,W)+g(\sigma(X,Z),\sigma(Y,W))-g(\sigma(X,W),\sigma(Y,Z))$

(4.8) $$(\bar{R}(X,Y)Z)^{\perp} = (\nabla\sigma)_X(Y,Z)-(\nabla\sigma)_Y(X,Z)$$

in both equations $X,Y,Z,W\in\chi(M)$ and $\perp$ denotes the normal component.

If $R^{\perp}$ denotes the curvature tensor of $\nabla^{\perp}$, that is,

(4.9) $R^{\perp}(X,Y)\xi = ([\nabla^{\perp}_X,\nabla^{\perp}_Y]-\nabla^{\perp}_{[X,Y]})\xi$ $X,Y\in\chi(M)$ and $\xi\in\chi(M)^{\perp}$

then the equation of Ricci is

(4.10) $$\bar{R}(X,Y,\xi,\eta) = R^{\perp}(X,Y,\xi,\eta)-g([A_\xi,A_\eta]X,Y)$$

for all $X,Y\in\chi(M)$ and $\xi,\eta\in\chi(M)^{\perp}$.

5. SPECTRAL GEOMETRY OF SUBMANIFOLDS IN EUCLIDEAN SPACE. If M is a Riemannian manifold, for each $\lambda_k\in\mathrm{Spec}(M,g)$, we consider its corresponding eigenspace V_k and recall that $\sum_k V_k$ is dense in $C^\infty(M)$.

For each $f\in C^\infty(M)$, we denote by f_t the projection of f onto the subspace V_t $(t=0,1,2,\dots)$, then one has the following spectral decomposition

(5.1) $$f=\sum_{t=0}^{\infty} f_t \quad \text{in } L^2\text{-sense } (\Delta f_t=\lambda_t f_t).$$

in particular, since $\dim V_0=1$, f_0 is constant.

Now we assume that $\psi:M^n\longrightarrow E^m$ is an isometric immersion of M in an m-dimensional Euclidean space. We will denote by D the canonical connection over E^m. If Δ is the Laplacian of M acting over $C^\infty(M)$, one can extend Δ to E^m-valued functions over M in natural way, if $F:M\longrightarrow E^m$ with $F=(F_1,\dots,F_m)$ with respect to a fixed basis on E^m, then one puts

(5.2) $$\Delta F = (\Delta F_1,\dots,\Delta F_m)$$

it is clear that this definition does not depend over the chosen basis in E^m. Moreover if a denotes a fixed vector in E^m, one has

(5.3) $\Delta\langle a,F\rangle = \langle a,\Delta F\rangle$ $\langle\,,\rangle$ being the Euclidean inner product.

In particular, if we consider $\psi:M\longrightarrow E^m$, from (5.1) and (5.2) one get

(5.4) $$\psi = \sum_{t=0}^{\infty}\psi_t\ ;\quad \psi_t:M\longrightarrow E^m \quad \text{with } \Delta\psi_t = \lambda_t\psi_t$$

this sequence is convergent coordinate to coordinate in the L^2-sense over $C^\infty(M)$. Moreover ψ_0 is constant and it is called the gravity center of ψ.

(5.5) $$\psi_0 = (1/\mathrm{vol}(M))\int_M \psi(M)dv$$

Since the decomposition $\sum k\ V_k$ is orthogonal, one has too

(5.6) $$\int_M \langle\psi_u,\psi_v\rangle = 0 \qquad \text{for } u\neq v$$

Now we will establish a very interesting theorem due to T.Takahashi ([54]), but before we will consider tne following

LEMMA 5.1.- Let $\psi:M^n\longrightarrow E^m$ an isometric immersion of an n-dimensional Riemannian manifold in the Euclidean space. if H denotes the mean curvature vector of ψ, then

(5.7) $$\Delta\psi = -nH$$

Proof.- Let a be a fixed vector in E^m and consider $\langle a,\psi\rangle:M^n\longrightarrow R$. If $X,Y\in\chi(M)$, direct computations give us

$$(d\langle a,\psi\rangle)(X) = X(\langle a,\psi\rangle) = \langle a,X\rangle$$

$$(\mathrm{Hess}\,\langle a,\psi\rangle)(X,Y) = XY\langle a,\psi\rangle - (\nabla_X Y)(\langle a,\psi\rangle) = \langle a,D_X Y\rangle - \langle a,\nabla_X Y\rangle =$$
$$= \langle a,\sigma(X,Y)\rangle$$

now from (2.4), one has

$$-\Delta\langle a,\psi\rangle = \sum_{i=1}^{n} \mathrm{Hess}\langle a,\psi\rangle(E_i,E_i) = \langle a, \sum_{i=1}^{n}\sigma(E_i,E_i)\rangle = \langle a,nH\rangle$$

because last formula is true for every a in the Euclidean space, one obtains (5.7).

It is easy to prove the following relations

$$\Delta\psi = -nH = \sum_{t\geq 1} \lambda_t\psi_t \tag{5.8}$$

$$\Delta^k\psi = \sum_{t\geq 1} \lambda_t^k\psi_t \quad \text{for all } k\in N \tag{5.9}$$

Definition.- Let ω a finite subset of N. We will say that the isometric immersion $\psi:M\longrightarrow E^m$ is of order ω if $\psi_t=0$ for all $t\in N-(\omega\cup\{0\})$. In particular if $\omega=\{u\}$ and ψ is an isometric immersion of order ω, we will say that ψ is of order u. These isometric immersion or their corresponding submanifolds have been called by B.Y.Chen ([16]) submanifolds of finite type. According the number of elements in ω we will use the terminology: mono-order, bi-order,...

Since our manifolds are assumed to be compact, we will give the compact version of Takahashi's theorem

THEOREM 5.2.- Let $\psi:M^n\longrightarrow E^m$ be an isometric immersion of an n-dimensional Riemannian manifold in the Euclidean space. Then ψ is of order u, for some $u\geq 1$ (mono-order) if and only if ψ is minimal in some hypersphere of E^m. Moreover if ψ is of order u, then $\lambda_u=n/r^2$ being r the radius of the corresponding hypersphere.

Proof.- Suppose that ψ is of order u, for some $u\geq 1$, then

$$\psi = \psi_0+\psi_u \quad \text{with} \quad \Delta\psi_u = \lambda_u\psi_u \tag{5.10}$$

Now from (5.7) and (5.10) one get

$$\lambda_u(\psi-\psi_0) = -nH \tag{5.11}$$

First we will prove that $\psi(M)$ is contained in some hypersphere S of center ψ_0. Let X be any vector field tangent to M, then

$$X\langle\psi-\psi_0,\psi-\psi_0\rangle = 2\langle\psi-\psi_0,X\rangle = -2(n/\lambda_u)\langle H,X\rangle = 0$$

If we denote by σ' (respectively $\bar\sigma$) the second fundamental form of M in S (respectively of S in E^m) and by H' (respectively $\bar H$) the mean curvature vector of M in S (respectively of S in E^m), then

$$H = H' + (1/n)\sum_{i=1}^{n} \bar{\sigma}(E_i,E_i) \tag{5.12}$$

From (5.11) we see that H is normal to S, then (5.12) implies that $H'=0$ and so M is minimal in S.

The converse is trivial.

Let r be the radius of S and $\xi=(\psi-\psi_0)/r$ a unit vector normal to S. Since $A_\xi=-(1/r)I$, from (5.12) one get

$$H = -(\psi-\psi_0)/r^2 \tag{5.13}$$

now from (5.11) and (5.13) one obtains that $\lambda_u=n/r^2$.

Remark.- The above result proves us that the spectrum of a Riemannian manifold M containes information about the possible radius of spheres in which M admits a minimal isometric immersion.

In order to finish this section we will give two results ([16], [47]). The first one can be looked like a formal generalization of the situation which we have found in Takahashi's theorem. The second one will be very useful along this paper.

PROPOSITION 5.3.- Let $\psi:M^n\longrightarrow E^m$ be an isometric immersion, then the following conditions are equivalents:

(i) ψ is of order ω, for some finite subset ω of N, with $\#(\omega)=k\geq 1$.

(ii) There exists real numbers $a_0,a_1,\dots,a_{k-1}$ such that

$$\Delta^k\psi = a_0(\psi-\psi_0)+a_1\Delta\psi+\dots+a_{k-1}\Delta^{k-1}\psi \tag{5.14}$$

ψ_0 being the center of gravity of ψ.

Proof.- Suppose $\omega=\{u_1,\dots,u_k\}$ and consider the elementary symmetric functions $s_1,\dots,s_k$ of $(\lambda_{u_1},\dots,\lambda_{u_k})$. From $\psi=\psi_0+\psi_{u_1}+\dots+\psi_{u_k}$ one has

$$\Delta^r\psi = \lambda_{u_1}^r\psi_{u_1}+\dots+\lambda_{u_k}^r\psi_{u_k} \quad \text{for all } r$$

so some direct computations give us

$$\Delta^k\psi = s_1\Delta^{k-1}\psi-s_2\Delta^{k-2}\psi+\dots+(-1)^{k-1}s_k(\psi-\psi_0)$$

Conversely, assume $\Delta^k\psi=a_0(\psi-\psi_0)+a_1\Delta\psi+\dots+a_{k-1}\Delta^{k-1}\psi$ fcr certain real numbers $a_0,a_1,\dots,a_{k-1}$, from (5.4) and (5.9) one obtains

$$\sum_{u\geq 1}\lambda_u^k\psi_u = a_0\sum_{u\geq 1}\psi_u+a_1\sum_{u\geq 1}\lambda_u\psi_u+\dots+a_{k-1}\sum_{u\geq 1}\lambda_u^{k-1}\psi_u \text{ and}$$

$$\sum_{u\geq 1}(\lambda_u^k-a_0-a_1\lambda_u-\dots-a_{k-1}\lambda_u^{k-1})\psi_u = 0$$

since this sequence is L^2-convergent, for any $t\in\mathbb{N}$, $t\geq 1$, we apply $\int_M\langle\psi_t,-\rangle dv$ to last formula and so

$$(\lambda_t^k-a_0-a_1\lambda_t-\dots-a_{k-1}\lambda_t^{k-1})\int_M\langle\psi_t,\psi_t\rangle dv = 0$$

which implies that $\psi_t=0$ for all $t\geq 1$ except at most for k different values of t (just for the solutions of the equation $\lambda^k-a_0-a_1\lambda-\dots-a_{k-1}\lambda^{k-1}=0$).

PROPOSITION 5.4.- Let $\psi:M^n\longrightarrow E^m$ be an isometric immersion. For any two natural numbers $k,t\geq 1$, one has

$$\int_M\{\langle\Delta^{k+t}\psi,\psi\rangle-s_1\langle\Delta^{k+t-1}\psi,\psi\rangle+s_2\langle\Delta^{k+t-2}\psi,\psi\rangle+\dots+(-1)^k s_k\langle\Delta^t\psi,\psi\rangle\}dv \geq 0$$

s_r being the elementary symmetric functions of $(\lambda_1,\dots,\lambda_k)$. The equality holds if and only if ψ is of order $\{1,2,\dots,k\}$.

Proof.- From (5.9) and direct computations, one has:

$$\Delta^{k+t}\psi-s_1\Delta^{k+t-1}\psi+s_2\Delta^{k+t-2}\psi+\dots+(-1)^k s_k\Delta^t\psi =$$

$$= \sum_{u\geq 1}\lambda_u^t(\lambda_u-\lambda_1)(\lambda_u-\lambda_2)\dots(\lambda_u-\lambda_k)\psi_u =$$

$$= \sum_{u\geq k+1}\lambda_u^t(\lambda_u-\lambda_1)\dots(\lambda_u-\lambda_k)\psi_u$$

By appliying $\int_M\langle\psi,-\rangle dv$ to the last formula, one get

$$\int_M\{\langle\Delta^{k+t}\psi,\psi\rangle-s_1\langle\Delta^{k+t-1}\psi,\psi\rangle+s_2\langle\Delta^{k+t-2}\psi,\psi\rangle+\dots+(-1)^k s_k\langle\Delta^t\psi,\psi\rangle\}dv =$$

$$= \sum_{u\geq k+1}\lambda_u^t(\lambda_u-\lambda_1)\dots(\lambda_u-\lambda_k)\int_M\langle\psi_u,\psi_u\rangle dv \geq 0$$

and the equality holds if and only if $\psi_u=0$ for all $u\in N$ with $u>k$, that is, if and only if ψ is of order $\{1,2,...,k\}$.

6. SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD. Let $(\bar{M},J,g)$ be a Kaehlerian manifold with complex structure J. With respect to a Riemannian manifold, nov one has an aditional structure J and so it seems natural to study submanifolds in $\bar{M}$ according their behaviours with respect to J.

In this sense, the first kind of submanifolds of a Kaehler manifold, vhich has been studied, is the family of complex submanifolds. If M is a submanifold of $(\bar{M},J,g)$, M is called complex if its tangent space at any point is J-invariant

$$J(T_xM) = T_xM \quad \text{for all } x\in M \tag{6.1}$$

in this case J can be induced over M and so, M is already a Kaehlerian manifold. Therefore, ve will say Kaehler submanifolds.

If a submanifold M of $(\bar{M},J,g)$ satisfies

$$J(T_xM) \subset T_x^{\perp}M \quad \text{for all } x\in M \tag{6.2}$$

then M is called a totally real submanifold. The family of totally real submanifolds of a Kaehlerian manifold has been studied for many authors too.

Let M be a real hypersurface of a 2m-dimensional Kaehlerian manifold, then one can choose a local unit normal vector field ξ and so, one can put

$$T_xM = \langle J\xi_x\rangle \oplus D(x) \tag{6.3}$$

where D(x) denotes the orthogonal complement of $\langle J\xi_x\rangle$ in T_xM and it is obvious that it is the greatest holomorphic subspace on T_xM.

We see that a real hypersurface of a Kaehlerian manifold is never a Kaehlerian (respectively, totally real) submanifold. By extending the behaviour of a real hypersurface with respect to the complex structure of a Kaehlerian manifold, A.Bejancu ([10])defined the concept of CR-submanifold of a Kaehlerian manifold as follows: A submanifold M of a Kaehlerian manifold $(\bar{M},J,g)$ is called a CR-submanifold if it admits a pair of distributions $(D,D^{\perp})$ such that

(1) $D^{\perp}$ is the orthogonal complement of D
(2) D is holomorphic and $D^{\perp}$ is totally real.

It is obvious that Kaehler submanifolds, totally real submanifolds and real hypersurfaces are CR-submanifolds.

Take $\bar{M}=CP^{n+p}$, the (n+p)-dimensional complex projective space endowed of its canonical complex structure J_o and the Fubini-Study metric g_o of holomorphic sectional curvature 1. Let M^n be a Kaehler submanifold of CP^{n+p} with complex dimension n. We choose an orthonormal basis of vector fields over CP^{n+p} $\{E_1,\ldots,E_n,JE_1=E_{1*},\ldots,JE_n=E_{n*}$ $\xi_1,\ldots,\xi_p,J\xi_1=\xi_{1*},\ldots,J\xi_p=\xi_{p*}\}$ which is adapted to M. We will use the following convention on the range of indices unless otherwise state:

$a,b,..=1,\ldots,n$ $\qquad i^*=\begin{cases} a^* & \text{if } i=a \\ a & \text{if } i=a^* \end{cases}$

$i,j,k,r,s,=1,\ldots,n,1^*,\ldots,n^*$

$\alpha,\beta,=1,\ldots,p$ $\qquad \lambda^*=\begin{cases} \alpha^* & \text{if } \lambda=\alpha \\ \alpha & \text{if } \lambda=\alpha^* \end{cases}$

$\lambda,\mu,=1,\ldots,p,1^*,\ldots,p^*$

We also use the following nomenclature:

$\nabla=$ the Levi-Civita connection of M

$\sigma=$ the second fundamental form of M in CP^{n+p}

$\nabla^{\perp}=$ the normal connection over $T^{\perp}M$ (the normal bundle of M in CP^{n+p}

A= the Weingarten endomorphism of M in CP^{n+p}.

The normal bundàe $T^{\perp}M$ of M is holomorphic and we will denote by J the complex structures over $T^{\perp}M$ and also over TM. Since CP^{n+p} is a Kaehlerian manifold, the Gauss equation implies

(6.4) $\qquad \sigma(JX,Y) = \sigma(X,JY) = J\sigma(X,Y) \quad \text{for all } X,Y\in\chi(M)$

therefore one has

(6.5) $\qquad A_{J\xi} = JA_{\xi} \quad \text{and} \quad JA_{\xi} = -A_{\xi}J \quad \text{for all } \xi\in\chi(M)^{\perp}$

as consequence M is minimal (Every complex submanifold of a Kaehlerian manifold is minimal).

If R denotes the curvature tensor of M, the Gauss equation is:

(6.6) $$R(X,Y,Z,W) = g(\sigma(X,W),\sigma(Y,Z))-g(\sigma(X,Z),\sigma(Y,W))+(1/4)\{g(X,W) g(Y,Z)-g(X,Z)g(Y,W)+g(JX,W)g(JY,Z)-g(JX,Z)g(JY,W)+2g(X,JY)g(JZ,W)\}$$

Now the Ricci curvature S of M is:

(6.7) $$S(X,Y) = \frac{n+1}{2}\, g(X,Y) - \sum_\lambda g(A_\lambda^2 X,Y),$$

and the scalar curvature ρ of M is:

(6.8) $$\rho = n(n+1) - |\sigma|^2,$$

where $|\sigma|$ denotes the length of the second fundamental form σ,

(6.9) $$|\sigma|^2 = \operatorname{tr} \sum_\lambda A_\lambda^2 = \sum_{i,j,\lambda} h_{ij}^\lambda h_{ij}^\lambda,$$

$\sigma(E_i,E_j) = \sum_\lambda h_{ij}^\lambda \xi_\lambda$. Finally, the holomorphic sectional curvature determined by a unit vector x of M is given by:

(6.10) $$H(x) = 1 - 2|\sigma(x,x)|^2 .$$

The following two lemmas are well-known (any-way their proofs are very direct):

LEMMA 6.1.- Let M^n be an n-dimensional complex submanifold of CP^{n+p}, then:

(6.11) $$|S|^2 = \tfrac{1}{2}n(n+1)^2 - (n+1)|\sigma|^2 + \operatorname{tr}(\sum_\lambda A_\lambda^2)^2,$$

(6.12) $$|R|^2 = 2n(n+1) - 4|\sigma|^2 + 2\sum_{\lambda\mu} (\operatorname{tr} A_\lambda A_\mu)^2,$$

(6.13) $$-\tfrac{1}{2}\Delta|\sigma|^2 = |\nabla\sigma|^2 + \frac{n+2}{2}|\sigma|^2 - 2\operatorname{tr}(\sum_\lambda A_\lambda^2)^2 - \sum_{\lambda\mu} (\operatorname{tr} A_\lambda A_\mu)^2.$$

LEMMA 6.2.- Let N^m be a Kaehler manifold of complex dimension m, then:

(6.14) $$\tfrac{1}{2}(m+1)m|R|^2 \geqslant 2m|S|^2 \geqslant \rho^2.$$

Moreover:

(i) the first equality holds if and only if N has constant holomorphic sectional curvature.

(ii) the second equality holds if and only if N is Einstein.

From (6.8), (6.11) and the second inequality of (6.13) one get

(6.15) $$\operatorname{tr}(\sum_\lambda A_\lambda^2)^2 \geqslant \frac{1}{2n}|\sigma|^4,$$

the equality holds if and only if M is Einstein.

Also we will consider the following result due to B. Y. Chen and K. Ogiue [19] :

THEOREM 6.3.- Let M be an algebraic hypersurface of degree d in CP^{n+1}. Then we have

$$\int_M \{3|R|^2-4|S|^2+{}^2\}\, dv = 2n(n-1)\{\binom{n+2}{2}-(n+2)d+d^2\}\ vol(M) \tag{6.16}$$

7. EXTRINSIC CHARACTERIZATIONS OF SOME COMPLEX SUBMANIFOLDS IN THE COMPLEX PROJECTIVE SPACE. In section 3 we gave two interesting results about the so called "inverse problem" for the complex quadric of small dimension. These results was intrinsic.

In order to study the inverse problem for any complex quadric one firstly must solve the following special problem: Let M^n be an n-dimensional Kaehler submanifold of CP^{n+q} such that $Spec^p(M,g) = Spec^p(Q_n,g_o)$ for some p, is it true that M is congruent to Qn ?. Of course, here the Kaehler structures over M and Q_n are the induced structures like complex submanifolds of CP^{n+q}. In this sense, we will call extrinsic characterization of Q_n by its spectrum to any answer of the above problem.

Some partial answer to this problem have been obtained, [2], [3], [4], [5], we will give here some of them.

THEOREM 7.1, [5].- Let (M,J,g) be an n-dimensional Kaehler manifold with $Spec^p(M,g)=Spec^p(Q_n,g_o)$ for some p (p=0,1,...,2n). If (M,J,g) can be holomorphically isometrically imbedded in CP^m for some m, then (M,J,g) is holomorphically isometric to (Q_n,J_o,g_o).

Proof.- The main idea of the proof is to use some arguments of Algebraic Geometry in order to reduce our problem to consider complex hypersurfaces. In fact, because (M,J,g) is holomorphically isometrically imbedded in some CP^m, from a well known result due to Chow,

M is a nonsingular algebraic variety. As usual, we shall identify M with its image in CP^m.

Let L be an $(m-n)$-dimensional linear subspace of CP^m and let λ be the number of points in $M \cap L$. From the dimensiona1 theory for algebraic manifolds, one obtains that λ does not depend on the choice of L, if L is in general position; λ is called the degree of M. Moreover from a classica1 theorem due to W.Wirtinger,[59], one has,

$$\mathrm{vol}(M) = \lambda(4\pi)^n/n! \tag{7.1}$$

On the other hand, since Q_n is an algebraic hypersurface of degree 2 in CP^{n+1}, we have

$$\mathrm{vol}(Q_n) = 2(4\pi)^n/n! \tag{7.2}$$

and so $\lambda=2$ because the volume is a spectral invariant. That is, M is an algebraic submanifold of degree 2 in CP^m. Now we use another well known result due to W.Barth,[7], to obtain M is contained in an $(n+1)$ dimensiona1 linear subspace of CP^m. Therefore, from theorem 6.3, one has,

$$\int_M \{|R|^2-4|S|^2+\rho^2\}dv = \int_{Q_n} \{|R_o|^2-4|S_o|^2+\rho_o^2\}dv_o \tag{7.3}$$

Now we use the coefficients of Minakshisundaram-Pleijel-Gaffney formula joint the Schwartz inequality and the Einsteinian character of Q_n to obtain

$$\int_M \{2n|S|^2-\rho^2\}dv \leqslant 0 \tag{7.4}$$

but this formula joint lemma 6.2 imply that M is Einstein. Finally one uses a well-known result due to B.Smyth,[52], to conclude (M,J,g) is holomorphically isometric to (Q_n,J_o,g_o).

When one only assumes that M is immersed, the extrinsic characterizations are weaker that the above case. In a direct attack, it seems natura1 to try find spectral invariants for Kaehler submanifolds of CP^m in terms of elements close to the immersion (for instance, in terms of the second fundamental form). In this line one has the following:

LEMMA 7.2. .- Let M_1 and M_2 be two Kaehler submanifolds in CP^m. If $\mathrm{Spec}^p(M_1,g_1)=\mathrm{Spec}^p(M_2,g_2)$ for some p, then one has

$$(7.5)\qquad \int_{M_1}\{2|\nabla\sigma_1|^2-4|S_1|^2-|R_1|^2\}dv_1 = \int_{M_2}\{2|\nabla\sigma_2|^2-4|S_2|^2- -|R_2|^2\}dv_2$$

This result allows us to give some extrinsic characterizations of complex quadrics by its spectrum, for instance:

THEOREM 7.3.[5].- Let (M,J,g) be an n-dimensional Kaehler manifold ($n\geq 5$) with $\mathrm{Spec}^2(M,g)=\mathrm{Spec}^2(Q_n,g_o)$. If (M,J,g) can be holomorphically isometrically immersed in CP^m, for some m, then (M,J,g) is holomorphically isometric to (Q_n,J_o,g_o).

Remark.- Since the spectral invariant obtained in lemma 7.2, already only depends on the volume and the tota1 scalar curvature, one can characterize the Veronese imbedding (see 12.2) by its volume and its tota1 scalar curvature (which are two spectral invariants) among all parallel Kaehler submanifolds of the complex projective space,[6]. In particular this imbedding is characterized by its spectrum among all the above submanifolds.

8. THE STANDARD IMBEDDING OF THE COMPLEX PROJECTIVE SPACE IN EUCLIDEAN SPACE. In this section we will describe with some details a very nice isometric immersion of the complex projective space in a Euclidean space which will play an important role in this paper. We will call it the standard imbedding. This imbedding has been used for some authors to study different problems,[17],[34],[46],[50], [53] etc.

Let C^{m+1} be the usual $(m+1)$-dimensional complex Euclidean space endowed with the usual inner product $<z,w> = \mathrm{Real}\, z.\bar{w}^t$ for $z = (z_0,..,z_m)$, $w = (w_0,..,w_m)$ in C^{m+1} ($\bar{w}$ means conjugate and w^t transpose). C^{m+1} can be considered like vector space and so CP^m is defined as the set of all complex lines of C^{m+1}. Let $\mathrm{End}(C^{m+1})$ be the space of all complex endomorphisms of C^{m+1}.

One defines a mapping $\phi: CP^m \longrightarrow \mathrm{End}(C^{m+1})$ as follows: If Π is a complex line of C^{m+1}, then $\phi(\Pi): C^{m+1} \longrightarrow C^{m+1}$ is simply the orthogonal projection over Π. One notices the following facts on ϕ :

(1) ϕ is one-to-one

(2) $(\phi(\Pi))^2 = \phi(\Pi)$ for all $\Pi \in CP^m$

(3) rank $(\phi(\Pi)) = 1$ for all $\Pi \in CP^m$

(4) $\phi(\Pi)$ is self-adjoint with respect to the inner product on C^{m+1}, for all $\Pi \in CP^m$.

Conversely, any element in $\mathrm{End}(C^{m+1})$ satisfying conditions (2),(3) and (4) is the orthogonal projection over some complex line of C^{m+1}. Moreover, in this context, condition (3) is equivalent to

(3)' $\mathrm{Trace}(\phi(\Pi)) = 1$ for all $\Pi \in CP^m$.

Remark.- If the homogeneous coordinates of $\Pi \in CP^m$ are $(z_0,..,z_m)$ for a chosen basis on C^{m+1}, then $\phi(\Pi)$ with respect to such basis is given by

$$(8.1) \qquad \frac{1}{z.\bar{z}^t}\, \bar{z}^t.z$$

Consider the space of all Hermitian matrices of order $m+1$

$$(8.2) \qquad HM(m+1,C) = \{A \in gl(m+1,C) \quad \bar{A}^t = A\}$$

which is a linear subspace of $gl(m+1,C)$ with dimension $(m+1)^2$.

On $HM(m+1,C)$ one defines the metric

(8.3) $g(A,B) = 2\mathrm{Trace}AB$ for all $A, B \in HM(m+1,C)$.

We define the following subset of $HM(m+1,C)$

(8.4) $CP^m = \{ A \in HM(m+1,C) \quad AA = A$ and $\mathrm{Trace}(A) = 1 \}$

LEMMA 8.1.- CP^m is a submanifold of $HM(m+1,C)$ which is diffeomorphic to the homogeneous space $U(m+1)/U(1)xU(m)$, $U(m)$ being the unitary group of order m.

Proof.- If $A \in CP^m$, it is a Hermitian matrix and so there exists P in $U(m+1)$ with

$$PAP^{-1} = \begin{pmatrix} h_0 & & \\ & \ddots & \\ & & h_m \end{pmatrix}$$

From (8.4), one obtains that $h_i = 1$ for a fixed index and $h_j = 0$, $i \neq j$. Therefore, we can assume

$$PAP^{-1} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = B.$$

We will say that B is the origin of CP^m and so we can look CP^m like the orbit of B under the $U(m+1)$-action over $HM(m+1,C)$ given by $(P,A) \quad PAP^{-1}$ for all (P,A) in $U(m+1)xHM(m+1,C)$. Moreover it is easy to see that the isotropy subgroup of B is $U(1)xU(m)$ which proves the lemma.

For all $A \in CP^m$, we denote by $T_A(CP^m)$ the tangent space of CP^m at A identified by means of ϕ with a subspace of $HM(m+1,C)$. Similarly we have the normal space at A, $T_A^{\perp}(CP^m)$. Then one has

LEMMA 8.2.- For each point $A \in CP^m$, one has

$$T_A(CP^m) = \{ X \in HM(m+1,C) \ / \ XA+AX=X \},$$

$$T_A^{\perp}(CP^m) = \{ Z \in HM(m+1,C) \ / \ AZ = ZA \}.$$

Proof.- Let $\alpha : I \longrightarrow CP^m$ be a curve with $\alpha(0) = A$ and $\alpha'(0) = X$, then $\alpha(t)\alpha(t) = \alpha(t)$ and so, $\alpha'(t)\alpha(t) + \alpha(t)\alpha'(t) = \alpha'(t)$, in particular when $t=0$, one get $XA+AX=X$ which proves the first inclusion.
Because for each P in $U(m+1)$ the mapping $HM(m+1,C) \longrightarrow HM(m+1,C)$ given by $A \text{-} \ PAP^{-1}$ is an isometry, it is enough to establish the equality

at the origin $B \in CP^m$. Moreover, it is easy to see that $\dim\{X \in HM(m+1,C) / XB + BX = X\} = 2m$, which proves the first equality. the second equality can be proved using a direct argument.

Remark.- The vector fields along CP^m given by $A \mapsto A$ and $A \longmapsto I$ (where 1 denotes the identity matrix of order $m+1$) are obviously normal to CP^m. Also the vector fields given by $A - AQ + QA - 2AQA$ are tangent to CP^m for all Q in $HM(m+1,C)$.
The following relations admit a direct proof

$$(8.5)\begin{cases} AXY = XYA \\ AXA = 0 \\ X(I-2A) = -(I-2A)X \\ (I-2A)^2 = 1 \\ (I-2A)XY = XY(I-2A) \end{cases}$$

for all $A \in CP^m$ and $X, Y \in T_A(CP^m)$, and they will be very useful along this paper.

LEMMA 8.3.- Let D <u>be the Levi-Civita connection of</u> $HM(m+1,C)$, $\tilde\nabla$ <u>the induced connection over</u> CP^m, $\tilde\sigma$ <u>the second fundamental form of</u> $\phi : CP^m \longmapsto HM(m+1,C)$, $\tilde\nabla^\perp$ <u>the normal connection,</u> $\tilde A$ <u>the Weingarten endomorphism and</u> $\tilde H$ <u>the mean curvature vector, then</u>

$$(8.6) \qquad \tilde\nabla_X Y = A(D_X Y) + (D_X Y)A - 2A(D_X Y)A$$

$$(8.7) \qquad \tilde\sigma(X,Y) = (XY + YX)(I-2A)$$

$$(8.8) \qquad \tilde\nabla^\perp_X Z = D_X Z + 2A(D_X Z)A - (D_X Z)A - A(D_X Z)$$

$$(8.9) \qquad \tilde A_Z X = (XZ - ZX)(I-2A)$$

$$(8.10) \qquad \tilde H = \frac{1}{2m}(I - (m+1)A)$$

<u>where</u> X, Y <u>are tangent vector fields of</u> CP^m, Z <u>is a normal vector field of</u> CP^m, A <u>any point of</u> CP^m <u>and the above formulas are computed at</u> A.

Proof.- Assume $\tilde\nabla$ and $\tilde\sigma$ defined respectively by (8.6) and (8.7). Take $A \in CP^m$, $X \in T_A(CP^m)$ and $Y \in \mathfrak{X}(CP^m)$. Let $\alpha : I \longrightarrow CP^m$ be a curve with $\alpha(0) = A$ and $\alpha'(0) = X$. Since $Y(t) \in T_{\alpha(t)}(CP^m)$ one has $\alpha(t)Y(t) + Y(t)\alpha(t) = Y(t)$, therefore

$$D_X Y = \frac{dY(t)}{dt}(0) = XY + YX + A(D_X Y) + (D_X Y)A$$

where we put Y by $Y(0)$. Hence

$$\tilde{\sigma}(X,Y) = (XY+YX)(I-2A) = (D_X Y - A(D_X Y) - (D_X Y)A)(I-2A) =$$

$$= -(A(D_X Y) + (D_X Y)A - 2A(D_X Y)A) + D_X Y = D_X Y - \tilde{\nabla}_X Y.$$

So it is enough to prove that $\tilde{\nabla}_X Y$ is tangent to CP^m and $\tilde{\sigma}(X,Y)$ is normal to CP^m which is trivial.

In order to prove (8.8) and (8.9) we take a normal vector field Z over CP^m, then $\alpha(t)Z(t) = Z(t)\alpha(t)$ and so

$$XZ + A(D_X Z) = (D_X Z)A + ZX, \text{ and}$$

$$\tilde{A}_Z X = (XZ - ZX)(I-2A) = \tilde{\nabla}^{\perp}_X Z - D_X Z.$$

Now from Weingarten formula it is enough to prove that $\tilde{A}_Z X$ is tangent to CP^m and $\tilde{\nabla}^{\perp}_X Z$ is normal to CP^m, but it is also trivial.

Finally, to prove (8.10), it is enough to prove that at the origin B of CP^m and use the argument that the mappings $A \longmapsto PAP^{-1}$ are isometries of $HM(m+1,C)$ for all P in $UM(m+1)$. Then one takes the orthonormal basis $\{E_1,\dots,E_m,E_{1*},\dots,E_{m*}\}$ of $T_B(CP^m)$ given by

$$E_k = \frac{1}{2}\ \begin{array}{c} \\ k \\ \\ \end{array}\begin{pmatrix} 0 & 0\,.\,.\,0 \overset{k}{1} 0\,.\,.\,0 \\ 0 & \\ \vdots & \\ 1 & 0 \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix} \qquad E_{k*} = \frac{\sqrt{-1}}{2}\begin{pmatrix} 0 & 0\,.\,.\,0 1 0\,.\,.\,0 \\ 0 & \\ \vdots & \\ 1 & 0 \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}$$

then, a direct computation proves that

$$\tilde{H}_B = \frac{1}{2m}\begin{pmatrix} -m & 0 \dots 0 \\ 0 & 1 \\ \vdots & & \ddots \\ 0 & & & 1 \end{pmatrix} = \frac{1}{2m}(I - (m+1)B).$$

LEMMA 8.5.- Let f <u>be the diffeomorphism obtained in lemma</u> 8.1. Then f <u>is already an isometry when one considers</u> on $U(m+1)/U(1)xU(m)$ the <u>metric of Fubini-Study with holomorphic sectional curvature</u> $c = 1$ <u>and on</u> CP^m <u>the metric indueed via</u> ϕ <u>from the Euclidean metric on</u> $HM(m+1,C)$.

Proof.- Since both metrics are $U(m+1)$-invariants, it is enough to prove that the diferential of f at the origin is a linear isometry between the corresponding tangent spaces

$$f:\ U(m+1)/U(1)\times U(m) \longrightarrow CP^m, \qquad f([P]) = PBP^{-1}.$$

If $o = [I]$, one knows that

$$T_o(\ U(m+1)/U(1)\times U(m)\ ) = \left\{ \begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix} / \ a \in C^m \right\}$$

and the Fubini-Study metric of holomorphic sectional curvature $c=1$ is given by

$$g_o\left(\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ -\bar{b}^t & 0 \end{pmatrix} \right) = 2\,\mathrm{Trace} \begin{pmatrix} a\bar{b}^t & 0 \\ 0 & \bar{a}^t b \end{pmatrix}$$

We take a curve $\alpha: I \longrightarrow U(m+1)$ with $\alpha(0) = 1$ and

$$\alpha'(0) = \begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix},$$

and consider $\beta: I \longrightarrow U(m+1)/U(1)\times U(m)$ given by $\beta(t) = [\alpha(t)]$. Then

$$df_o \begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix} = (f\circ\beta)'(0) = (\alpha(t)B\alpha(t)^{-1})'(0) =$$

$$= \alpha'(0)B\bar{\alpha}^t(0) + \alpha(0)B\,\overline{\alpha'(0)}^t = \begin{pmatrix} 0 & -a \\ -\bar{a}^t & 0 \end{pmatrix} \quad \text{and so}$$

$$g\left(df_o \begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix},\ df_o \begin{pmatrix} 0 & b \\ -\bar{b}^t & 0 \end{pmatrix} \right) = 2\,\mathrm{Trace} \begin{pmatrix} a\bar{b}^t & 0 \\ 0 & \bar{a}^t b \end{pmatrix}$$

which proves the lemma.

LEMMA 8.6.- Let J <u>be the natura1 complex structure over</u> $U(m+1)/U(1)\times U(m)$. <u>The induced complex structure over</u> CP^m <u>via the diffeomorphism</u> f <u>is given by</u>

$$J(X) = \sqrt{-1}\,(I-2A)X \qquad \underline{\text{for all}}\ X\ \underline{\text{in}}\ T_A(CP^m).$$

Proof.- We will prove the formula at the origin. As it is well-known the complex structure $\tilde{J}$ at the origin of $U(m+1)/U(1)\times U(m)$ is given by

$$\tilde{J}\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix} = -\sqrt{-1}\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix}.$$

if $\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix} \in T_B(CP^m)$, the induced complex structure is

$$J\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix} = df_o.\tilde{J}.(df_o)^{-1}\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix} = \sqrt{-1}\begin{pmatrix} 0 & -a \\ \bar{a}^t & 0 \end{pmatrix} =$$

$$= \sqrt{-1}\,(I-2A)\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix}.$$

The following proposition gives some geometrical properties of the imbedding $\phi : CP^m \longrightarrow HM(m+1,C)$ which will be very useful.

PROPOSITION 8.7 <u>The isometric immersion</u> $\phi : CP^m \longrightarrow HM(m+1,C)$ <u>satisfies</u>
(a) <u>It is an</u> $U(m+1)$<u>-equivariant isometric imbedding.</u>
(b) $\tilde{\sigma}(JX,JY) = \tilde{\sigma}(X,Y)$ <u>for all</u> X, Y <u>in</u> $T_A(CP^m)$.
(c) $\tilde{\nabla}\tilde{\sigma} = 0$, <u>that is the imbedding is</u> parallel.
(d) <u>It is minimal in a hypersphere of center</u> $(1/m+1)I$ <u>and radius</u> $\sqrt{2m/(m+1)}$. <u>As consequence it is mono-order and the order is one</u>.

Proof.- (a) It is an immediate consequence from lemma 8.1 and 8.5 if we recall that $U(m+1)$-equivariant means that every isometry of CP^m determined by the elements of $U(m+1)$ can be obtained as the restriction to CP^m of some isometry of $HM(m+1,C)$.
(b) It follows from (8.5), (8.7) and lemma 8.6.
(c) Let X, Y_1, Y_2 be three vector fields tangent to CP^m, then by using (b) and the fact that CP^m is Kaehlerian, one get

$$(\tilde{\nabla}\tilde{\sigma})_X(JY_1,JY_2) = (\tilde{\nabla}\tilde{\sigma})_X(Y_1,Y_2).$$

As consequence $(\tilde{\nabla}\tilde{\sigma})_X(Y,JY) = 0$ for all X,Y tangent to CP^m. Now from Codazzi's equation one has

$$(\tilde{\nabla}\tilde{\sigma})_X(X,X) = 0 \text{ for all } X \text{ tangent to } CP^m,$$ which proves (c).

(d) Let A be any point of CP^m, then

$$g\left(A - \frac{1}{m+1}I,\ A - \frac{1}{m+1}I\right) = 2\,\mathrm{Trace}\left(A - \frac{1}{m+1}I + \frac{1}{(m+1)^2}I\right) = \frac{2m}{m+1}.$$

This proves that CP^m is contained in a sphere S of center $\frac{1}{m+1}I$ and radius $\sqrt{2m/(m+1)}$.
Moreover the mean curvature vector $\tilde{H}$ of $CP^m \longrightarrow HM(m+1,C)$ in A is given by

$$H_A = \frac{1}{2m}(I-(m+1)A) = -\frac{m+1}{2m}\left(A - \frac{1}{m+1}I\right)$$

which proves that $\tilde{H}_A$ is normal to S at A, therefore the mean curvature vector of CP^m in S is trivial.
From the Takahashi's theorem, ϕ is mono-order, of order $u \geqq 1$, with $\lambda_u = \dim CP^m/(\text{radius } S)^2 = m+1$. But the first eigenvalue of CP^m is $m+1$, then $u = 1$.

We will finish this section by given two technical lemmas which admit a direct proof

LEMMA 8.8.- <u>For the imbedding</u> $\phi : CP^m \longrightarrow HM(m+1,C)$ <u>one has:</u>

(8.11) $g(\tilde\sigma(X,Y),\tilde\sigma(V,W)) = \frac{1}{2}g(X,Y)g(V,W) + \frac{1}{4}\{g(X,W)g(Y,V) + g(X,V)g(Y,W) + g(X,JW)g(Y,JV) + g(X,JV)g(Y,JW)\}$

(8.12) $\tilde A_{\tilde\sigma(X,Y)}V = \frac{1}{2}g(X,Y)V + \frac{1}{4}\{g(Y,V)X + g(X,V)Y + g(JY,V)JX + g(JX,V)JY\}$.

(8.13) $g(\tilde\sigma(X,Y),I) = 0$.

(8.14) $g(\tilde\sigma(X,Y),A) = -g(X,Y)$.

LEMMA 8.9.- <u>Let</u> X, Y <u>be two orthonormal vectors in</u> $T_A(CP^m)$. Then

(8.15) $g(\tilde\sigma(X,X),\tilde\sigma(X,X)) = 1$,

(8.16') $\frac{1}{2} \leq g(\tilde\sigma(X,X),\tilde\sigma(Y,Y)) \leq 1$.

<u>If one has</u> $g(X,JY) = 0$, then

(8.17) $g(\tilde\sigma(X,X),\tilde\sigma(Y,Y)) = \frac{1}{2}$,

(8.17') $g(\tilde\sigma(X,Y),\tilde\sigma(X,Y)) = \frac{1}{4}$.

9. A NEW WAY TO STUDY SPECTRAL GEOMETRY OF SUBMANIFOLDS IN THE COMPLEX PROJECTIVE SPACE. The theorem of Takahashi (theorem 5.2) gives a very nice spectral characterization for minimal submanifolds of a sphere. But it implicitly depends on the standard immersion of the sphere in the Euclidean space (as the geometrical locus of points having constant distance to a fixed point). Let $S^{m-1}(r)$ be an $(m-1)$-dimensional sphere of radius r in E^m, if $\langle , \rangle$ denotes the Euclidean metric on E^m, then

$$S^{m-1}(r) = \{a \in E^m / \langle a,a\rangle = r^2\}.$$

Consider the inclusion $i : S^{m-1}(r) \longrightarrow E^m$ or standard imbedding. If M^n denotes an n-dimensional (compact) Riemannian manifold, then M admits a minimal isometric immersion τ in $S^{m-1}(r)$ if and only if the isometric immersion $\psi = i.\tau : M^n \longrightarrow E^m$ has the simplest spectral behaviour, that is, it is mono-order of order u

$$\psi = \psi_0 + \psi_u ,$$

with

$$\psi_0 = \frac{1}{vol(M)} \int_M \psi(M).dv,$$

and $\Delta\psi_u = \lambda_u \psi_u$. Moreover $\lambda_u = n/r^2$.

This results sugests us [46] to consider the following way to study spectral geometry of submanifolds in the complex projective space. First, it seems natural to try to find a good isometric immersion of CP^m in a Euclidean space, say for instance

$$\phi : CP^m \longrightarrow E^N$$

and so study submanifolds M in CP^m ($\tau : M^n \longrightarrow CP^m$) by looking the spectral behaviour of the isometric immersion $\psi = \phi . \tau : M^n \longrightarrow E^N$. Of course this idea can be extended to other symmetric spaces (see for instance [17]).

In this sense, we will use the isometric imbedding $\phi : CP^m \longrightarrow HM(m+1,C)$ defined in the last section.

For example, one can put the following problem, which is formally like in the theorem of Takahashi: Assume that $\tau : M^n \longrightarrow CP^m$ is an isometric immersion of a Riemannian manifold M^n in CP^m. Using $\phi : CP^m \longrightarrow HM(m+1,C)$ one can see M^n like a submanifold in the Euclidean space HM(m+1,C)

$$\psi : M^n \longrightarrow HM(m+1,C).$$

The spectral behaviour of ψ must be closely related to the geometry of the immersion $\tau : M^n \longrightarrow CP^m$. Is the minimality of τ equivaient to the simplest spectral behaviour of ψ (to be mono-order)?

We will see in section 11 that this problem is not true because if M is a Kaehler submanifold of CP^m, then it is minimal. On the other hand , we will prove that the only complex submanifolds for which ψ is mono-order are totally geodesic and so linear varieties.

Therefore a more solid first problem is: To try to study the submanifolds $\tau : M^n \longrightarrow CP^m$ such that the corresponding isometric immersion $\psi : M^n \longrightarrow HM(m+1,C)$ is mono-order (of order $u \geqq 1$). In section 11, we will classify all CR-minimal submanifolds in CP^m satisfying last condition. As we will see the solution obtained indicates that complex submanifolds admit a more involved behaviour with respert to the spectral character of ψ and so we will study in a second problem complex submanifolds in CP^m whose isometric immersion ψ is bi-order.

10. THE VOLUME OF A MINIMAL SUBMANIFOLD IN THE COMPLEX PROJECTIVE SPACE. We recall that a submanifold $\tau: M \longrightarrow CP^m$ is called a CR-submanifold if it admits a pair of complementary distribution $(D, D^{\perp})$ such that D is holomorphic and $D^{\perp}$ is totally real. In general we will denote M by M^{2n+q} where 2n indicates the real dimension of D and q the dimension of $D^{\perp}$.
Given a submanifold $\tau: M^n \longrightarrow CP^m$, one consider $\psi: M^n \longrightarrow HM(m+1, C)$, where $\psi = \phi . \tau$. Let $H^{\perp}$ be the normal component to CP^m of the mean curvature vector of ψ. We will start this section by giving the best posible estimation for the length of $H^{\perp}$, [17], [46].

LEMMA 10.1.- (a) Let M^n be an n-dimensional submanifold of CP^m, then

$$\frac{n+1}{2n} \leqq g(H^{\perp}, H^{\perp}) \leqq \frac{n+2}{2n} . \tag{10.1}$$

Moreover, the first equality (respectively the second equality) holds if and only if M^n is totally real (respectevily n is even and M^n is Kaehlerian).

(b) In particular if M^{2n+q} is a CR-submanifold of CP^m, then

$$g(H^{\perp}, H^{\perp}) = \frac{(2n+q)^2 + 4n + q}{2(2n+q)^2} . \tag{10.2}$$

Proof.- Take any point A in M and an orthonormal basis $\{E_1, \ldots, E_n\}$ of $T_A M$, then trivially one has

$$H^{\perp} = \frac{1}{n} \sum_{i=1}^{n} \tilde{\sigma}(E_i, E_i).$$

Now we use lemma 8.9 to obtain

$$g(H^{\perp}, H^{\perp}) = \frac{1}{n^2} \sum_{ij} g(\tilde{\sigma}(E_i, E_i), \tilde{\sigma}(E_j, E_j)) = \frac{n+1}{2n} + \frac{1}{2n^2} \sum_{ij} g(E_i, JE_j)^2 .$$

Define the endomorphism P on $T_A M$ by $\langle PX, Y \rangle = \langle JX, Y \rangle$ for all X, Y in $T_A M$, then

$$g(H^{\perp}, H^{\perp}) = \frac{n+1}{2n} + \frac{1}{2n^2} g(P, P). \tag{10.3}$$

Because P is nothing but the tangential component of J, one has

$$0 \leqq g(P, P) \leqq n, \tag{10.4}$$

where the first equality (respectively, the second equality) holds if and only if M is totally real at A (respectively, n is even and M^n is Kaehlerian at A). Now (10.1) follows from (10.3) and (10.4) and the equalities (10.1) are also characterized.

(b) If M^{2n+q} is a CR-submanifold, at any point A in M, one can choose and orthonormal basis of T_AM, say for instance $\{E_1,\ldots,E_n,JE_1,\ldots,JE_n,F_1,\ldots,F_q\}$ where $\{E_1,\ldots,E_n,JE_1,\ldots,JE_n\}$ span D_A and $\{F_1,\ldots,F_p\}$ span $D_A^{\perp}$. Now we use lemma 8.9 and a direct computation like in the proof of (a) to obtain (10.2).

LEMMA 10.2.- Let M^{2n+q} be a CR-submanifold of CP^m and $\tilde{\sigma}_M$ the restriction of the second fundamental form of $\phi:CP^m \longrightarrow HM(m+1,C)$ to the tangent bundle of M. Then

$$g(\tilde{\sigma}_M,\tilde{\sigma}_M) = \frac{1}{4}\{(2n+q)^2 + 4n + 3q\}. \tag{10.5}$$

The proof is again an immediate consequence of lemma 8.9.

COROLLARY 10.3.- Let M^{2n+q} be a CR-submanifold of CP^m. Denote by ρ the scalar curvature of M and by H (respectively by σ) the mean curvature vector (respectively the second fundamental form) of M in CP^m. Then

$$\rho = \frac{(2n+q)^2 + 4n - q}{4} + (2n+q)^2 g(H,H) - |\sigma|^2. \tag{10.6}$$

Proof.- It is easy to see that

$$\rho = (2n+q)^2 g(H,H) + (2n+q)^2 g(H^{\perp},H^{\perp}) - |\sigma|^2 - |\tilde{\sigma}_M|^2.$$

Now (10.6) follows from last formula joint formulas (10.2) and (10.5).

In order to obtain an estimate for the volume of any minima1 submanifold of CP^m, we start by recalfing the following proposition [14]

PROPOSITION 10.4.- Let M^n be an n-dimensiona1 submanifold in the Euclidean space E^m, then

$$\int_M |H'|^n dv \geqq c_n , \tag{10.7}$$

where H' denotes the mean curvature vector of M and c_n the volume of the n-dimensiona1 unit sphere. Moreover the equality in (10.7) holds if and only if M is imbedded as the standard sphere in a (n+1)-dimensional afin subspace of E^m.

Now we obtain the following theorem,[46] THEOREM 10.5.- Let M^n be an n-dimensional minimal submanifold of CP^m, then

(10.8) $$\mathrm{vol}(M) \geqq c_n\left(\frac{2n}{n+2}\right)^{\frac{n}{2}}$$

Proof.- Because M is minimal in CP^m, $H^\perp$ is essentially the mean curvature vector of M in $HM(m+1,C)$. Therefore from (10.1) and (10.7) one get (10.8).
If $M = S^2 = CP^1$ it is obvious that the equality in (10.8) follows. Conversely, assume that equality in (10.8) holds, then one obtains:
(i) from the equality in (10.7), M^n is imbedded as a standard sphere of certain radius R in an (n+1)-dimensional Euclidean space, and
(ii) from equality in the second part of (10.11, n is even, say n=2k and M^{2k} is a complex submanifold of CP^m.
In particular $M^n = S^{2k}(R)$ must be a Kaehler manifold and so k=l.
But $\mathrm{vol}(M^2) = R^2c_2$ and so from the equality in (10.8) one get R=l.
Furthemore M^2 is already a CR-minimal submanifold of CP^m and so from (10.6) one get $\rho = 2-|\sigma|^2$, which implies that $\sigma = 0$ and the proof is finish.

Remark.- The stimate obtained in the last theorem has been improved by B. Y. Chen (see [17]).

11. THE FIRST EIGENVALUE OF A MINIMAL SUBMANIFOLD IN THE COMPLEX PROJECTIVE SPACE.

The main purpose of this section is to obtain an upper bound for the first eigenvalue of a minimal submanifold of CP^m. The estimate obtained will be the best posible.
We start be solving the following problem : Let M^{2n+q} be a CR-minimal submanifold of CP^m ($\tau: M \longrightarrow CP^m$) then M can be looked as a submanifold of a certain Euclidean space ($\psi: M \longrightarrow HM(m+1,C)$, $\psi = \phi.\tau$). From Takahashi's Theorem it seems natural to ask: When M is minimal in some hypershere of $HM(m+1,C)$?. The complete solution to this problem is given in the following theorem, [46]

THEOREM 11.1.- Let M^{2n+q} be a CR-minimal submanifold of CP^m. Then M is minimal in some sphere of $HM(m+1,C)$ if and only if either:
(a) q=0 and M^{2n} is a totally geodesic complex submanifold of CP^m, or
(b) n=0 and M^q is a totally totally real submanifold of CP^m for which there exists a totally geodesic complex submanifold CP^q of CP^m

<u>such that</u> M^q <u>is a totally real submanifold of</u> CP^q.

Proof.- First we prove the sufficient condition. Let M^{2n} be like in (a), then M^{2n} must be some linear variety CP^n and so it admits an isometric imbedding $\phi : CP^n \longrightarrow HM(n+1,C)$ which is certainly minimal in a sphere (see Proposition 8.7). Suppose now that M^q is in situation (b). For each point A in M^q, one takes an orthonormal basis $\{E_1,\dots,E_q\}$ in T_AM, then $\{E_1,\dots,E_q,JE_1,\dots,JE_q\}$ is an orthonormal basis of T_ACP^q. If H' (respectively σ') denotes the mean curvature vector (respectively the second fundamental form) of $\psi : M^q \longrightarrow HM(q+1,C)$ and because $\tau : M^q \longrightarrow CP^q$ is minimal, then

$$H' = \frac{1}{q}\sum_{i=1}^{q}\sigma'(E_i,E_i) = \frac{1}{q}\sum_{i=1}^{q}\tilde\sigma(E_i,E_i).$$

Now from proposition 8.7 and lemma 8.3, one obtains

$$H'_A = \frac{1}{2q}\Big(\sum_{i=1}^{q}\tilde\sigma(E_i,E_i) + \sum_{i=1}^{q}\tilde\sigma(JE_i,JE_i)\Big) = \tilde H_A =$$

$$= \frac{1}{2q}(I - (q+1)A) = -\frac{q+1}{2q}\Big(A - \frac{1}{q+1}I\Big) \text{ for all } A \text{ in } M.$$

As M^q is immersed through CP^q in a sphere of $HM(q+1,C)$, whose center is $\frac{1}{q+1}I$, last formula proves us that M^q must be minimal in such a sphere.

Now we will prove the necessary condition. Assume that M^{2n+q} is a CR-minimal submanifold of CP^m which is minimal in some sphere, say S of $HM(m+1,C)$. Let Q be the center of S, it is clear that one can assume that Q is a diagonal matrix, otherwise one uses the repeated argument of the isometry of type $A \longmapsto PAP^{-1}$ for some P in $U(m+1)$. Let H' be the mean curvature vector of $\psi : M \longrightarrow HM(m+1,C)$. Because M is minimal in S, H' must be normal to S at each point of M and so $H' = h(A-Q)$ where h is a non-zero constant and A denotes any point of M in which is computed H'.

On the other hand and because M is minimal in CP^m, H' is also normal to CP^m at each point A in M. Therefore one get $Q \in T_A^{\perp}(CP^m)$ at each point $A \in M$. Consequently M is contained in the linear subspace L of $HM(m+1,C)$ defined by

$$L = \{ A \in HM(m+1,C) \,/\, AQ = QA \}.$$

Suppose

$$Q = \begin{pmatrix} a_1 & & & & \\ & \ddots a_1 & & & \\ & & \ddots a_r & & \\ & & & \ddots & a_r \end{pmatrix}.$$

Then

$$CP^m \cap L = \left\{ \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{pmatrix} \text{ such that } A_i^2 = A_i \text{ and } \sum_{i=1}^{r} \operatorname{Trace} A_i = 1 \right\}.$$

Because each A_i is a Hermitian matrix with $A_i^2 = A_i$, then for all A in M there exists a index $j \in \{1,..r\}$ such that $\operatorname{Trace} A_j = 1$ and $\operatorname{Trace} A_i = 0 \quad i \neq j$. Therefore

$$CP^m \cap L = \bigcup_{j=1}^{r} \left\{ \begin{pmatrix} 0 & & & \\ & \ddots A_j & & \\ & & \ddots & 0 \end{pmatrix} \text{ such that } A_j^2 = A_j \text{ and } \operatorname{Trace} A_j = 1 \right\}$$

But M is connected and so it must be contained in some component of CP^m L, that is $M \subset CP^k$, $k \leq m$. On the other hand M is also minima1 in the great sphere S L. Therefore the problem is reduced to study CR-minima1 submanifolds of CP' which are minima1 in a sphere of $HM(r+1,C)$ whose center is aI, with $a \in R$ and 1 being the identity matrix of order $r+1$.

The mean curvature vector H' of M in $HM(r+1,C)$ is nothing but $H^\perp$ because M is minima1 in CP', then

$$H^\perp = \frac{1}{2n+q} \sum \tilde{\sigma}(E_i, E_i),$$

which joint lemma 8.8 imply

(11.1) $\quad g(H^\perp, A - aI) = -1.$

Now by using (10.2), (11.1) and since $H^\perp = H' = h(A - aI)$, one get

$$h = -g(H^\perp, H^\perp) = -\frac{(2n+q)^2 + 4n + q}{2(2n+q)^2},$$

and so

(11.2) $\quad g(A - aI, A - aI) = -\dfrac{1}{h} = \dfrac{2(2n+q)^2}{(2n+q)^2 + 4n+q}$ for all A in M.

Notice that formula (11.2) gives us the radius of the sphere S..

On the other hand

(11.3) $\quad g(A-aI, A-aI) = 2 - 4a + 2a^2(r+1).$

From (11.2) and (11.3) one get the following equation

(11.4) $(r+1)\{(2n+q)^2+4n+q\}a^2 - 2\{(2n+q)^2+4n+q\}a + 4n+q = 0,$

and so one has

(11.5) $(2n+q)^2 \geqq r(4n+q)$ and $4nq \geqq 5nq,$

which implies $n=0$ and $q=r$ (case (b)) or $q=0$ and $n=r$ (case (a)).

The following lemma can beloolced as a special case of a formula due to R. C. Reilly [45]. Any way we obtain it by using a more simple method, [47].

LEMMA 11.2.- Let $\psi: M^n \longrightarrow E^N$ be an isometric immersion of a Riemannian manifold in the Euclidean space. If H' denotes the mean curvature vector of ψ and λ_1 the first eigenvalue of M, then

(11.6) $$n\int_M g(H',H')\,dv - \lambda_1 \mathrm{vol}(M) \geqq 0.$$

Proof.- First one takes $k=t=1$ in proposition 5.4 to obtain

(11.7) $$\int_M g(\Delta^2\psi,\psi)\,dv - \lambda_1\int_M g(\Delta\psi,\psi)\,dv \geqq 0,$$

the equality holding if and only if ψ is of order 1.
Now it is not difficult to see that

(11.8) $$\Delta g(\psi,\psi) = -2n + 2g(\psi,\Delta\psi) = -2n(1+g(H',\psi)),$$

where we used formula (5.7).
From (5.7) and (11.7) one get

$$n^2\int_M g(H',H')\,dv + \lambda_1 n\int_M g(H',\psi)\,dv \geqq 0,$$

which combined with (11.8) gives us (11.6).

COROLLARY 11.3, [17].- Let M^n be a minima1 submanifold of CP^m. Then

(11.9) $$\lambda_1 \leqq \frac{n+2}{2}.$$

Moreover the equality holds if and only if n is even and $M = CP^{n/2}$ imbedded in CP^m as a totally geodesic complex submanifold.

Proof.- Since M^n is minima1 in CP^m, the mean curvature vector of M^n in HM(m+l,C) is H'. Furthermore, from (10.1),

$$g(H^\perp,H^\perp) \leqq \frac{n+2}{2},$$

now (11.9) follows from (10.1) and (11.6).

Assume now $\lambda_1 = (n+2)/2$, then one has the second equality in (10.1) which implies that n is even, say n= 2k and M^{2k} is a complex submanifold of CP^m. Moreover one has equality in (11.6) gives us that $\psi: M^{2k} \longrightarrow HM(m+1,C)$ is of order one, in particular it must be mono-order and so from Takahashi's theorem ψ is minimal at some sphere in $HM(m+1,C)$, from theorem 11.1 it is totally geodesic.

Using the same method B. Y. Chen,[17], showed the following

PROPOSITION 11.4.- Let M^n be a minimal submanifold of RP^m (the m-dimensional real projective space with sectional curvature 1). Then

$$\lambda_1 \leqq 2(n+1).$$

Moreover the equality holds if and only if M^n is a RP^n imbedded as a totally geodesic submanifold in RP^m.

PROPOSITION 11.5.- Let M^n be a minimal submanifold of QP^m (the m-dimensional quaternion projective space with quaternion sectional curvature 1). Then

$$\lambda_1 \leqq \frac{n+4}{2} .$$

Moreover the equality holds if and only if n is a multiple of 4 and M is a $QP^{n/4}$ imbedded in QP^m as a totally geodesic submanifold.

COROLLARY 11.6,[46].- Let M^{2n+q} be a CR-minimal submanifold of CP^m. Then

$$\lambda_1 \leqq \frac{(2n+q)^2 + 4n + q}{2(2n+q)} . \tag{11.10}$$

Moreover if the equality holds, then either: (a) q=0 and M^{2n} is a totally geodesic complex submanifold of CP^m, or (b) n=0 and M^q is a minimal totally real submanifold of some CP^q totally geodesic complex submanifold in CP^m.

Proof.- The formula (11.10) is an immediate consequence from (11.6) and (10.2). The equality in (11.10) holds if and only if the corresponding immersion $\psi: M \longrightarrow HM(m+1,C)$ is of order 1, in particular it is mono-order and so from theorem 11.1 one obtains the second part.

COROLLARY 11.7, [46].- Let M^{2n+q} be a CR-minimal submanifold of CP^m which is minimal at some sphere of $HM(m+1,C)$. Then

$$\frac{(2n+q)^2+4n+q}{2(2n+q)} \in \mathrm{Spec}(M,g).$$

Proof.- If R denotes the radius of the sphere in which M is minimal then for Takahashi's theorem, one get

$$(2n+q)/R^2 \in \mathrm{Spec}(M,g).$$

Moreover formula (11.2) gives us that

$$R^2 = \frac{2(2n+q)^2}{(2n+q)^2+4n+q}.$$

COROLLARY 11.8,[46],[23].- Let M^{2n} be a complex submanifold of CP^m. Then

$$\lambda_1 \leqq n+1.$$

Moreover the equality holds if and only if M^{2n} is totally geodesic.

COROLLARY 11.9,[46].- Let M^q be a minimal totally real submanifold of CP^m.

(i) If there exists a linear subvariety CP^q such that M^q is a totally real submanifold of CP^q, then $(q+1)/2 \in \mathrm{Spec}(M,g)$.

(ii) If $\lambda_1 = (q+1)/2$, then there exists a linear subvariety CP^q of CP^m such that M is a totally real submanifold of CP^q.

Proof.- (i) It follows from corollary 11.7 because in this case $n=0$.
(ii) It follows from the second part of corollary 11.6.

12. PARALLEL COMPLEX SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE.- In last section we knew that the only complex submanifolds of CP^m which are minimal in some sphere of $HM(m+1,C)$ are totally geodesic and therefore, under conditions we are assuming in this paper, they are linear subvarieties. Consequently these submanifolds are the unique complex submanifolds of CP^m whose corresponding isometric immersions $\psi : M \longrightarrow HM(m+1,C)$ admit the simplest spectral behaviour (they are mono-order).

After totally geodesic complex submanifolds, parallel complex submanifolds (that is, complex submanifolds with parallel second fundamental form) are the more simple complex submanifolds. Therefore it seems natural to study the spectral behaviour of $\psi : M \longrightarrow HM(m+1,C)$ when M is a parallel complex submanifold of CP^m.

In this section we study parallel complex submanifolds of CP^m which are Einstein. First we will give some examples of such kind of submanifolds which are not totally geodesic

12.1. <u>The complex quadric</u>.- Consider the imbedding $\phi: CP^{n+1} \longrightarrow HM(n+2,C)$ given in section 8. The standard projection of $C^{n+2}-\{0\}$ over CP^{n+1} is given as follows: For all $z=(z_0,..,z_{n+1}) \in C^{n+2}-\{0\}$

$$z \longmapsto \phi(z) = (1/z\bar{z}^t)\bar{z}^t.z$$

On CP^{n+1} one consider the homogeneous coordinate system $(z_0,..,z_{n+1})$ determined from the above projection. As we know the complex quadric is defined by $Q_n = \{(z_0,..,z_{n+1}) \in CP^{n+1} / \sum z_i^2 = 0\}$. Q_n is a complex hypersurface of CP^{n+1} which is holomorphically isometric to the Hermitian symmetric space $SO(n+2)/SO(2)xSO(n)$. In order to identify Q_n in $HM(n+2,C)$ by means of the imbedding ϕ one notices that

$$\phi(A) = A = (1/z\bar{z}^t)\bar{z}^t z \quad \text{and} \quad AA^t = (1/z\bar{z}^t)^2\bar{z}^t(\sum z_i^2)\bar{z}$$

for all $A \in CP^{n+1}$. Therefore

$$Q_n = \{A \in CP^{n+1} / AA^t = 0\}.$$

We also recall that Q_n has parallel second fundamental form. Moreover a well-known result due to Smyth,[52], affirms that the only Einstein complex hypersurfaces (of course complete) of the complex projective space are (up congruence) linear subvarieties and complex quadrics.

12.2 <u>The Veronese imbedding</u>.- Let $S_k(C^{n+1})$ be the complex vector space whose elements are homogeneous polynomials of degree k over C^{n+1}. Denote by S_k^* its dual space. If $d=\dim S_k - 1$, one considers the complex projective space CP^d over S_k^* . For each point $z \in C^{n+1}$, one can define $F(z) \in S_k^*$ by

$$F(z): S_k \longrightarrow C \qquad F(z)(p) = P(z) \quad \text{for all polynomials } P \in S_k.$$

It is clear that $F(\lambda z)(P) = \lambda^k F(z)(P)$ for all $\lambda \in C$. It proves that $F: C^{n+1} \longrightarrow S_k^*$ defines a rational mapping $f: CP^n \longrightarrow CP^d$ which gives a holomorphic imbedding.
If one considers over CP^d the Fubini-Study metric with constant holomorphic sectional curvature 1, then the standard metric via f

induces over CP^n the Fubini-Study metric with constant holomorphic curvature $c = 1/k$. In particular when k= 2, the isometric imbedding $f:CP^n(1/2) \longrightarrow CP^d$ has parallel second fundamental form and it is called the Veronese imbedding.

12.3. The Segre imbedding.- One starts by taking the tensorial product mapping

$$C^{n+1} \times C^{n+1} \longrightarrow C^{n+1} \otimes C^{n+1} = C^{nm+n+m+1},$$

defined by $(z,w) \longmapsto z \otimes w$.
If $\lambda, \mu \in C$, then $(\lambda z, \mu w) \longmapsto \lambda\mu(z \otimes w)$ and so this mapping induces a holomorphic one which is already holomorpnic imbedding from $CP^n x CP^m$ in CP^{nm+n+m}. This imbedding is called the Segre imbedding. If one takes on CP^{nm+n+m} the Fubini-Study metric with holomorphic sectional curvature one, the induced metric over $CP^n x CP^m$ is the Riemannian product of the Fubini-Study metrics on CP" and CP^m both with holomorphic sectional curvature one. This imbedding is also parallel. Moreover $CP^n x CP^m$ is Einstein if and only if n=m.

12.4. The Plücker imbedding.- Consider the complex Grassmannian of 2-dimensional planes, $G(2,n,C) = \{ \Pi \in C^{n+2} /$ nis a 2-dimensiomal complex plane $\}$. If $\Lambda^2(C^{n+2}) = C^{n+2} \wedge C^{n+2}$, one defines

$$C^{n+2} x C^{n+2} \longrightarrow \Lambda^2(C^{n+2}) \quad \text{by} \quad (z,w) \longmapsto z \wedge w.$$

If (z,w) and $(\tilde{z},\tilde{w})$ are two basis of the same element $\Pi \in G(2,n,C)$ then

$$\tilde{z} \wedge \tilde{w} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} z \wedge w, \quad \text{with } \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0,$$

where $\tilde{z} = az + bw$, $\tilde{w} = cz + dw$. Consecuently the above mapping induces a holomorphic imbedding from $G(2,n,C)$ in $CP^{\frac{1}{2}n(n+1)-1}$. It is called the Plücker imbedding. The induced metric over $G(2,n,C)$ from the Fubini-Study metric with holomorphic sectional curvature one via the above imbedding is nothing but the standard metric over the Grassmannian and the imbedding is also parallel.

12.5. The rank two Hermitian symmetric spaces $M_5 = SO(10)/U(5)$ and $M_6 = E_6/Spin(10)xT$ admit holomorphic standard imbedding in CP^{15} and CP^{26} respectively which have parallel second fundamental form. They are obtained by using the representation theory of Lie groups (to see [37],[55]).

The complete classification for Einstein parallel complex submanifold in complex projective space is consequence of results in [37]and [55]. If M is an Einstein complex submanifold of a complex projective space then M is parallel if and only if M is totally geodesic or M is animbedded submanifold which is congruent with the standard of some of the submanifolds described before, that is, with the standard imbedding of some M_i, $1 \leqq i \leqq 6$, where

$$M_1 = CP^n(\tfrac{1}{2});\ M_2 = Q_n,\ n \geqq 3;\ M_3 = CP^n \times CP^n;\ M_4 = G(2,s,C),\ s \geqq 3;$$
$$M_5 = SO(10)/U(5);\quad M_6 = E_6/Spin(10) \times T.$$

The first eigenvalue of M_i, $1 \leqq i \leqq 6$, can be computed by using the following fact, [39]: If M is an Einstein homogeneous Kaehler manifold with positive scalar curvature ρ, then $\lambda_1 = \rho/n$, n being the complex dimension of M.

In [36], T. Nagano give a theoretical method to compute the spectrum of classica1 symmetric spaces. In particular one obtains the second eigenvalue of M_i, $1 \leqq i \leqq 5$.

Rerently S. Udagawa, ("Spectral geometry of Kaehler submanifolds of a complex projective space" preprint) computed the complete spectrum of $M_6 = E_6/Spin(10) \times T$, namely

$$\mathrm{Spec}(M_6) = \{\, 2m_1(m_1+m_2+8) + m_2(m_2+1) / \ m_1, m_2 \in Z^+ \} =$$
$$= \{\, 0 < \lambda_1 = 12 < \lambda_2 = 18 \quad \ldots \}.$$

Table 1. Einstein parallel complex submanifolds

Submanifold	n	p	ρ	λ_1	λ_2
$M_1 = CP^n(\frac{1}{2})$	n	$\frac{1}{2}n(n+1)$	$\frac{1}{2}n(n+1)$	$\frac{1}{2}(n+1)$	n+2
$M_2 = Q_n,\ n \geqq 3$	n	1	n^2	n	n+2
$M_3 = CP^n \times CP^n$	2n	n^2	2n(n+1)	n+l	2n+2
$M_4 = G(2,s,C),\ s \geqq 3$	2s	$\frac{1}{2}s(s-1)$	2s(s+2)	s+2	2s+2
$M_5 = SO(10)/U(5)$	10	5	80	8	12
$M_6 = E_6/Spin(10) \times T$	16	10	192	12	18

In this table n denotes complex dimension, p full complex dimension, ρ is the scalar curvature and λ_i the i-th eigenvalue.

13. COMPLEX SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE WHICH ARE BI-ORDER. To motivate the study of complex submanifolds M in CP^m whose corresponding imbedding $\psi : M \longrightarrow HM(m+1,C)$ is bi-order, we sall see what is the spectral behaviour of ψ when M is the complex quadric

$$Q_n = \{ A \in CP^{n+1} / AA^t = 0 \}.$$

It is clear that the imbedding of Q_n in $HM(n+2,C)$ is $SO(n+2)$-equivariant. One takes the point $C \in Q_n$ gives by

$$C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad c = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix}$$

It is not difficult to see that $T_C(Q_n) = \{ X \in T_C(CP^{n+1}) / XC^t + CX^t = 0 \}$. Moreover by choosing an appropiate basis of $T_C(Q_n)$ one obtains the mean curvature vector of $\psi : Q_n \longrightarrow HM(n+2,C)$ at C to be

$$\eta = \frac{1}{2n}\{ I - (n+1)C - C^t \}$$

and since ψ is $SO(n+2)$-equivariant, last formula is true at any point $A \in Q_n$, that is,

(13.1) $$\eta = \frac{1}{2n}\{ I - (n+1)\psi - \psi^t \}.$$

Finally it is easy to see that $\psi = \psi_0 + \psi_1 + \psi_2$, where $\psi_0 = \frac{1}{n+2} I$, is the center of gravity of ψ.

$$\psi_1 = \tfrac{1}{2}(\psi - \psi^t) \quad \text{with} \quad \Delta\psi_1 = n\psi_1 = \lambda_1\psi_1,$$

$$\psi_2 = \tfrac{1}{2}(\psi + \psi^t) - \frac{1}{n+2} I \quad \text{with} \quad \Delta\psi_2 = \lambda_2\psi_2 = (n+2)\psi_2.$$

which proves us that $\psi : Q_n \longrightarrow HM(n+2,C)$ is bi-order of order $\{1,2\}$.

Before give a complete classification for complex s'ubmanifolds of CP^m whose corresponding imbeddings in $HM(m+1,C)$ are bi-order, we will give a few technical lemmas whose proofs can be obtained directly.

LEMMA 13.1.- Let M^n be a complex submanifold of CP^{n+p}. If η denotes the mean curvature vector of $\psi : M^n \longrightarrow HM(n+p+1,C)$, then

(13.2) $$\eta = \frac{1}{2n} \sum_{i=1}^{n} \tilde{\sigma}(E_i, E_i),$$

(13.3) $$\Delta\eta = (n+1)\eta + \frac{1}{n}\sum_{ij} \tilde{\sigma}(A_{\sigma(E_i,E_j)}E_i, E_j) - \frac{1}{n}\sum_{ij} \tilde{\sigma}(\sigma(E_i,E_j), \sigma(E_i,E_j)).$$

where σ and Λ denote respectively the second fundamental form and the Weingarten endomorphism of $\tau: M^n \longrightarrow CP^{n+p}$ and $\{E_i\}$ is a loca1 orthonorma1 basis of vertor fields over M.

Remark.- One notices that the vector fields along M defined by η and $\Delta\eta$ are norma1 to CP^{n+p}. Moreover if one puts

$\sigma(E_i, E_j) = \sum_\lambda h^\lambda_{ij}$ then (13.3) can be written as

$$\Delta\eta = (n+1)\eta + \sum_{ijk\lambda} h^\lambda_{ij} h^\lambda_{ik} \tilde{\sigma}(E_j, E_k) - \frac{1}{n} \sum_{ij\lambda\mu} h_{ij} h_{ij} \tilde{\sigma}(\xi_\lambda, \xi_\mu) \tag{3.14}$$

LEMMA 13.2.- Let M^n be a camplex submanifold of CP^{n+p} and $\psi: M^n \longrightarrow HM(n+p+1, C)$. Then

$$g(\psi,\psi) = 2 \tag{13.5}$$

$$g(\psi,\eta) = -1 \tag{13.6}$$

$$g(\psi,\Delta\eta) = -(n+1) \tag{13.7}$$

$$g(\eta,\eta) = \frac{n+1}{2n} \tag{13.8}$$

$$g(\eta,\Delta\eta) = \frac{(n+1)^2}{2n} + (1/2n^2)\|\sigma\|^2 \tag{13.9}$$

$$g(\Delta\eta,\Delta\eta) = \frac{(n+1)^3}{2n} + \frac{(n+1)}{n^2}\|\sigma\|^2 + \frac{1}{n^2}\sum_{\lambda,\mu}(\mathrm{Tr} A_\lambda A_\mu)^2 + \frac{1}{n^2}\Big(\sum_\lambda A_\lambda^2\Big)^2 \tag{13.10}$$

Let $T^\perp_A M$ the norma1 space of $\tau: M^n \longrightarrow CP^{n+p}$ at A and define a tensor T: $T^\perp_A M \times T^\perp_A M \longrightarrow R$ by $T(\xi,\nu) = \mathrm{Tr}(\Lambda_\xi \Lambda_\nu)$ for all ξ $\nu \in T^\perp_A M$. The length of T is estimated in the following lemma.

LEMMA 13.3.- Let M^n be a comolex submanifold of CP^{n+p}. then

$$\frac{1}{2p}|\sigma|^4 \leq |T|^2 \leq \frac{1}{2}|\sigma|^4. \tag{13.11}$$

Moreover first equality holds if and only if T=kg (k being some constant and g the restricted metric over $T^\perp M$)

Proof.- $|T|^2 = \sum_{\lambda\mu}(\mathrm{Tr}(\Lambda_\lambda \Lambda_\mu))^2$, $|\sigma|^2 = \sum_\lambda \mathrm{Tr}\, \Lambda_\lambda^2$.
Let G be the metric over the space of symmetric bilinear forms on $T^\perp_A M$ via the metric g over $T^\perp_A M$. Since T and g belong to such space, from the Schwartz inequality one has

$$G(T,g)^2 \leq |T|^2 |g|^2,$$

with $|g|^2 = 2p$ and $G(g,T) = \sum_{\lambda\mu} T_{\lambda\mu} g_{\lambda\mu} = \mathrm{Tr}(T) = \sum_\lambda \mathrm{Tr}(\Lambda_\lambda^2) = |\sigma|^2$ which proves the first inequality. The equality holding if and only

$T = kg$ for some constant k. The second inequality is wee-known,[41].

THEOREM 13.4. 4.- Let M^n be a full complex submanifold of CP^{n+p}. Then $\psi: M^n \longrightarrow HM(n+p+1,C)$ is of order $\{u_1,u_2\}$ (for certain natural number u_1, u_2) if and only if
(1) M is Einstein, and
(2) $T = kg$ for some real number k.

Proof.- Assume ψ is of order $\{u_1,u_2\}$, then $\Delta\eta = a\eta + b(\psi - Q)$ for some real number a, b where Q is the center of gravity of ψ and η its mean curvature vector.
If $b=0$, $\Delta^2\psi = a\Delta\psi$, that is $\Delta\psi - a\psi = C$ (constant) which implies that ψ is monô-order and from Theorem 11.1 M^n is totally geodesic, satisfying (1) and (2).
We now consider $b \neq 0$, because ψ, η and $\Delta\eta$ define vector fields along M which are normal to CP^{n+p} at each point of M, one obtains that Q is in $T_A^{\perp}(CP^{n+p})$ for all $A \in M$, which proves that M is contained in the linear subspace L of $HM(n+p+1,C)$ defined by

$$L = \{A \in HM(n+p+1,C) / AQ = QA\}.$$

From a well-known argument one can consider Q to be diagonal. Moreover by using a similar argument as in the proof of Theorem 11.1, one get

$$Q = \frac{1}{n+p+1} I,$$ 1 being de identity matrix of order n+p+l.
Therefore

$$\Delta\eta = a\eta + b\left(\psi - \frac{1}{n+p+1} I\right). \tag{13.12}$$

Now we use some technical arguments to see the behaviour of the second fundamental form σ of $\psi: M^n \longrightarrow CP^{n+p}$. Take $r \neq s, s^*$ and use (8.13) and (8.14) to obtain

$$g(\tilde{\sigma}(E_r,E_s),\Delta\eta) = a\, g(\tilde{\sigma}(E_r,E_s),\eta). \tag{13.13}$$

Now from (8.11), (13.4) and (13.13), one get

$$\sum_{i\lambda} h^{\lambda}_{ir} h^{\lambda}_{is} = 0. \tag{13.14}$$

Since any complex submanifolds satisfies

$$\sum_{i\lambda} h^{\lambda}_{ir} h^{\lambda}_{ir^*} = 0, \tag{13.15}$$

from (13.14) and (13.15) one get that $\sum_{\lambda} A_{\lambda}^2$ is diagonal. Now we apply $g(\tilde{\sigma}(E_r,E_r),-)$ to (13.12) to obtain

$$\frac{(n+1)^2}{2n} + \frac{1}{n}\sum_{i\lambda} h_{ir}^{\lambda} h_{ir}^{\lambda} = \frac{n+1}{2n}a - b,$$

which proves

$$\sum_{\lambda} A_{\lambda}^2 = [\frac{n+1}{2}a - nb - \frac{(n+1)^2}{2}]I \tag{13.16}$$

Finally from (6.15) and (13.16) one get M is Einstein.
In order to prove (2) it is enough to apply the same argument over the norma1 bundle. The sufficient condition is essentially technical and no difficult.

The following corollary,[47], gives us precisely the eigenvalues λ_{u_1} and λ_{u_2} corresponding to the order $\{u_1,u_2\}$ in last Theorem.

COROLLARY 13.5.- Let M^n <u>be a full complex submanifold of</u> CP^{n+p}. <u>If</u> M <u>is Einstein and</u> T = kg, then

$$\frac{1}{2}\left\{n+1+\frac{p+n}{pn}|\sigma|^2 \pm\sqrt{(n+1-\frac{p+n}{pn}|\sigma|^2)^2+\frac{4}{n}|\sigma|^2}\right\} \tag{13.17}$$

<u>are eigenvalues of</u> M.

Proof.- According last theorem, one has

$$\Delta\eta = a\eta + b(\psi - \frac{1}{n+p+1}I) \quad \text{for certain } a,\ b \in R$$

$$g(\psi,\Delta\psi) = ag(A,\eta) + bg(\psi,\psi - \frac{1}{n+p+1}I)$$

$$g(\eta,\Delta\eta) = ag(\eta,\eta) + bg(\eta,\psi - \frac{1}{n+p+1}I)$$

Now we use lemma 13.2 to obtain

$$a = n+l + \frac{p+n}{pn}|\sigma|^2$$

$$b = \frac{p+n+1}{2pn}|\sigma|^2$$

Since $\psi : M \longrightarrow HM(n+p+1,C)$ is of order $\{u_1,u_2\}$, one has

$$\psi - \frac{1}{n+p+1}I = \psi_{u_1} + \psi_{u_2},$$

$$-2n\eta = \lambda_{u_1}\psi_{u_1} + \lambda_{u_2}\psi_{u_2},$$

$$-2n\Delta\eta = \lambda_{u_1}^2\psi_{u_1} + \lambda_{u_2}^2\psi_{u_2},$$

which joint (13.12) gives us

$$\left(\frac{\lambda_{u_1}^2}{2n} - \frac{a\lambda_{u_1}}{2n} + b\right)\psi_{u_1} + \left(\frac{\lambda_{u_2}^2}{2n} - \frac{a\lambda_{u_2}}{2n} + b\right)\psi_{u_2} = 0,$$

that is λ_{u_1} and λ_{u_2} are roots of the equation $\lambda^2 - a\lambda + 2nb = 0$.

14. PARALLEL COMPLEX SUBMANIFOLDS AND BI-ORDER COMPLEX SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE ARE VERY CLOSE. In last section we shown that complex submanifolds of the complex projective space which are Einstein with $T = kg$ have a good spectral behaviour for their corresponding immersions in Euclidean space.They are just the second step in the study of the Spectral Geometry of complex submanifolds in the complex projective space.

Now we will see that these conditions are close to a simple behaviour of the second fundamental form.

THEOREM 14.1,[47].- <u>Let M^n be a complex submanifold of CP^{n+p} whose immersion is full.</u>

(i) If M is Einstein with $T = kg$, then

$$|\sigma|^2 \geqq \frac{np(n+2)}{2p+n}. \tag{14.1}$$

<u>The equality holds if and only if</u> $\nabla\sigma = 0$, with $\sigma \neq 0$.

(ii) <u>If</u> $|\sigma|^2 = \frac{np(n+2)}{2p+n}$, <u>then</u> $\nabla\sigma = 0$ <u>if and only if</u> M <u>is Einstein</u> with $T=kg$.

Proof.- (i) If M is Einstein, from (6.15) one has $\mathrm{Tr}\,(\sum_\lambda A_\lambda^2)^2 = \frac{|\sigma|^4}{2n}$. From $T = kg$ and lemma 13.3, one get $|T|^2 = \sum_{\lambda\mu}(\mathrm{Tr}\, A_\lambda A_\mu)^2 = \frac{1}{2p}|\sigma|^4$.

Now one uses last formulas in (6.13) to obtain

$$\tfrac{1}{2}\Delta|\sigma|^2 = |\nabla\sigma|^2 + \frac{n+2}{2}|\sigma|^2 - \frac{2p+n}{2np}|\sigma|^4.$$

Since $|\sigma|^2$ is constant and so $\Delta|\sigma|^2 = 0$, one get

$$|\nabla\sigma|^2 = \frac{2p+n}{2pn}|\sigma|^2 - \frac{n+2}{2}|\sigma|^2,$$

which proves (i).

(ii) It is clear that $\Delta|\sigma|^2 = 0$, now from (6.13), (6.15) and lemma 13.3 one get

$$|\nabla\sigma|^2 = 2\mathrm{Tr}\,(\sum_\lambda A_\lambda^2)^2 + \sum_{\lambda\mu}(\mathrm{Tr}\, A_\lambda A_\mu)^2 - \frac{n+2}{2}|\sigma|^2 \geq$$

$$\frac{1}{n}|\sigma|^4 + \frac{1}{2p}|\sigma|^4 - \frac{n+2}{2}|\sigma|^2 = \frac{n+2p}{2np}|\sigma|^2 - \frac{n+2}{2}|\sigma|^2 = 0.$$

Remark.- If M is Einstein with $T=kg$ and we assume that $\nabla\sigma = 0$, with $\sigma \neq 0$, then from the above lemma one has

$$|\sigma|^2 = \frac{np(n+2)}{2p+n}$$

and so the eigenvalues in corollary 13.5 are

$$\frac{n(n+p+1)}{2p+n},\ n+2 \in \mathrm{Spec}(M).$$

As consequence, we will obtain a nice characterization of the Veronese imbedding. First we shall see the Veronese imbedding is of order $\{1,2\}$. In fact, $\nabla\sigma = 0$ and

$$|\sigma|^2 = n(n+1) - \rho = \tfrac{1}{2}n(n+1)$$

Since $p = \frac{1}{2}n(n+1)$, we can write

$$|\sigma|^2 = \frac{np(n+2)}{2p+n}.$$

Therefore from theorem 14.1, M is Einstein (well-known), $T=kg$ and the corresponding eigenvalues are

$$\lambda_{u_1} = \frac{n(n+p+1)}{2p+n} = \frac{n+1}{2} = \lambda_1, \quad \lambda_{u_2} = n+2 = \lambda_2,$$

which proves us that tha Veronese imbedding $CP^n(1) \to CP^{n+\frac{1}{2}n(n+1)}$ is of order $\{1,2\}$ in $HM(n+\frac{1}{2}n(n+1)+1,C)$.

LEMMA 14.2,[47].- Let M^n be a complex submanifold of CP^{n+p} whose immersion is full. Suppose that M is Einstein with $T=kg$. Then

$$\tfrac{1}{2}n(n+1) \geqq p. \tag{14.2}$$

Moreover the equality holds if and only if $\tau: M^n \longrightarrow CP^{n+p}$ is the Veronese imbedding.

Proof.- Because $|T|^2 = \frac{1}{2p}|\sigma|^4$ one has $|R|^2 = 2n(n+1) - 4|\sigma|^2 + \frac{1}{p}|\sigma|^4$. On the other hand $\rho = n^2(n+1)^2 + |\sigma|^4 - 2n(n+1)|\sigma|^2$. Since $\frac{1}{2}n(n+1)|R|^2 \geq \rho^2$ and equality holds if and only if M has constant holomorphic sectional curvature. One get

$$\left\{\frac{n(n+1)}{2p} - 1\right\}|\sigma|^4 \geqq 0, \text{ that is } p \geqq \tfrac{1}{2}n(n+1).$$

Moreover equality holds if and only if M has constant holomorphic sectional curvature. But with codimension $\frac{1}{2}n(n+1)$ we know that $\tau: M \longrightarrow CP^{n+p}$ must be the Veronese imbedding, [11].

15. THEFIRST AND THE SECOND EIGENVALUES OF COMPLEX SUBMANIFOLDS IN THE COMPLEY PROJECTIVE SPACE. We already know that the first eigenvalue of an n-dimensiona1 complex submanifold M of CP^m satisfies: $\lambda_1 \leqq n+1$ and the equality holds if and only if M is totally geodesic. In this section we will obtain a spectral inequality for complex submanifolds of the complex projective space in terms of the five simplest spectral invariants: dimension, volume, tota1 scalar curvature, first eigenvalue and second eigenvalue.

We start by taking k=2 and t=l in Proposition 5.4 to obtain

$$(15.1)\quad \int_M g(\Delta^3\psi,\psi)\,dv - (\lambda_1+\lambda_2)\int_M g(\Delta^2\psi,\psi)\,dv + \lambda_1\lambda_2\int_M g(\Delta\psi,\psi)\,dv \geqq 0,$$

and the equality holds if and only if the immersion $\psi: M \longrightarrow HM(m+1,C)$ is of order (1.2). Because $\Delta\psi=-2n\eta$, one can write from (15.1)

$$(15.2)\quad 4n^2\int_M g(\Delta\eta,\eta)\,dv - 4n^2(\lambda_1+\lambda_2)\int_M g(\eta,\eta)dv - 2n\lambda_1\lambda_2\int_M g(\eta,\psi)dv \geqq 0.$$

THEOREM 15.1,[47].- Let M^n be a compact Kaehler submanifold of CP^m, then

$$(15.3)\qquad n[n+1+(n+1-\lambda_1)(n+1-\lambda_2)]\mathrm{vol}(M) \geqq \int_M \rho\,dv.$$

Moreover if the equality holds, then M is Einstein with T= kg (if the immersion is full).

Proof.- Inequality (15.3) follows from (15.2) joint lemma 13.2. If equality holds, then $\psi: M^n \longrightarrow HM(m+1,C)$ is of order {1,2} and so from theorem 13.4 one obtains the second part.

COROLLARY 15.2,[23],[47].- Let M^n be an n-dimensiona1 complex submanifold of CP^m, then $\lambda_1 \leqq n+l$. Moreover equality holds if and only if M is totally geodesic.

Proof.- Ih proposition 5.4, one takes k=1, t=1 and k=1, t=2 respectively to obtain

$$\int_M g(\Delta^2\psi,\psi)\,dv - \lambda_1\int_M g(\Delta\psi,\psi)\,dv \geqq 0, \text{ and}$$

$$\int_M g(\Delta^3\psi,\psi)\,dv - \lambda_1\int_M g(\Delta^2\psi,\psi)\,dv \geqq 0.$$

Both equalities occur if and only if $\psi: M \longrightarrow HM(m+1,C)$ is of order one. Since $\Delta\psi=-2n\eta$, Δ is self-adjoint, using lemma 13.2 one get

(15.4) $\qquad 2n(n+1-\lambda_1)\mathrm{vol}(M) \geqq 0,$

(15.5) $\qquad 2n(n+1)(n+1-\lambda_1)\mathrm{vol}(M) + 2\int_M |\sigma|^2 dv \geqq 0.$

Now (15.4) implies $\lambda_1 \geqq n+1$. If the equality holds in (15.4), one also has the equality in (15.5) which implies $\sigma = 0$.

COROLLARY 15.3, [47].- Let M^n be a complex submanifold of CP^m. Suppose that

(15.6) $\qquad \lambda_1 = \frac{1}{n\,\mathrm{vol}(M)} \int_M \rho\, dv$

and M is not totally geodesic. Then

(15.7) $\qquad \lambda_2 \leqq n+2.$

Moreover if the equality holds, then M is Einstein with parallel second fundamental form.

Proof.- From (15.3) and (15.6) one has

$$n+1 + (n+1-\lambda_1)(n+1-\lambda_2) \geqq \frac{1}{n\,\mathrm{vol}(M)}\int_M \rho\, dv = \lambda_1,$$

and so $(n+1-\lambda_1)(n+2-\lambda_2) \geqq 0$. Because M is not totally geodesic $\lambda_1 < n+1$, then $\lambda_2 \leqq n+2$ which proves (15.7).
The second part is very easy.

Remark.- Notice that condition (15.6) is natura1 because, for instance, every Einstein homogeneous Kaehler with positive scalar curvature satisfies it.

16. APPLICATIONS.

16.1.- Complete intersections.- Let M^n be a complex submanifold of CP^{n+p} determined by p polinomials $P_1,\dots,P_p$. If the Jacobian matrix associated with $P_1,\dots,P_p$ over $\pi^{-1}(M)$ (π being the standard projection) is always of maximun rank, then we say that M^n is a complete intersection. In this case one has the following

THEOREM 16.1,[47].- Let M^n be a complex submanifold imbedded in CP^{n+p}. Assume that M is a complete intersection of p homogeneous polynomials of degrees $a_1,\dots,a_p$. Then

(16.1) $$(n+1-\lambda_1)(n+1-\lambda_2) \geqq p - \sum_\alpha a_\alpha .$$

Moreover the equality holds if and only if $M = CP^n$ or $M = Q_n$ imbedded canonically.

Proof.- For complete intersections, the tota1 scalar curvature was computed by K. Ogiue, [41] to be

(16.2) $$\int_M \rho dv = n(n+p+1-\sum_\alpha a_\alpha)\mathrm{vol}(M).$$

Now (16.1) follows from (15.3) and (16.2).

Assume equality in (16.1), then one has equality in (15.3) and so from theorem 15.1 one get M must be Einstein. In these conditions a result due to J. Hano, [26], proves us that $M = CP^n$ (totally geodesic) or $M = Q_n$ (canonica1 imbedding). The converse is obvious.

16.2.- The inverse problem for M_i.- Because al1 elements in formulas (15.6) and (15.7) are spectral invariants, one can prove the so called inverse problem for M_i. The following theorem was proved in [47] except for M_6, which was recently proved by S. Udagawa ("Spectral geometry of Kaehler submanifolds of a complex projective space" preprint).

THEOREM 16.2.- Let M^n be an n-dimensiona1 complex submanifold of CP^m. Assume that

$$\mathrm{Spec}(M) = \mathrm{Spec}(M_i) \quad \text{for some } i=1,..,6.$$

Then M is imbedded and it is congruent with the standard imbedding of the corresponding M_i (of course Spec(M) means the spectrum of the induced metric).

Proof.- First we notice from table 1 (by looking at the dimension and λ_1, for instance) that $\mathrm{Spec}(M_i) = \mathrm{Spec}(M_j)$, $i,j = 1,..,6$ implies $i=j$.

On the other hand, submanifolds Mi satisfy (15.6) and the equality in (15.7). Since they are spectral equalities, if M has the same spectrum of some M_i, it must satisfy also (15.6) and equality in (15.7). Therefore M is Einstein and parallel but it is not totally geodesic ($\lambda_1 \neq n+1$). Consequently M is congruent with the standard imbedding of some M_j, $j=1,..,6$. Now first remark we did proves us the theorem.

REFERENCES

[1] AUBIN T..:"Fonction de Green et valeurs propes du Laplacian" J. Math. Pure et Appl. (1974),347-371.

[2] BARROS M..:"Algunas caracterizaciones de la cuádrica compleja por el espectro de su Laplaciano" Pub.Mat. UAB 21 (1980),11-14.

[3] BARROS M..:"Some extrinsic characterizations of complex quadrics by its spectrum" Tensor N.S. 36 (1982),315-318.

[4] BARROS M.:"Some characterizations of complex immersions in the complex projective space by its spectrum" Conf.Univ. Limoges 1980.

[5] BARROS M. and CHEN B.Y.:"Complex quadrics and spectral geometry" Geometriae Dedicata 11(1981),337-340.

[6] BARROS M. et ROMERO A.:"Sur un invariant spectral pour les sous--variétés complexes de l'espace projectif complexe" C.R.Acad.Sci. Paris 292 (1981),523-525.

[7] BARTH W.:"Submanifolds of low dimensions in projective space" Proc.Int.Cong.Math. Vancouver 1974 vol 1,409-413.

[8] BERGER M.:"Eigenvalues of the Laplacian" Proc.Symp. Pure Math. AMS 1970,121-125.

[9] BERGER M.,GAUDUCHON P. et MAZET E.:"Le spectre d'une variété Riemannienne" Lecture Notes in Math. 194, Springer-Verlag 1971.

[10] BEJANCU A..:"CR-submanifolds of a Kaehler manifold"I,II, Proc.AMS 69 (1978),135-142; Trans.AMS 250 (1979),21-24.

[11] CALABI E.:"Isometric imbedding of complex manifolds" Ann. of Math. 58 (1953),1-23.

[12] CHEEGER J..:"A lower bound for the smallest eigenvalue of the Laplacian" A Symp. in honor of S.Bochner, Princeton Univ. Press 1970.

[13] CHEN B.Y.: "Geometry of submanifolds" Marcel Dekker, 1973.

[14] CHEN B.Y..:"On the total curvature of immersed manifolds" II,Amer. J.Math. 94 (1972),899-907.

[15] CHEN B.Y."CR-submanifolds of a Kaehler manifold" 1, J.Diff.Geom. 16 (1981),305-322.

[16] CHEN B.Y.: "On the total curvature of immersed manifolds,IV:Submanifolds of finite type and their applications" Bull.Acad. Sinica 11 (1983),309-328.

[17] CHEN B.Y.: "On the first eigenvalue of Laplacian of compact minimal submanifolds of rank one symmetric spaces" preprint.

[18] CHEN B.Y. and OGIUE K.: "A characterization of the complex sphere" Mich.Math.J. 21 (1974), 231-232.

[19] CHEN B.Y. and OGIUE K.: "Some implications of second Chern class for complete intersections manifolds" Tamkang J.Math. 6 (1975) 15-23.

[20] CHEN B.Y. and VANHECKE L.: "The spectrum of the Laplacian of Kaehler manifolds" Proc. AMS 79 (1980), 82-86.

[21] CHENG S.Y.: "Eigenvalue comparison theorems and its geometric applications" Math.Z. 143 (1975), 289-297.

[22] DONELLY H.: "Symmetric Einstein spaces and spectral geometry" Indiana Univ. Math.J. 24 (1974), 603-606.

[23] EJIRI N.: "The first eigenvalue of the Laplacian for compact minimal submanifolds in a complex projective space" preprint.

[24] GALLOT S. et MEYER D.: "Operateur de courbure et Laplacien des formes differentielles d'une variété Riemannienne" J.Math. Pures et Appl. 54 (1975), 259-284.

[25] GUILLEMIN V. and KAZHDAN D.: "Some inverse spectral results for negatively curved n-manifolds" AMS Symp.Geom. Laplace operator 36 Hawaii, 1979, 153-180.

[26] HANO J.: "Einstein complete intersections in complex projective space" Math.Ann. 216 (1975), 197-208.

[27] HELGASON S.: "Differential geometry, Lie groups and symmetric spaces" Academic Press, N.Y., 1978.

[28] KAC M.: "Can one hear the shape of a drum" Amer.Math. Monthly 73 (1966), 1-23.

[29] KNESER M.: "Lineare relationen zwischen quadratischer formen" Math. Ann. 168 (1967), 31-39.

[30] KOBAYASHI S.: "On compact Kaehler manifolds with positive definite Ricci tensor" Ann. of Math. 74 (1961), 570-574.

[31] LI P.: "A lower bound for the first eigenvalue of the Laplacian on a compact Riemannian manifold" Indiana Univ.J.Math. 28 (1979) 1013-1019.

[32] LI P. and YAU S.T.: "Estimates of eigenvalues of a compact Riemannian manifold" Proc.Symp.Pure Math. 36 (1980), 205-239.

[33] LICHNEROWICZ A.:"Geometrie des groupes de tranaformations" Dunod, Paris, 1958.

[34] LITTLE J.A.:"Manifolds with planar geodesics" J.Diff.Geom. 11 (1976),265-285.

[35] MILNOR J.:"Eigenvalues of the Laplace operator on certain manifolds" Proc.Nat.Acad.Sci. USA 51 (1964),542.

[36] NAGANO T.: "On the minimum eigenvalues of the Laplacian in Riemannian manifolds" Sci. Papers Coll.Gen.Ed.Univ. Tokyo 11 (1961) 177-182.

[37] NAKAGAWA H. and TAKAGI R.:"On locally symmetric Kaehler submanifolds in a complex projective space" J.Math.Soc. Japan 28 (1976),638-667.

[38] OBATA M.: "Certain conditions for a Riemannian manifold to be isometric with a sphere" J.Math.Soc. Japan 14 (1963).

[39] OBATA M.:"Riemannian manifolds admitting a solution of a certain system of differential equations" Proc. U. States-Japan Sem. Diff.Geom. Kyoto, Japan 1965.

[40] OGIUE K.:"Generalized scalar curvatures of cohomological Einstein Kaehler manifolds" J.Diff .Geom. 10 (1975),201-205.

[41] OGIUE K.:"Differential geometry of Kaehler submanifolds" Adv. in Math. 13 (1974),73-114.

[42] PERRONE D.:"On the minima1 eigenvalue of the Laplacian operator for p-forms in conformally flat Riemannian manifolds" Proc AMS 86 (1982),103-108.

[43] PERRONE D.: "Cohomological Einstein Kaehler manifolds characterized by the spectrum" Math.Z. 185 (1984),179-183.

[44] REILLY R.C.:"Applications of the integra1 of an invariant of the Hessian" Bull. AMS 82 (1974).

[45] REILLY R.C.:"On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space" Comm.Math.Helv. 52 (1977) 525-533.

[46] ROS A.:"Spectral geometry of CR-minima1 submanifolds in the complex projective space" Kodai Math.J. 6 (1983),88-99.

[47] ROS A.:"On spectral geometry of Kaehler submanifolds" J.Math. Soc. Japan (to appear).

[48] ROS A.: "Eigenvalue inequalities for minimal submanifolds and P-manifolds" Math.Z. (to appear).
[49] SAKAI T.: "On eigenvalues of Laplacian and curvature of Riemannian manifolds" Tohoku Math.J. 23 (1971(,589-603.
[50] SAKAMOTO K.: "Planar geodesic immersions" Tohoku Math.J. 29 (1977) 25-56.
[51] SIMON u. and WISSNER H.: "Geometry of the Laplace operator" Proc. Conf. Algebra Geom. Kuwait Univ. 1981, Alden Press Oxford 171-191.
[52] SMYTH B.: "Differential geometry of complex hypersurfaces" Ann. of Math. 85 (1967),246-266.
[53] TAI S.S.: "Minimum imbedding of compact symmetric spaces of rank one" J.Diff.Geom. 2 (1968),55-66.
[54] TAKAHASHI T.: "Minimal immersions of Riemannian manifolds" J.Math. Soc. Japan 18 (1966),380-385.
[55] TAKEUCHI M.: "Homogeneous Kaehler submanifolds in complex projective space" Japan J. Math. 4 (1978),171-219.
[56] TANNO S.: "The spectrum of the Laplacian for 1-forms" Proc. AMS 45 (1974),125-129.
[57] TSUKADA K.: "Locally symmetric Einstein Kaehler manifolds and spectral geometry" Tohoku Math.J. 31 (1979),255-259.
[58] VIGNERAS M.: "Variétés Riemaniennes isospectrales et non isometriques" Ann. of Math. 112 (1980),21-32.
[59] WIRTINGER W.: "Eine determinantem densität und ihre anwendung auf analytische gebilde in Euclidischer und Hermitischer massbestimmung" Monafschefte für Math. und Physik 44 (1936),343-365.
[60] YAU S.T.: "Isoperimetric constants of the first eigenvalue of a compact Riemannian manifold" Ann.Scient.Ec.Norm.Sup. 8 (1975), 487-507.
[61] YANO K. and KON M.: "Anti-invariant submanifolds" Marcel Dekker 1976.

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