SPECTRAL GEOMETRY OF SUBMANIFOLDS

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PREFACE. These notes reflect the lectures given by the first author at the University of Lecce in June-July 1984. The work deals with geometry of Laplacian's spectrum of submanifolds in the complex projective space. The second part contains a new way to study this subject given by the second author in his Ph. Dr. (Granada, 1984). This idea can be extended to other symmetric spaces.

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For convenience to the reader we give the content of this paper as follows:

1. Introduction.
2. Some problems related with the spectrum of a Riemannian manifold.
3. The inverse problem
4. Basic formulas in theory of submanifolds.
5. Spectral geometry of submanifolds in Euclidean space.
7. Extrinsic characterizations of some complex submanifolds in the complex projective space by its spectrum.
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1. INTRODUCTION. Let \( \Omega \) be a very regular bounded domain with \( \partial \Omega \) boundary \( \Omega \) in \( \mathbb{R}^d \) (for \( d=2 \), one can think of \( \Omega \) as a vibrating membrane fixed along \( \partial \Omega \)). The vibrations of \( \Omega \) are the functions \( F: \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) \( (p,t) \rightarrow F(p,t) \) with

\[
\Delta F + \frac{\partial^2 F}{\partial t^2} = 0 \quad F(\partial \Omega \times \mathbb{R}) = 0
\]

being \( \Delta = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \) (the Laplacian in \( \mathbb{R}^d \)).

It is important to study solutions of the type \( F(p,t) = f(p)e(t) \) with \( f: \Omega \rightarrow \mathbb{R} \) and \( e: \mathbb{R} \rightarrow \mathbb{R} \). For these solutions one obtains

\[
\Delta f/f = -\frac{e''}{e} = \lambda
\]

where \( \lambda \) must be constant and it is very related to the frequencies of our vibrations since \( e'' + \lambda e = 0 \). That is a good reason to be interested in the spectrum of \( \Omega \),

\[
\text{Spec}(\Omega) = \{0 = \lambda_0 < \lambda_1 \ldots < \lambda_2 < \lambda_3 \ldots \}
\]

consisting of all \( \lambda \)'s (real numbers) such that there exists some \( f \neq 0 \) with \( \Delta f = \lambda f, f/\partial \Omega = 0 \). Each \( \lambda \) written in \( \text{Spec}(\Omega) \) as many times as its multiplicity indicates. Being multiplicity of \( \lambda = \dim (f/\Delta f = f) \).

Despite its simplicity and its physical background, \( \text{Spec}(\Omega) \) is almost unknown. For example \( \text{Spec}(\Omega) \) is known when \( \Omega \) is a ball or a rectangular parallelepiped and few other \( \Omega \).

A natural and very old problem is the following: Let \( \Omega_1 \) and \( \Omega_2 \) be two plane regions bounded by curves \( r_1 \) and \( r_2 \) respectively and consider the eigenvalue problems:
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\[ \Delta f = \lambda f \text{ in } \Omega_1 \quad \Delta g = \mu g \text{ in } \Omega_2 \]
\[ f = 0 \text{ in } \partial \Omega_1 \quad g = 0 \text{ in } \partial \Omega_2 \]

Suppose that for each \( n \in \mathbb{N} \) the eigenvalue \( \lambda_n \) for \( \Omega_1 \) is equal to the eigenvalue \( \mu_n \) for \( \Omega_2 \): Are the regions \( \Omega_1 \) and \( \Omega_2 \) congruent in the sense of Euclidean geometry?

This problem is called the inverse problem because it is well-known that: If \( \phi \) is an isometry of the plane then \( \text{Spec}(\phi(\Omega)) = \text{Spec}(\Omega) \) for any domain \( \Omega \) in \( \mathbb{R}^2 \).

The inverse problem was firstly posed for Bochner and Bers said about it that it can be posed in the following amusing way: Can one hear the shape of a drum?

Of course this problem is still unsolved but it is clear that the spectrum of \( \Omega \) contains an ordinate information about the geometry of \( \Omega \). \( \text{Spec}(\Omega) \) is like a "secret code" of information about the geometry of \( \Omega \) which we need to discover.

To be less greedy than asking to hear the complete shape of \( \Omega \), one can ask only deduce from \( \text{Spec}(\Omega) \) some information on the geometry of \( \Omega \). For example: One can hear the area of \( \Omega \). This is an old problem posed by H.A.Lorentz (1910) which was solved by H.Weyl. In fact he proved

\[ \lim_{\lambda \to +\infty} \frac{N(\lambda)}{\lambda} = \frac{\text{vol}(\Omega)}{2\pi} \]

where \( N(\lambda) \) is the number of eigenvalues less than \( \lambda \).

The length of \( \partial \Omega \) is also a spectral invariant and therefore we can prove that: You can hear the shape of a circular drum. In fact if \( L \) denotes the length of \( \partial \Omega \), the classical isoperimetric inequality states

\[ L^2 \geq 4\pi \text{vol}(\Omega) \]

with the equality holding if and only if \( \Omega \) is a circular disk.

This result joint the facts that \( \text{vol}(\Omega) \) and \( L \) are spectral invariants allow us to prove that: You can hear the shape of a circular drum.

The spectral problem appear in many parts of mathematics and physics: statistical mechanics, diffusion theory, quantum statistics etc. (see [28]).
In this paper we are interested in the spectral problem but for free vibrating membranes, that is, for Riemannian manifolds which are $C^\infty$, compact, connected and without boundary. All manifolds in this paper are assumed to have these properties.

2. Some Problems Related With the Spectrum of a Riemannian Manifold. Let $M$ be a compact, connected, without boundary Riemannian manifold (along this paper we will omit these properties), (also all geometric objects are assumed to be $C^\infty$). We will denote by $g$ the metric tensor of $M$. As usual $C^\infty(M)$ (respectively $\Lambda^p(M)$) will denote the real valued functions over $M$ (respectively the exterior p-forms over $M$). $\Lambda(M)$ will be the Grassmann algebra over $M$, $d$ (respectively $\delta$) will denote exterior derivative (respectively the coderivative).

The Laplace-Beltrami operator of $(M,g)$ is defined over $\Lambda(M)$ by

\[
\Delta : \Lambda(M) \rightarrow \Lambda(M) \quad \Delta(\omega) = (d\delta + \delta d)(\omega)
\]

it is clear that the degree of $\omega$ is preserved by $\Delta$.

In particular we are interested in the Laplacian acting over functions. In this case various definitions exist for $\Delta$ (of course equivalent), we recall some of them:

(a) Suppose $N=\dim M$, if $m \in M$ and $f \in C^\infty(M)$, we choose an orthonormal set of geodesics $\{\gamma_i\}$, $i=1,...,n$ parametrized by arc length and passing through $m$ at $t=0$, then

\[
(\Delta f)(m) = -\sum_{i=1}^{n} \frac{d^2(f \cdot \gamma_i)}{dt^2}(0)
\]

(b) Last definition is obviously equivalent to the following one, first we consider the Hessian of $f$ like the symmetric 2-times covariant tensor over $M$ given by

\[
\text{Hess} f(X,Y) = XYf - (\nabla_X Y)f \quad X,Y \in \chi(M)
\]

$\chi(M)$ being the Lie algebra of vector fields over $M$, then

\[
\Delta f = -\text{Trace(Hess } f) = -\sum_{i=1}^{n} \text{Hess } f(E_i,E_i)
\]

where $\{E_i\}$ $i=1,...,n$ is a local orthonormal frame of vector fields.

For other definitions of $\Delta$ see [9].

The canonical measure of $(M,g)$ will be denoted by $dv$. One defines
a pre-Hilbert structure over $C^\infty(M)$ by taking

$$<f,g> = \int_M f.g dv$$

for all $f,g \in C^\infty(M)$.

The completed of $C^\infty(M)$ with respect to this inner product is $L^2(M)$. An eigenvalue of $\Delta$ is a real number $\lambda$ such that $\Delta f = \lambda f$ for some non-trivial function $f \in C^\infty(M)$. The set of all eigenvalues of $\lambda$ will be called the spectrum of $(M,g)$ which will be denoted by $\text{Spec}(M,g)$.

Since $\Delta$ is elliptic, $\text{Spec}(M,g)$ is discrete and the multiplicity of each $\lambda$, that is, $\dim V_\lambda = \dim \{ f \in C^\infty(M) / \Delta f = \lambda f \}$ is finite. In particular, since $M$ is compact, $\dim V_0 = 1$ ($V_0$ consists only of constant functions). Therefore

$$\text{Spec}(M,g) = \{ 0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots \}$$

As it is well known, the decomposition $\bigoplus V_\lambda, \lambda \in \text{Spec}(M,g)$ is orthogonal with respect to (2.5). Moreover $\bigoplus V_\lambda$ is dense in $C^\infty(M)$ (in $L^2$-sense).

Similarly the spectrum of $\Delta$ acting over exterior $p$-forms will be denoted by $\text{Spec}^p(M,g)$. One has

$$\text{Spec}^p(M,g) = \{ p^0 \lambda_0 \ldots p^0 \lambda_1 \ldots p^0 \lambda_2 \ldots \}$$

From Hodge theory one has:

(i) If $n$ denotes the dimension of $M$, then $\text{Spec}^p(M,g) = \text{Spec}^{n-p}(M,g)$

(ii) $0$ belongs to $\text{Spec}^p(M,g)$ if and only if the $p$th Betti number of $M$ is non-zero (being this Betti number the multiplicity of $0$ in $\text{Spec}^p(M,g)$).

Now one can put the following general problems: Given a Riemannian manifold $(M,g)$, one has $\text{Spec}(M,g)$ which only depends on the Riemannian structure

(1) what is the behaviour of the elements of $\text{Spec}(M,g)$?

(2) what information can one obtain from $\text{Spec}(M,g)$ about the geometry and the topology of $(M,g)$ and $M$ respectively?

These kind of problems and others related are known under the general name of Spectral geometry.

Now, we will briefly treat about some of this problems:
A) How to determine $\text{Spec}(M,g)$ for a given Riemannian manifold?

This problem is very difficult and we only know a few about it:
- We can compute the spectrum of a Riemannian product from the spectra of the factors.
- We can also relate the spectra of two Riemannian manifolds connected by means of a Riemannian submersion with totally geodesic fibers.
- If $\dim M = 1$ and $L$ denotes the length of $(M,g)$ then one has

$$\text{Spec}(M,g) = \{4\pi^2 n^2/L^2 \text{ with } n \in N\}$$

each eigenvalue with multiplicity 2 except $\lambda_0 = 0$. Therefore, when $M$ is 1-dimensional, the spectrum characterizes the Riemannian structure and even $\lambda_1$ does.
- The spectrum is completely known for $(S^n, g_0)$, the $n$-dimensional sphere endowed with its canonical Riemannian structure. This is also true for compact symmetric spaces of rank one with their canonical structures.
- The spectrum is, in some sense, also completely known for symmetric spaces and some homogeneous spaces, at least in theory, [36].
- Consider a flat torus $M = \mathbb{R}^n/\Lambda$ where $\Lambda$ is any lattice of $\mathbb{R}^n$, endowed with $g_0/\Lambda$, $g_0$ being the canonical metric on $\mathbb{R}^n$ (this metric has no problems in going to the quotient since it is invariant by translations). We put

$$\Lambda^* = \{\xi \in \mathbb{R}^n/(\xi, \eta) \in Z, \text{for all } \eta \in \Lambda\}$$

the dual lattice of $\Lambda$. Then

$$\text{Spec}(\mathbb{R}^n/\Lambda, g_0/\Lambda) = \{4\pi^2 |\xi|^2/|\xi^\vee|^*\}$$

the multiplicity of $4\pi^2 |\xi|^2$ is the number of $\eta \in \Lambda^*$ with $|\eta|^2 = |\xi|^2$.

(B) Since it is very difficult to compute $\text{Spec}(M,g)$, one must obtain estimates of eigenvalues by means of geometrical invariants of $(M,g)$.

The more important results in this direction have been obviously obtained for the first non-trivial eigenvalue. Now we recall some of them.

The first result is due to A. Lichnerowicz [33] and M. Obata [38].
THEOREM 2.1. - Let \( M \) be an \( n \)-dimensional Riemannian manifold, \( n > 1 \). Suppose there exists a constant \( k > 0 \) such that the Ricci curvature \( S \) of \( M \) satisfies \( \frac{S}{n} \geq k \) (\( g \) being the metric tensor of \( M \)). Then:

(a) \( \lambda_1 \geq \frac{n}{(n-1)}k \) (Lichnerowicz)

(b) \( \lambda_1 = \frac{n}{(n-1)}k \) if and only if \( (M, g) \) is isometric to \( (S^n, g_o) \) (Obata).

This estimate of Lichnerowicz-Obata was extended later by R.C. Reilly ([44]) to manifolds with boundary where he treated the Dirichlet boundary value problem \( (\Delta u = F(u) \text{ with } u = 0 \text{ in } aM) \) under the additional assumption that \( aM \) has non-negative mean curvature.

There exist some classical papers due to Faber-Krahn, Polyá-Szegö, Payne, Weinberger etc. ([38]) which give estimates for the first eigenvalue of a domain \( \Omega \) in \( \mathbb{R}^n \). For instance an estimate due to Faber-Krahn for the first eigenvalue of a vibrating membrane \( \Omega \) in \( \mathbb{R}^2 \) is

\[
\lambda_1 \geq \frac{\pi}{\text{vol}(\Omega)} \]

the equality holding if and only if \( \Omega \) is a disk. Here \( \pi \) is the first zero of the first Bessel's function.

The estimate of Faber-Krahn can not be extended to Riemannian manifold, in fact one has the following examples:

- Consider the flat torus \( \mathbb{R}^2/\Lambda \) where \( \Lambda \) is the lattice of \( \mathbb{R}^2 \) of lengths \( (t, 1/t) \). Its area is 1, but for small values of \( t \), \( \lambda_1 = 4\pi^2 t^2 \) is close to zero.

- Consider two spheres \( E_1 \) and \( E_2 \) connected by means of a cylinder of radius \( r \) and length \( L \). One defines a function of measure zero \( f \) such that: \( f \) is constant \( k \) over \( E_1 \), \( f \) is constant-k over \( E_2 \) and \( f \) is linear over the cylinder being constant over its sections. Then one obtains

\[
|\nabla f|^2/|f|^2 \leq \frac{4\pi r}{(L\text{vol}(M))} \quad (\nabla f \text{ being the gradient of } f)
\]

but the inequality of Poincaré gives

\[
\int_M |\nabla f|^2 dv \geq \frac{1}{\lambda_1} \int_M f^2 dv \quad \text{for all } f \text{ with measure zero}
\]
therefore one obtains the following estimate

\[(2.12) \quad \lambda_1 \leq 4\pi r/(L\text{vol}(M))\]
	his estimate, for small values of \(r\), is close to zero while \(\text{vol}(M)\) is close to \(2\text{vol}(S^2, g_0)\).

These examples prove us that in a possible lower bound of \(\lambda_1\), must participate other geometrical invariants apart from the volume and the diameter.

After the work of Lichnerowicz and Obata, J.Cheeger ([12]) obtained a lower bound of \(\lambda_1\) by a constant which is involved in a certain type of isoperimetric inequality.

In [l] T.Aubin gave a lower bound of \(\lambda_1\) in terms of the following elements: a lower bound of the volume, an upper bound of the diameter, a lower bound of the sectional curvature, an upper bound of the Ricci curvature and a lower bound of the injectivity radius.

It seems obvious to relax the dependency of the Aubin estimate on the geometric quantities. In this sense, S.T.Yau ([60] showed that one can estimate \(\lambda_1\) from below by a lower bound involving the following elements: a lower bound of the volume, an upper bound of the diameter and a lower bound of the Ricci curvature. Moreover, basing on an upper bound of \(\lambda_1\) obtained by S.Y.Cheng ([21]), Yau conjectured that one should be able to drop the dependency of the volume in the last estimate. Therefore by combining with Cheng's result this would give the best possible estimate of \(\lambda_1\) for a general compact Riemannian manifold.

This conjecture was solved by P.Li ([31]). His method depended basically on a gradient estimate of the first eigenfunctions. In particular, for compact Riemannian manifolds with non-negative Ricci curvature, he obtained \(\lambda_1 \geq \pi^2/(4d^2)\), \(d\) being the diameter of the manifold. Because for these kind of spaces, the upper estimate of Cheng is \(\lambda_1 \leq \pi^2/\text{d}^2\), \(\text{n}\) being the dimension of the manifold. One has the following estimation of \(\lambda_1\) for a compact Riemannian manifold with non-negative Ricci curvature

\[(2.13) \quad \pi^2/4d^2 \leq \lambda_1 \leq \pi^2/\text{d}^2\]
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Recently, P.Li and S.T.Yau ([32]) extended the above estimates to compact manifolds with boundary:

(a) For Dirichlet boundary valued problem, the estimate also depends on the lower bound of the mean curvature of \(\mathfrak{M}\).

(b) For Neumann boundary valued problem, they need to assume that the second fundamental form of \(\mathfrak{M}\) is positive semidefinite.

When one considers \(\Delta\) acting over exterior p-forms, there exist some interesting estimates of \(P_{\lambda_1}\), in this way one can see for instance [24],[42],[51].

(c) To find the regularities of the eigenvalues sequence in the infinity (to study the behaviour of the spectrum in the limit).

This problem has been intensely studied and one has some satisfactory answers for it. The central idea can be explained as follows: Given a compact Riemannian manifold, one considers the function

\[
Z(t) = \sum_{\lambda=1}^{\infty} e^{-\lambda_1 t} \lambda_{1,\lambda} \text{Spec}(M,g)
\]

(the same problem can be considered for Spec\(^P\)(M,g)).

The function \(Z(t)\) is well defined for each \(t>0\) (by means of the fundamental solution of heat equation) and its behaviour when \(t \to 0^+\) is completely described by the well-known asymptotic expansion formula of Minakshisundaram-Pleijel-Gaffney

\[
\sum_{\lambda=1}^{\infty} e^{-\lambda_1 t} \approx (4\pi t)^{-n/2} (a_0 + a_1 t + a_2 t^2 + \ldots)
\]

the coefficients \(a_\lambda\) involving only Riemannian invariants of corresponding degrees. For example, one knows a few coefficients:

\[
a_0 = \text{vol}(M)
\]

\[
a_1 = 1/6 \int_M \rho \, dv
\]

\[
a_2 = 1/360 \int_M (2|R|^2 - 2|S|^2 + 5p^2) \, dv
\]

where \(R, S\) and \(\rho\) denotes respectively the Riemannian curvature tensor, the Ricci curvature and the scalar curvature (see for instance [9]).

The coefficient \(a_3\) has been computed for T.Sakai [49].

3. THE INVERSE PROBLEM. As we said before, one can think in the
following problem: Can one hear the shape of a manifold? (manifolds are considered like vibrating membranes because they are not boundary). This problem can be formulated with respect to the injectivity of a map whose image looks too formidable today (to see the first problem). In fact one considers the map

$$\text{Spec} : \{\text{Riemannian structures}\} \rightarrow \mathbb{R}^N$$

is it injective?.

We know that the injectivity of this map is false in general. In fact Witt discovered in 1941 two lattices $\Lambda_1$ and $\Lambda_2$ in $\mathbb{R}^{16}$ which are not isometric but with the same number of elements of any given norm. Therefore the two corresponding flat torus have the same spectrum (2.10) and they are not isometric [35].

In 1967, M. Kneser, [29], obtained the same result for two flat torus of dimension 12.

Recently, M. Vigneras, [58], obtained two surfaces of Poincaré (that is, surfaces with constant curvature $-1$) which have the same spectrum but they are not isometric.

Since these examples have a discrete character, an open problem is to decide whether or not the exists a nontrivial family $(M, g(t))$ of Riemannian structures over $M$ with $\text{Spec}(M, g(t)) = \text{Spec}(M, g(o))$ for every $t$ (isospectral deformation). In this sense, V. Guillemin and D. Kazhdan ([25]) showed that under certain conditions, every isospectral deformation of a negatively curved Riemannian manifold is isometric (and so the deformation is trivial).

For Kaehlerian isospectral deformations, one knows that the complex projective space $\mathbb{CP}^n$ with its canonical Kaehlerian structure does not admit non-trivial Kaehlerian isospectral deformations, [9], (here Kaehlerian deformations means that each metric in the deformation must be Kaehlerian with the fixed complex structure). In the same way M. Barros, [3], showed that the complex quadric of complex dimension 3 with its canonical Kaehlerian structure (which is obtained from the fact that it is a complex hypersurface of $\mathbb{CP}^4$) does not admit non-trivial Kaehlerian isospectral deformation.

Also, we can think of some special cases of injectivity, for ins-
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tance, one can study the following problem: Whence a manifold can be characterized by its spectrum?. It is first that the first candidate to be characterized by its spectrum. It be the sphere with its canonical metric. Many authors obtained partial answers to the following problem: If $\text{Spec}^p(M, g) = \text{Spec}^p(s^n, g_o)$ for some $p$, then $(M, g)$ is isometric to $(s^n, g_o)$.

For instance $(s^2, g_o)$ is characterized by its spectrum over functions ([9]). S.Tanno ([56]), by using the spectrum of $A$ acting over 1-forms gave an affirmative answer to the above problem for $n=2, 3, 16, \ldots 93$.

Other candidate to be characterized by its spectrum is the complex projective space $\mathbb{C}P^n$ endowed with the Fubini-Study metric $g_o$, which, as it is well-known is the corresponding model in Kaehlerian geometry with constant positive holomorphic sectional curvature. Therefore the corresponding problem is: Let $(M, J, g)$ a Kaehlerian manifold and assume that $\text{Spec}^p(M, g) = \text{Spec}^p(\mathbb{C}P^n, g_o)$. Can we say that $(M, J, g)$ is holomorphically isometric to $(\mathbb{C}P^n, J_o, g_o)$? (here $J_o$ denotes the canonical complex structure over $\mathbb{C}P^n$).

Probably the most complete answer to this problem was obtained by B.Y. Chen and L.Vanhecke, ([20]), which gave an affirmative answer to the above problem when $p=2$ and $n\neq 8$.

An important contribution to this problem was obtained by M.Barras and B.Y.Chen, ([5]): Let $Q_2$ be the complex quadric of complex dimension 2. In general $Q_n$ is defined extrinsically as a complex hypersurface of $\mathbb{C}P^n$ and after linear submanifolds it is the most important submanifold of $\mathbb{C}P^{n+1}$, (it will play an important role in this paper), then

**Theorem 3.1.** - Let $(M, J, g)$ be a Kaehler surface with $\text{Spec}(M, g) = \text{Spec}(Q_2, g_o)$. If $(M, J, g)$ satisfies one of the following conditions:

1. the arithmetic genus $a(M) \geq 1$
2. the Hirzebruch index $t(M) \leq 0$
3. the Euler characteristic $\chi(M) \geq 4$

then $(M, J, g)$ is holomorphically isometric to $(Q_2, J_o, g_o)$. 
Proof.- We will denote by $S$ (respectively $S_0$) the Ricci tensor of $M$ (respectively $Q_2$) and by $\rho$ (respectively $\rho_0$) the scalar curvature of $M$ (respectively of $Q_2$). Each assumption (1), (2) or (3) implies an integral inequality for $M$ and $Q_2$ in terms of their corresponding quadratic invariants. For instance (3) implies

\[ \int_M (|R|^2 - 4|S|^2 + \rho_2^2)\,dv \geq \int_{Q_2} (|R_0^2|^2 - 4|S_0|^2 + \rho_0^2)\,dv_0 \]

therefore combining this inequality with the equality $a_2 = a_2^0$ where $a_2$ (respectively $a_2^0$) denotes the corresponding coefficient in the Minasian-Pleijel-Gaffney asymptotic expansion for $M$ (respectively for $Q_2$), one obtains

\[ \int_M (2|S|^2 + \rho_2^2)\,dv \leq \int_{Q_2} (2|S_0|^2 + \rho_0^2)\,dv_0 \]

We recall that in general $|S|^2 \geq \rho^2/4$ and the equality holds if and only if $(M,J,g)$ is Einstein. Then by using this fact joint (3.2) and the fact that $Q_2$ is Einstein (and so $\rho_0$ is constant) joint the Schwartz inequality, one can prove that $(M,J,g)$ is an Einstein Kaehler surface. Now we use a well-known result due to H. Donelly, ([22]) to prove that $(M,g)$ is a symmetric space which must be simply connected because its Ricci tensor is definite positive, ([30]). Therefore $(M,J,g)$ is a Hermitian symmetric space. If it is irreducible, it must be $CP^2$ ([27]) but this contradicts every assumption (1), (2) or (3). Thus $(M,J,g)$ is a reducible Hermitian symmetric space and so the Riemannian product of two complex projective lines and holomorphically isometric to $(Q_2,J_0,g_0)$.

Remark.- (1) For general dimension, the problem of obtaining intrinsic characterizations of $Q_2$ by its spectrum looks formidable today. (2) $Q_2$ is the first space characterized by its spectrum which has not constant curvature (in the correct sense). (3) The above result has important consequences, for instance, $(Q_2,J_0,g_0)$ is completely characterized by its spectrum among all rational surfaces. Also among all topological reducible Kaehler surfaces. In these cases, $a(M) = 1$ and $t(M) = 0$, respectively.

There exists other important contributions to the inverse problem, to see for instance [43] and [57].
4. BASIC FORMULAS IN THEORY OF SUBMANIFOLDS. Now we will recall some fundamental facts in theory of submanifolds (a fundamental reference for that is [13]). Let $M^n$ and $\mathbb{M}^{n+p}$ be two Riemannian manifolds with dimension $n$ and $n+p$ respectively. We will denote by $\nabla$ and $\nabla^\perp$ the Levi-Civita connections of $M$ and $\mathbb{M}$ respectively. Now we assume that $f : M \to \mathbb{M}$ is an isometric immersion, then for any point $x \in M$ one has: The tangent space of $M$ at $x$, $T_x M$; the normal space of $M$ via $f$ at $x$, $(T_x^\perp M)_f$ (sometimes we will omit $f$) and the connection $\nabla^\perp$ determined by $f$ in the normal vector bundle $T_x^\perp M$. Also $\mathfrak{X}(M)$ will denote the Lie algebra of vector fields tangent to $M$ and $\mathfrak{X}^\perp(M)$ the space of vector fields normal to $M$ via $f$.

As it is well-known, the most important geometric object associated to $f$ is its second fundamental form $\sigma$, which is well defined via the Gauss equation
\begin{equation}
\sigma(X,Y) = \nabla_X Y - \nabla^\perp_X Y
\end{equation}
for all $X,Y \in \mathfrak{X}(M)$. That is, $\sigma(X,Y)$ is the normal component of $\nabla_X Y$. Properties of symmetry and bilinearity of $\sigma$ are well-known.

The Weingarten endomorphisms of $f$ are defined by means of Weingarten's formula
\begin{equation}
A_X \xi = \nabla_X \xi - \nabla^\perp_X \xi
\end{equation}
for all $X,Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}(M)^\perp$. That is $A_X \xi$ is, up the sign, the tangential component of $\nabla_X \xi$.

One has the relation
\begin{equation}
g(A_X \xi,Y) = g(\sigma(X,Y),\xi) \quad \text{for all } X,Y \in \mathfrak{X}(M) \text{ and } \xi \in \mathfrak{X}(M)^\perp
\end{equation}
g denoting the metric over $M$ and $\mathbb{M}$.

The mean curvature vector of $f$ is a vector field $H \in \mathfrak{X}(M)^\perp$ given by
\begin{equation}
H = \left(1/n\sum_{i=1}^{n} \sigma(E_i, E_i)\right)
\end{equation}
where $\{E_i\}i=1,...,n$ denotes an orthonormal basis for the tangent space of $M$ at the corresponding point.
One can define the covariant derivative of $\sigma$ by
\[(4.5) \quad (\nabla \sigma)_X(Y,Z) = \nabla_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)\]
for all $X, Y, Z \in \mathcal{X}(M)$.

- If $\sigma = 0$, the isometric immersion $f$ is called totally geodesic.
- If $H = 0$, the isometric immersion $f$ is called minimal.
- If $\nabla \sigma = 0$, the isometric immersion $f$ is called parallel.

We will use the following sign for the Riemannian curvature $R$ of $M$ (the same for others curvatures)
\[(4.6) R(X,Y,Z,W) = g(((\nabla_X \nabla_Y - \nabla_Y \nabla_X)z, W) for all X, Y, Z, W \in \mathcal{X}(M)\]

The equations of Gauss and Codazzi for the curvature are given by
\[(4.7) \quad (\nabla \sigma)_X(Y,Z) = (\nabla \sigma)_Y(X,Z) \quad \text{in both equations} \quad X, Y, Z, W \in \mathcal{X}(M)\]
and
\[(4.8) \quad (\nabla \sigma)_X(Y,Z) = \nabla _X \sigma(Y,Z) - \nabla _Y \sigma(Y,Z) \quad \text{for all} \quad X, Y \in \mathcal{X}(M)\]

If $R^\perp$ denotes the curvature tensor of $\psi^\perp$, that is,
\[(4.9) \quad R^\perp(X,Y) = \nabla _X \nabla _Y - \nabla _Y \nabla _X - [A, A]_X \text{ and } X, Y \in \mathcal{X}(M)\]
then the equation of Ricci is
\[(4.10) \quad R(X,Y,\xi,\eta) = R^\perp(X,Y,\xi,\eta) - g([A_\xi, A_\eta]X,Y) \quad \text{for all} \quad X, Y \in \mathcal{X}(M)\]
connection over $E^m$. If $A$ is the Laplacian of $M$ acting over $C^\infty(M)$, one can extend $A$ to $E^m$-valued functions over $M$ in a natural way, if $F: M \to E^m$ with $F = (F^1, \ldots, F^m)$ with respect to a fixed basis on $E^m$, then one puts

$$\Delta F = (\Delta F^1, \ldots, \Delta F^m)$$

it is clear that this definition does not depend over the chosen basis in $E^m$. Moreover if $a$ denotes a fixed vector in $E^m$, one has

$$\Delta \langle a, F \rangle = \langle a, \Delta F \rangle \quad \langle , \rangle \text{ being the Euclidean inner product.}$$

In particular, if we consider $\psi: M \to E^m$, from (5.1) and (5.2) one get

$$\psi = \lim_{t \to 0} \psi_t \quad \psi_t: M \to E^m \quad \text{with} \quad \Delta \psi_t = \lambda \psi_t$$

this sequence is convergent coordinate to coordinate in the $E^m$-sense over $C^\infty(M)$. Moreover $\psi_0$ is constant and it is called the gravity center of $\psi$.

$$\psi_0 = \left(1/\text{vol}(M)\right) \int_M \psi(M) \, dv$$

Since the decomposition $\sum k V_k$ is orthogonal, one has too

$$\int_M \langle \psi_u, \psi_v \rangle = 0 \quad \text{for} \quad u \neq v$$

Now we will establish a very interesting theorem due to T. Takahashi ([54]), but before we will consider the following

**Lemma 5.1.** Let $\psi: M^n \to E^m$ an isometric immersion of an n-dimensional Riemannian manifold in the Euclidean space. If $H$ denotes the mean curvature vector field, then

$$\Delta \psi = -nH$$

**Proof.** Let $a$ be a fixed vector in $E^m$ and consider $\langle a, \psi \rangle: M^n \to k$. If $X, Y \in \chi(M)$, direct computations give us

$$(d\langle a, \psi \rangle)(X) = X(\langle a, \psi \rangle) = \langle a, X \rangle$$

$$(\text{Hess} \langle a, \psi \rangle)(X, Y) = XY\langle a, \psi \rangle - (\nabla_X Y)(\langle a, \psi \rangle) = \langle a, D_X Y \rangle - \langle a, \nabla_Y X \rangle = \langle a, \sigma(X, Y) \rangle$$
now from (2.4), one has

\[ -\Delta \langle a, \psi \rangle = \sum_{i=1}^{n} \text{Hess} \langle a, \psi \rangle(E_i, E_i) = \langle a, \sum_{i=1}^{n} \sigma(E_i, E_i) \rangle = \langle a, nH \rangle \]

because last formula is true for every \( a \) in the Euclidean space, one obtains (5.7).

It is easy to prove the following relations

\[ \Delta \psi = -nH = \sum_{t \geq 1} \lambda_t \psi_t \]  \hspace{1cm} (5.8)
\[ \Delta^k \psi = \sum_{t \geq 1} \lambda_t^k \psi_t \]  \hspace{1cm} (5.9)

Definition.- Let \( \omega \) a finite subset of \( N \). We will say that the isometric immersion \( \psi: M \rightarrow E^m \) is of order \( \omega \) if \( \psi_t = 0 \) for all \( t \in N - (\omega \cup \{0\}) \).

In particular if \( \omega = \{u\} \) and \( \psi \) is an isometric immersion of order \( u \), we will say that \( \psi \) is of order \( u \). These isometric immersion or their corresponding submanifolds have been called by B.Y.Chen ([16]) submanifolds of finite type. According the number of elements in \( \omega \) we will use the terminology: mono-order, bi-order,...

Since our manifolds are assumed to be compact, we will give the compact version of Takahashi's theorem

**Theorem 5.2.** Let \( \psi: M \rightarrow E^m \) be an isometric immersion of an \( n \)-dimensional kiemannian manifold in the Euclidean space. Then \( \psi \) is of order \( u \), for some \( u \geq 1 \) (mono-order) if and only if \( \psi \) is minimal in some hypersphere of \( E^m \). Moreover if \( \psi \) is of order \( u \), then \( \lambda_u = \frac{n}{r^2} \) being \( r \) the radius of the corresponding hypersphere.

**Proof.** Suppose that \( \psi \) is of order \( u \), for some \( u \geq 1 \), then

\[ \psi = \psi_0 + \psi_u \]  \hspace{1cm} (5.10)
\[ \Delta \psi_u = \lambda_u \psi_u \]

Now from (5.7) and (5.10) one get

\[ \lambda_u (\psi - \psi_0) = -nH \]  \hspace{1cm} (5.11)

First we will prove that \( \psi(M) \) is contained in some hypersphere \( S \) of center \( \psi_0 \). Let \( X \) be any vector field tangent to \( M \), then

\[ X < \psi - \psi_0, \psi - \psi_0 > = 2 < \psi - \psi_0, X > = -2(\mu / \lambda_u) < H, X > = 0 \]

If we denote by \( \sigma \) (respectively \( \sigma' \)) the second fundamental form of \( M \) in \( S \) (respectively of \( S \) in \( E^m \)) and by \( H' \) (respectively \( R \)) the
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mean curvature vector of $M$ in $S$ (respectively of $S$ in $E^m$), then

\[(5.12)\quad H = H' + (1/n) \sum_{i=1}^{n} \bar{g}(E_i, E_i)\]

From (5.11) we see that $H$ is normal to $S$, then (5.12) implies that $H'=0$ and so $M$ is minimal in $S$.

The converse is trivial.

Let $r$ be the radius of $S$ and $\xi = (v - v_0)/r$ a unit vector normal to $S$. Since $A_\xi = -(1/r)I$, from (5.12) one get

\[(5.13)\quad H = -(v - v_0)/r^2\]

now from (5.11) and (5.13) one obtains that $\lambda_u = n/r^2$.

Remark.- The above result proves us that the spectrum of a Riemannian manifold $M$ contains information about the possible radius of spheres in which $M$ admits a minimal isometric immersion.

In order to finish this section we will give two results ([16], [47]). The first one can be looked like a formal generalization of the situation which we have found in Takahashi's theorem. The second one will be very useful along this paper.

**Proposition 5.3.** Let $\psi: M^n \rightarrow E^m$ be an isometric immersion, then the following conditions are equivalent:

(i) $\psi$ is of order $\omega$, for some finite subset $\omega$ of $N$, with $\#(\omega) = k \geq 1$.

(ii) There exists real numbers $a_0, a_1, \ldots, a_{k-1}$ such that

\[(5.14)\quad \Delta^k \psi = a_0 (v - v_0) + a_1 \Delta \psi + \ldots + a_{k-1} \Delta^{k-1} \psi\]

$\psi_0$ being the center of gravity of $\psi$.

**Proof.** Suppose $\omega = \{u_1, \ldots, u_k\}$ and consider the elementary symmetric functions $s_1, \ldots, s_k$ of $(\lambda_{u_1}, \ldots, \lambda_{u_k})$. From $\psi_0 = \psi + \psi_1 + \ldots + \psi_k$ one has

\[(r^k \psi)_r = \lambda_u^{r_1} u_1^{r_1} + \ldots + \lambda_u^{r_k} u_k^{r_k}\]

for all $r$.

So some direct computations give us

\[\Delta^k \psi = s_1 \Delta^{k-1} \psi - s_2 \Delta^{k-2} \psi + \ldots + (-1)^{k-1} s_k (v - v_0)\]

Conversely, assume $\Delta^k \psi = a_0 (v - v_0) + a_1 \Delta \psi + \ldots + a_{k-1} \Delta^{k-1} \psi$ for certain
real numbers $a_0, a_1, \ldots, a_{k-1}$, from (5.4) and (5.9) one obtains

\[
\int_{u \geq 1} \lambda^k u \psi u = a_0 \int_{u \geq 1} \psi u + a_1 \sum_{u \geq 1} \lambda u \psi u + \ldots + a_{k-1} \sum_{u \geq 1} \lambda^{k-1} u \psi u \quad \text{and}
\]

\[
\int_{u \geq 1} (\lambda^k a_0 - a_1 \lambda u - \ldots - a_{k-1} \lambda^{k-1} u) \psi u = 0
\]

since this sequence is $L^2$-convergent, for any $t \in \mathbb{N}$, $t \geq 1$, we apply

\[
\int_{M} \langle \psi, \psi \rangle \, dv
\]

to last formula and so

\[
(\lambda^k t - a_0 - a_1 \lambda t - \ldots - a_{k-1} \lambda^{k-1}) \int_{M} \langle \psi, \psi \rangle \, dv = 0
\]

which implies that $\psi_t = 0$ for all $t \geq 1$ except at most for $k$ different values of $t$ (just for the solutions of the equation $k^t - a_0 - a_1 t - \ldots - a_{k-1} \lambda^{k-1} t = 0$).

**PROPOSITION 5.4.** Let $\psi : M^n \rightarrow E^m$ be an isometric immersion. For any two natural numbers $k, t \geq 1$, one has

\[
\int_{M} \{ \langle \psi, \psi \rangle - s_1 \langle \psi, \psi \rangle + s_2 \langle \psi, \psi \rangle + \ldots + (-1)^k s_k \langle \psi, \psi \rangle \} \, dv \geq 0
\]

$s_i$ being the elementary symmetric functions of $(\lambda_1, \ldots, \lambda_k)$. The equality holds if and only if $\psi$ is of order $(1, 2, \ldots, k)$.

**Proof.** From (5.9) and direct computations, one has:

\[
\begin{align*}
\Delta^{k+t} \psi - s_1 \Delta^{k+t-1} \psi + s_2 \Delta^{k+t-2} \psi + \ldots + (-1)^k s_k \Delta^t \psi = \\
= \int_{u \geq 1} \lambda^k u (\lambda u - \lambda_1)(\lambda u - \lambda_2) \ldots (\lambda u - \lambda_k) \psi u = \\
= \int_{u \geq 1} \lambda^k u (\lambda u - \lambda_1) \ldots (\lambda u - \lambda_k) \psi u
\end{align*}
\]

By applying $\int_{M} \langle \psi, \psi \rangle \, dv$ to the last formula, one get

\[
\begin{align*}
\int_{M} \{ \langle \psi, \psi \rangle - s_1 \langle \psi, \psi \rangle + s_2 \langle \psi, \psi \rangle + \ldots + (-1)^k s_k \langle \psi, \psi \rangle \} \, dv = \\
= \int_{u \geq 1} \lambda^k u (\lambda u - \lambda_1) \ldots (\lambda u - \lambda_k) \int_{M} \langle \psi, \psi \rangle \, dv \geq 0
\end{align*}
\]
and the equality holds if and only if $\psi_u = 0$ for all $u \in \mathbb{N}$ with $u > k$, that is, if and only if $\psi$ is of order $\{1, 2, \ldots, k\}$.

6. SUBMANIFOLDS OF A KAHLERIAN MANIFOLD. Let $(\tilde{M}, J, g)$ be a Kaehlerian manifold with complex structure $J$. With respect to a Riemannian manifold, now one has an additional structure $J$ and so it seems natural to study submanifolds in $\tilde{M}$ according to their behaviours with respect to $J$.

In this sense, the first kind of submanifolds of a Kaehler manifold, which has been studied, is the family of complex submanifolds. If $M$ is a submanifold of $(\tilde{M}, J, g)$, $M$ is called complex if its tangent space at any point is $J$-invariant

$$J(T_x M) = T_x M \quad \text{for all} \ x \in M$$

in this case $J$ can be induced over $M$ and so, $M$ is already a Kaehlerian manifold. Therefore, we will say Kaehler submanifolds.

If a submanifold $M$ of $(\tilde{M}, J, g)$ satisfies

$$J(T_x M) \subset T_x^\perp M \quad \text{for all} \ x \in M$$

then $M$ is called a totally real submanifold. The family of totally real submanifolds of a Kaehlerian manifold has been studied by many authors too.

Let $M$ be a real hypersurface of a $2m$-dimensional Kaehlerian manifold, then one can choose a local unit normal vector field $\xi$ and so, one can put

$$T_x M = <J\xi_x> \oplus D(x)$$

where $D(x)$ denotes the orthogonal complement of $<J\xi_x>$ in $T_x M$ and it is obvious that it is the greatest holomorphic subspace on $T_x M$.

We see that a real hypersurface of a Kaehlerian manifold is never a Kaehlerian (respectively, totally real) submanifold. By extending the behaviour of a real hypersurface with respect to the complex structure of a Kaehlerian manifold, A. Bejancu ([10]) defined the concept of CR-submanifold of a Kaehlerian manifold as follows: A submanifold $M$ of a Kaehlerian manifold $(\tilde{M}, J, g)$ is called a CR-submanifold if it admits a pair of distributions $(D, D^\perp)$ such that
(1) $D^\perp$ is the orthogonal complement of $D$

(2) $D$ is holomorphic and $D^\perp$ is totally real.

It is obvious that Kaehler submanifolds, totally real submanifolds and real hypersurfaces are CR-submanifolds.

Take $\tilde{M}=\mathbb{CP}^{n+p}$, the $(n+p)$-dimensional complex projective space endowed of its canonical complex structure $J_0$ and the Fubini-Study metric $g_0$ of holomorphic sectional curvature $1$. Let $M^n$ be a Kaehler submanifold of $\mathbb{CP}^{n+p}$ with complex dimension $n$. We choose an orthonormal basis of vector fields over $\mathbb{CP}^{n+p}$ $\{E_1, \ldots, E_n, JE_1 = E_1^*, \ldots, JE_n = E_n^*\}$ which is adapted to $M$. We will use the following convention on the range of indices unless otherwise state:

- $i, j, k, r, s = 1, \ldots, n^n$
- $i^* = \{a^* \text{ if } i = a\}$
- $i^* = \{a \text{ if } i = a^*\}$
- $\lambda, \mu, = 1, \ldots, p^n$
- $\lambda^* = \{a^* \text{ if } \lambda = a\}$
- $\lambda^* = \{a \text{ if } \lambda = a^*\}$

We also use the following nomenclature:

- $\nabla$ = the Levi-Civita connection of $M$
- $\sigma$ = the second fundamental form of $M$ in $\mathbb{CP}^{n+p}$
- $\nabla^\perp$ = the normal connection over $T^\perp M$ (the normal bundle of $M$ in $\mathbb{CP}^{n+p}$)
- $A$ = the Weingarten endomorphism of $M$ in $\mathbb{CP}^{n+p}$.

The normal bundle $T^\perp M$ of $M$ is holomorphic and we will denote by $J$ the complex structures over $T^\perp M$ and also over $\mathcal{T} M$. Since $\mathbb{CP}^{n+p}$ is a Kaehlerian manifold, the Gauss equation implies

\[(6.4) \quad \sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y) \quad \text{for all } X, Y \in \mathcal{T} M\]

therefore one has

\[(6.5) \quad A_{\xi^*} = JA_\xi \quad \text{and} \quad JA_\xi = -A_{\xi^*} J \quad \text{for all } \xi \in \mathcal{T} M\]

as consequence $M$ is minimal (Every complex submanifold of a Kaehlerian manifold is minimal).

If $R$ denotes the curvature tensor of $M$, the Gauss equation is:

\[(6.6) \quad R(X, Y, Z, W) = \tau(X, W, Y, Z) - \tau(X, Z, Y, W) + (1/4)\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) + \} g(X, JY)g(JZ, W)\]
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Now the Ricci curvature $S$ of $M$ is:

\begin{equation}
S(X,Y) = \frac{n+1}{2} g(X,Y) - \sum_{\lambda} g(A^2_{\lambda}X,Y),
\end{equation}

and the scalar curvature $\rho$ of $M$ is:

\begin{equation}
\rho = n(n+1) - |\sigma|^2,
\end{equation}

where $|\sigma|$ denotes the length of the second fundamental form $\sigma$,

\begin{equation}
|\sigma|^2 = \text{tr} \sum_{\lambda} A^2_{\lambda} = \sum_{i,j} h^i_j h^j_i,
\end{equation}

$\sigma(E_i,E_j) = \sum_{\lambda} h^i_j h^j_i$. Finally, the holomorphic sectional curvature determined by a unit vector $x$ of $M$ is given by:

\begin{equation}
H(x) = 1 - 2|\sigma(x,x)|^2.
\end{equation}

The following two lemmas are well-known (any-way their proofs are very direct):

**Lemma 6.1.** Let $M^n$ be an $n$-dimensional complex submanifold of $\mathbb{C}P^{n+p}$, then:

\begin{equation}
|S|^2 = \frac{1}{2} n(n+1)^2 - (n+1) |\sigma|^2 + \text{tr}(\sum_{\lambda} A^2_{\lambda}),
\end{equation}

\begin{equation}
|R|^2 = 2n(n+1) - 4 |\sigma|^2 + 2 \sum_{\lambda, \mu} (\text{tr} A_{\lambda A_{\mu}})^2,
\end{equation}

\begin{equation}
-\frac{1}{2} |\sigma|^2 = |\nu| + \frac{n+2}{2} |\sigma|^2 - 2 \sum_{\lambda, \mu} (\text{tr} A_{\lambda A_{\mu}})^2 - \sum_{\lambda, \mu} (\text{tr} A_{\lambda A_{\mu}})^2.
\end{equation}

**Lemma 6.2.** Let $N^m$ be a Kaehler manifold of complex dimension $m$, then:

\begin{equation}
\frac{1}{2}(m+1)m |R|^2 \geq 2m |S|^2 \geq \rho^2.
\end{equation}

Moreover:

(i) the first equality holds if and only if $N$ has constant holomorphic sectional curvature.

(ii) the second equality holds if and only if $N$ is Einstein.

From (6.8), (6.11) and the second inequality of (6.13) one get

\begin{equation}
\text{tr}(\sum_{\lambda} A^2_{\lambda}) \geq \frac{1}{2n} |\sigma|^4,
\end{equation}

the equality holds if and only if $M$ is Einstein.

Also we will consider the following result due to B. Y. Chen and K. Ogiue [19]:
THEOREM 6.3.- Let $M$ be an algebraic hypersurface of degree $d$ in $\mathbb{C}P^{n+1}$. Then we have

$$\int_M |R|^2 - 4|S|^2 + 2} \, dv = 2n(n-1)\{(n^2 + (n+2)d + d^2) \, \text{vol}(M)$$

7. EXTRINSIC CHARACTERIZATIONS OF SOME COMPLEX SUBMANIFOLDS IN THE COMPLEX PROJECTIVE SPACE. In section 3 we gave two interesting results about the so called "inverse problem" for the complex quadric of small dimension. These results was intrinsic.

In order to study the inverse problem for any complex quadric one firstly must solve the following special problem: Let $M^n$ be an $n$-dimensional Kaehler submanifold of $\mathbb{C}P^{n+q}$ such that $\text{Spec}^p(M,g) = \text{Spec}^p(Q_n, g_0)$ for some $p$, is it true that $M$ is congruent to $Q_n$? Of course, here the Kaehler structures over $M$ and $Q_n$ are the induced structures like complex submanifolds of $\mathbb{C}P^{n+q}$. In this sense, we will call extrinsic characterization of $Q_n$ by its spectrum to any answer of the above problem.

Some partial answer to this problem have been obtained, [2, 1, 3, 4, 5], we will give here some of them.

'THEOREM 7.1, [5].- Let $(M, J, g)$ be an $n$-dimensional Kaehler manifold with $\text{Spec}^p(M,g) = \text{Spec}^p(Q_n, g_0)$ for some $p$ ($p=0,1,\ldots,n$). If $(M, J, g)$ can be holomorphically isometrically imbedded in $\mathbb{C}P^m$ for some $m$, then $(M, J, g)$ is holomorphically isometric to $(Q_n, J_0, g_0)$.

Proof.- The main idea of the proof is to use some arguments of Algebraic Geometry in order to reduce our problem to consider complex hypersurfaces. In fact, because $(M, J, g)$ is holomorphically isometrically imbedded in some $\mathbb{C}P^m$, from a well known result due to Chow,
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M is a nonsingular algebraic variety. As usual, we shall identify M with its image in $\mathbb{C}^m$.

Let $L$ be an $(m-n)$-dimensional linear subspace of $\mathbb{C}^m$ and let $\lambda$ be the number of points in $M \cap L$. From the dimensional theory for algebraic manifolds, one obtains that $\lambda$ does not depend on the choice of $L$, if $L$ is in general position; $\lambda$ is called the degree of $M$. Moreover from a classical theorem due to W. Wirtinger,[59], one has,

(7.1) \[ \text{vol}(M) = \lambda(4\pi)^n/n! \]

On the other hand, since $Q_n$ is an algebraic hypersurface of degree 2 in $\mathbb{C}^{n+1}$, we have

(7.2) \[ \text{vol}(Q_n) = 2(4\pi)^{n}/n! \]

and so $\lambda=2$ because the volume is a spectral invariant. That is, $M$ is an algebraic submanifold of degree 2 in $\mathbb{C}^m$. Now we use another well known result due to W. Barth,[7], to obtain $M$ is contained in an $(n+1)$ dimensional linear subspace of $\mathbb{C}^m$. Therefore, from theorem 6.3, one has,

(7.3) \[ \int_M \{ |R|^2-4|S|^2+\rho^2 \}dv = \int_{Q_n} \{ |R_0|^2-4|S_0|^2+\rho_0^2 \}dv_0 \]

Now we use the coefficients of Minakshisundaram-Pleijel-Gaffney formula joint the Schwartz inequality and the Einsteinian character of $Q_n$ to obtain

(7.4) \[ \int_M \{ 2n|S|^2-\rho^2 \}dv < 0 \]

but this formula joint lemma 6.2 imply that $M$ is Einstein. Finally one uses a well-known result due to B. Smyth,[52], to conclude $(M,J,g)$ is holomorphically isometric to $(Q_n,J_0,g_0)$.

When one only assumes that $M$ is immersed, the extrinsic characterizations are weaker than the above case. In a direct attack, it seems natural to try find spectral invariants for Kaehler submanifolds of $\mathbb{C}^m$ in terms of elements close to the immersion (for instance, in terms of the second fundamental form). In this line one has the following:
LEMMA 7.2. Let $M_1$ and $M_2$ be two Kaehler submanifolds in $\mathbb{CP}^m$. If $\text{Spec}^p(M_1, g_1) = \text{Spec}^p(M_2, g_2)$ for some $p$, then one has

\begin{equation}
\int_{M_1} (2|\sigma_1|^2 - 4|s_1|^2 - |R_1|^2) dv_1 = \int_{M_2} (2|\sigma_2|^2 - 4|s_2|^2 - |R_2|^2) dv_2
\end{equation}

This result allows us to give some extrinsic characterizations of complex quadrics by its spectrum, for instance:

THEOREM 7.3. [5]. Let $(M, J, g)$ be an $n$-dimensional Kaehler manifold $(n \geq 5)$ with $\text{Spec}^2(M, g) = \text{Spec}^2(Q_n, g_0)$. If $(M, J, g)$ can be holomorphically isometrically immersed in $\mathbb{CP}^m$, for some $m$, then $(M, J, g)$ is holomorphically isometric to $(Q_n, J_0, g_0)$.

Remark.- Since the spectral invariant obtained in lemma 7.2, already only depends on the volume and the total scalar curvature, one can characterize the Veronese imbedding (see 12.2) by its volume and its total scalar curvature (which are two spectral invariants) among all parallel Kaehler submanifolds of the complex projective space, [6]. In particular this imbedding is characterized by its spectrum among all the above submanifolds.
8. THE STANDARD IMBEDDING OF THE COMPLEX PROJECTIVE SPACE IN EUCLIDEAN SPACE. In this section we will describe with some details a very nice isometric immersion of the complex projective space in a Euclidean space which will play an important role in this paper. We will call it the standard imbedding. This imbedding has been used for some authors to study different problems, [17], [34], [46], [50], [53] etc.

Let \( \mathbb{C}^{m+1} \) be the usual \((m+1)\)-dimensional complex Euclidean space endowed with the usual inner product

\[
\langle z, w \rangle = \text{Re}(z \cdot \overline{w}^t)
\]

for \( z = (z_0, \ldots, z_m) \) and \( w = (w_0, \ldots, w_m) \) in \( \mathbb{C}^{m+1} \) (\( \overline{w} \) means conjugate and \( w^t \) transpose). \( \mathbb{C}^{m+1} \) can be considered like vector space and so \( \mathbb{C}P^m \) is defined as the set of all complex lines of \( \mathbb{C}^{m+1} \). Let \( \text{End}(\mathbb{C}^{m+1}) \) be the space of all complex endomorphisms of \( \mathbb{C}^{m+1} \).

One defines a mapping \( \phi: \mathbb{C}P^m \rightarrow \text{End}(\mathbb{C}^{m+1}) \) as follows: If \( \pi \) is a complex line of \( \mathbb{C}^{m+1} \), then \( \phi(\pi): \mathbb{C}^{m+1} \rightarrow \mathbb{C}^{m+1} \) is simply the orthogonal projection over \( \pi \). One notices the following facts on \( \phi \):

1. \( \phi \) is one-to-one
2. \( (\phi(\pi))^2 = \phi(\pi) \) for all \( \pi \in \mathbb{C}P^m \)
3. \( \text{rank}(\phi(\pi)) = 1 \) for all \( \pi \in \mathbb{C}P^m \)
4. \( \phi(\pi) \) is self-adjoint with respect to the inner product on \( \mathbb{C}^{m+1} \), for all \( \pi \in \mathbb{C}P^m \).

Conversely, any element in \( \text{End}(\mathbb{C}^{m+1}) \) satisfying conditions (2), (3) and (4) is the orthogonal projection over some complex line of \( \mathbb{C}^{m+1} \).

Moreover, in this context, condition (3) is equivalent to

\[
(3)' \quad \text{Trace}(\phi(\pi)) = 1 \text{ for all } \pi \in \mathbb{C}P^m.
\]

Remark.- If the homogeneous coordinates of \( \pi \in \mathbb{C}P^m \) are \( (z_0^m, z_m) \) for a chosen basis on \( \mathbb{C}^{m+1} \), then \( \phi(\pi) \) with respect to such basis is given by

\[
\phi(\pi) = \frac{1}{z^t \cdot z} z^t \cdot z
\]

Consider the space of all Hermitian matrices of order \( m+1 \)

\[
\mathbb{H}M(m+1, \mathbb{C}) = \{ A \in \mathfrak{gl}(m+1, \mathbb{C}) \mid A^t = A \}
\]

which is a linear subspace of \( \mathfrak{gl}(m+1, \mathbb{C}) \) with dimension \((m+1)^2\).
On $\text{HM}(m+1,\mathbb{C})$ one defines the metric

\[(8.3) \quad g(A,B) = 2\text{Trace}AB \quad \text{for all } A, B \in \text{HM}(m+1,\mathbb{C}).\]

We define the following subset of $\text{HM}(m+1,\mathbb{C})$

\[(8.4) \quad \text{CP}^m = \{ A \in \text{HM}(m+1,\mathbb{C}) \mid AA = A \text{ and } \text{Trace}(A) = 1 \}\]

**Lemma 8.1.** $\text{CP}^m$ is a submanifold of $\text{HM}(m+1,\mathbb{C})$ which is diffeomorphic to the homogeneous space $U(m+1)/U(1) \times U(m)$, $U(m)$ being the unitary group of order $m$.

**Proof.** If $A \in \text{CP}^m$, it is a Hermitian matrix and so there exists $P$ in $U(m+1)$ with

$$PAP^{-1} = \begin{pmatrix} h_0 & \cdot & \cdot & h_m \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ h_m & \cdot & \cdot & h_0 \end{pmatrix}$$

From (8.4), one obtains that $h_i = 1$ for a fixed index and $h_j = 0$, $i \neq j$. Therefore, we can assume

$$PAP^{-1} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} = B.$$

We will say that $B$ is the origin of $\text{CP}^m$ and so we can look $\text{CP}^m$ like the orbit of $B$ under the $U(m+1)$-action over $\text{HM}(m+1,\mathbb{C})$ given by $(P,A) \mapsto PAP^{-1}$ for all $(P,A)$ in $U(m+1) \times \text{HM}(m+1,\mathbb{C})$. Moreover it is easy to see that the isotropy subgroup of $B$ is $U(1) \times U(m)$ which proves the lemma.

For all $A \in \text{CP}^m$, we denote by $T_A(\text{CP}^m)$ the tangent space of $\text{CP}^m$ at $A$ identified by means of $\varphi$ with a subspace of $\text{HM}(m+1,\mathbb{C})$. Similarly we have the normal space at $A$, $N_A(\text{CP}^m)$. Then one has

**Lemma 8.2.** For each point $A \in \text{CP}^m$, one has

$$T_A(\text{CP}^m) = \{ X \in \text{HM}(m+1,\mathbb{C}) \mid XA + AX = X \},$$

$$N_A(\text{CP}^m) = \{ Z \in \text{HM}(m+1,\mathbb{C}) \mid AZ = ZA \}. $$

**Proof.** Let $\alpha : I \longrightarrow \text{CP}^m$ be a curve with $\alpha(0) = A$ and $\alpha'(0) = X$, then $\alpha(t) = \alpha(t)\alpha(t)$ and so, $\alpha'(t)\alpha(t) + \alpha(t)\alpha'(t) = \alpha'(t)$, in particular when $t=0$, one get $XA + AX = X$ which proves the first inclusion.

Because for each $P$ in $U(m+1)$ the mapping $\text{HM}(m+1,\mathbb{C}) \longrightarrow \text{HM}(m+1,\mathbb{C})$ given by $A \mapsto PAP^{-1}$ is an isometry, it is enough to establish the equality
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at the origin B ∈ CP^m. Moreover, it is easy to see that dim \{ X ∈ HM(m+1,C) / XB + BX = X \} = 2m, which proves the first equality. The second equality can be proved using a direct argument.

Remark. - The vector fields along CP^m given by A - A and A - I (where I denotes the identity matrix of order m+1) are obviously normal to CP^m. Also the vector fields given by A - QA+ QA - 2QAQ are tangent to CP^m for all Q in HM(m+1,C).

The following relations admit a direct proof

\[
\begin{align*}
AXY &= XAY \\
AXA &= 0 \\
X(I-2A) &= -(I-2A)X \\
(I-2A)^2 &= 1 \\
(I-2A)XY &= XY(I-2A)
\end{align*}
\]

for all A ∈ CP^m and X, Y ∈ TA(CP^m), and they will be very useful along this paper.

**Lemma 8.3.** - Let D be the Levi-Civita connection of HM(m+1,C), \( \tilde{\nabla} \) the induced connection over CP^m, \( \nabla \) the second fundamental form of \( \phi : CP^m \rightarrow HM(m+1,C) \), \( \nabla \) the normal connection, \( \lambda \) the Weingarten endomorphism and \( \bar{H} \) the mean curvature vector, then

\[
\begin{align*}
(8.6) & \quad \tilde{\nabla}_X Y = A(D_X Y) + (D_X A - 2A(D_X Y))A \\
(8.7) & \quad \nabla(X,Y) = (XY + YX)(I-2A) \\
(8.8) & \quad \tilde{\nabla}^\perp_X Z = D_X Z + 2A(D_X Z)A - (D_X Z)A - A(D_X Z) \\
(8.9) & \quad \lambda X = (XZ - ZX)(I-2A) \\
(8.10) & \quad \bar{H} = \frac{1}{2m}(I - (m+1)A)
\end{align*}
\]

where X, Y are tangent vector fields of CP^m, Z is a normal vector field of CP^m, A any point of CP^m and the above formulas are computed at A.

**Proof.** Assume \( \tilde{\nabla} \) and \( \nabla \) defined respectively by (8.6) and (8.7). Take A ∈ CP^m, X ∈ TA(CP^m) and Y ∈ X(CP^m). Let \( \alpha : I \rightarrow CP^m \) be a curve with \( \alpha(0) = A \) and \( \alpha'(0) = X \). Since Y(t) ∈ TA(t)(CP^m) one has \( \alpha(t)Y(t) + Y(t)\alpha(t) = Y(t) \), therefore
\[ D_X Y = \frac{dY(t)}{dt}(0) = XY + YX + A(D_X Y) + (D_X Y)A \]
where we put \( Y \) by \( Y(0) \). Hence
\[ \tilde{\sigma}(X, Y) = (XY + YX)(I - 2A) = (D_X Y - A(D_X Y) - D_X Y)A(I - 2A) = \]
\[ = -(A(D_X Y) + (D_X Y)A - 2A(D_X Y)A + D_X Y = D_X Y - \tilde{\sigma}(X, Y). \]

So it is enough to prove that \( \tilde{\sigma}(X, Y) \) is normal to \( \mathbb{C}P^m \) which is trivial.

In order to prove (8.8) and (8.9) we take a normal vector field \( Z \) over \( \mathbb{C}P^m \), then \( \alpha(t)Z(t) = Z(t)a(t) \) and so
\[ XZ + A(D_X Z) = (D_X Z)A + ZX, \]
and
\[ \tilde{\sigma}(X, Z) = (XZ - ZX)(I - 2A) = \tilde{\sigma}(X, Z). \]

Now from Weingarten formula it is enough to prove that \( \tilde{\sigma}(X, Z) \) is tangent to \( \mathbb{C}P^m \) and \( \tilde{\sigma}(X, Z) \) is normal to \( \mathbb{C}P^m \), but it is also trivial.

Finally, to prove (8.10), it is enough to prove that at the origin \( B \) of \( \mathbb{C}P^m \) and use the argument that the mappings \( A \rightarrow PAP^{-1} \) are isometries of \( H\text{M}(m+1, \mathbb{C}) \) for all \( P \) in \( \text{UM}(m+1) \). Then one takes the orthonormal basis \( \{E_1, \ldots, E_m, E_{m+1}, \ldots, E_{2m}\} \) of \( T_B(\mathbb{C}P^m) \) given by
\[
E_k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \end{pmatrix} \quad E_k^* = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \end{pmatrix}
\]
then, a direct computation proves that
\[
\tilde{H}_B = \frac{1}{2m} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ \end{pmatrix} = \frac{1}{2m} (I - (m+1)B). \]

**LEMMA 8.5.** - Let \( f \) be the diffeomorphism obtained in lemma 8.1. Then \( f \) is already an isometry when one considers on \( U(m+1)/U(1) \times U(m) \) the metric of Fubini-Study with holomorphic sectional curvature \( c = 1 \) and on \( \mathbb{C}P^m \) the metric induced via \( \phi \) from the Euclidean metric on \( H\text{M}(m+1, \mathbb{C}) \).

**Proof.** - Since both metrics are \( U(m+1)-\)invariants, it is enough to
prove that the differential of $f$ at the origin is a linear isometry between the corresponding tangent spaces

$$f: U(m+1)/U(1) \times U(m) \rightarrow \mathbb{C}P^m, \quad f([P]) = PB^{-1}.$$

If $o = [I]$, one knows that

$$T_o(U(m+1)/U(1) \times U(m)) = \{ \begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix} \mid a \in \mathbb{C}^m \}$$

and the Fubini-Study metric of holomorphic sectional curvature $c=1$ is given by

$$g_o\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ -b^t & 0 \end{pmatrix} = 2 \text{Trace} \begin{pmatrix} ab^t & 0 \\ 0 & a^tb \end{pmatrix}$$

We take a curve $\alpha: \mathbb{R} \rightarrow U(m+1)$ with $\alpha(0) = 1$ and

$$\alpha'(0) = \begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix},$$

and consider $\beta: \mathbb{R} \rightarrow U(m+1)/U(1) \times U(m)$ given by $\beta(t) = [\alpha(t)]$. Then

$$df_o\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix} = (f \circ \beta)'(0) = (\alpha(t)Ba(t)^{-1})'(0) =$$

$$= \alpha'(0)Ba^{-t}(0) + \alpha'(0)B a(0)^{-t} = \begin{pmatrix} 0 & -a \\ -a^t & 0 \end{pmatrix}$$

and so

$$g\left(df_o\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix}, df_o\begin{pmatrix} 0 & b \\ -b^t & 0 \end{pmatrix}\right) = 2 \text{Trace} \begin{pmatrix} ab^t & 0 \\ 0 & a^tb \end{pmatrix}$$

which proves the lemma.

**Lemma 8.6.** Let $J$ be the natural complex structure over $U(m+1)/U(1) \times U(m)$. The induced complex structure over $\mathbb{C}P^m$ via the diffeomorphism $f$ is given by

$$J(X) = \sqrt{-1} (I-2A)X \quad \text{for all } X \text{ in } T_A(\mathbb{C}P^m).$$

**Proof.** We will prove the formula at the origin. As it is well-known the complex structure $\hat{J}$ at the origin of $U(m+1)/U(1) \times U(m)$ is given by

$$\hat{J}\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix} = -\sqrt{-1}\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix}.$$ 

if $\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix} \in T_B(\mathbb{C}P^m)$, the induced complex structure is

$$J\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix} = df_o \hat{J} (df_o)^{-1}\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix} = \sqrt{-1} \begin{pmatrix} 0 & -a \\ -a^t & 0 \end{pmatrix} =$$
The following proposition gives some geometric properties of the imbedding $\phi : \mathbb{C}P^m \to \text{HM}(m+1, \mathbb{C})$ which will be very useful.

**Proposition 8.7** The isometric immersion $\phi : \mathbb{C}P^m \to \text{HM}(m+1, \mathbb{C})$ satisfies

(a) It is an $U(m+1)$-equivariant isometric imbedding.

(b) $\tilde{\sigma}(JX, JY) = \tilde{\sigma}(X, Y)$ for all $X, Y$ in $T_A(\mathbb{C}P^m)$.

(c) $\tilde{\psi} \circ \tilde{\sigma} = 0$, that is the imbedding is parallel.

(d) It is minimal in a hypersphere of center $(1/(m+1))I$ and radius $\sqrt{2m/(m+1)}$. As consequence it is mono-order and the order is one.

**Proof**.-(a) It is an immediate consequence from lemma 8.1 and 8.5 if we recall that $U(m+1)$-equivariant means that every isometry of $\mathbb{C}P^m$ determined by the elements of $U(m+1)$ can be obtained as the restriction to $\mathbb{C}P^m$ of some isometry of $\text{HM}(m+1, \mathbb{C})$.

(b) It follows from (8.5), (8.7) and lemma 8.6.

(c) Let $X, Y_1, Y_2$ be three vector fields tangent to $\mathbb{C}P^m$, then by using (b) and the fact that $\mathbb{C}P^m$ is Kaehlerian, one get

$$\tilde{\psi}_X (JY_1, JY_2) = \tilde{\psi}_X (Y_1, Y_2).$$

As consequence $(\tilde{\psi}_X)_{(Y, JY)} = 0$ for all $X, Y$ tangent to $\mathbb{C}P^m$. Now from Codazzi's equation one has

$$\tilde{\psi}_X (X, X) = 0$$

for all $X$ tangent to $\mathbb{C}P^m$, which proves (c).

(d) Let $A$ be any point of $\mathbb{C}P^m$, then

$$g(A - \frac{1}{m+1} I, A - \frac{1}{m+1} I) = 2\text{Trace} \left( A - \frac{1}{m+1} I + \frac{1}{(m+1)^2} I \right) = \frac{2m}{m+1}. $$

This proves that $\mathbb{C}P^m$ is contained in a sphere $S$ of center $\frac{1}{m+1} I$ and radius $\sqrt{2m/(m+1)}$.

Moreover the mean curvature vector $\tilde{H}$ of $\mathbb{C}P^m \to \text{HM}(m+1, \mathbb{C})$ in $A$ is given by

$$\tilde{H}_A = \frac{1}{2m} (I - (m+1)A) = -\frac{m+1}{2m} (A - \frac{1}{m+1} I)$$

which proves that $\tilde{H}_A$ is normal to $S$ at $A$, therefore the mean curvature vector of $\mathbb{C}P^m$ in $S$ is trivial.

From the Takahashi's theorem, $\phi$ is mono-order, of order $u \geq 1$, with $\lambda_u = \dim \mathbb{C}P^m / (\text{radius } S)^2 = m+1$. But the first eigenvalue of $\mathbb{C}P^m$ is $m+1$, then $u = 1$. 

$$= \sqrt{-1} (I-2A) \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}.$$
We will finish this section by giving two technical lemmas which admit a direct proof.

**Lemma 8.8.** For the imbedding $\phi : \mathbb{CP}^m \to \mathbb{H}^{m+1}(C)$ one has:

\[(8.11) g(\gamma^*(X,Y), \gamma^*(V,W)) = \frac{1}{2} g(X,Y) g(V,W) + \frac{1}{4} \{ g(X,W) g(Y,V) + g(X,V) g(Y,W) + g(Y,X) g(V,W) + g(Y,V) g(X,W) \} \]

\[(8.12) \lambda_\gamma^*(X,Y) \gamma^*(V) = \frac{1}{2} g(X,Y) V + \frac{1}{4} \{ g(Y,V) X + g(X,V) Y + g(JY,V) JX + g(JX,V) JY \} \]

\[(8.13) g(\gamma^*(X,Y), I) = 0. \]

\[(8.14) g(\gamma^*(X,Y), A) = - g(X,Y). \]

**Lemma 8.9.** Let $X, Y$ be two orthonormal vectors in $T_A(\mathbb{CP}^m)$. Then

\[(8.15) g(\gamma^*(X,X), \gamma^*(X,X)) = 1, \]

\[(8.16') \frac{1}{2} \leq g(\gamma^*(X,X), \gamma^*(Y,Y)) \leq 1. \]

If one has $g(X,Y) = 0$, then

\[(8.17) g(\gamma^*(X,X), \gamma^*(Y,Y)) = \frac{1}{2}, \]

\[(8.17') g(\gamma^*(X,X), \gamma^*(X,X)) = \frac{1}{4}. \]

9. A NEW WAY TO STUDY SPECTRAL GEOMETRY OF SUBMANIFOLDS IN THE COMPLEX PROJECTIVE SPACE. The theorem of Takahashi (theorem 5.2) gives a very nice spectral characterization for minimal submanifolds of a sphere. But it implicitly depends on the standard immersion of the sphere in the Euclidean space (as the geometrical locus of points having constant distance to a fixed point). Let $S^{m-1}(r)$ be an $(m-1)$-dimensional sphere of radius $r$ in $\mathbb{E}^m$, if $<,>$ denotes the Euclidean metric on $\mathbb{E}^m$, then

$S^{m-1}(r) = \{ a \in \mathbb{E}^m / <a,a> = r^2 \}.$

Consider the inclusion $i : S^{m-1}(r) \to \mathbb{E}^m$ or standard imbedding. If $M^n$ denotes an $n$-dimensional (compact) Riemannian manifold, then $M$ admits a minimal isometric immersion $\gamma$ in $S^{m-1}(r)$ if and only if the isometric immersion $\psi = i \circ \gamma : M^n \to \mathbb{E}^m$ has the simplest spectral behaviour, that is, it is mono-order of order $u$,

$\psi = \psi_0 + \psi_u.$
with
\[ \psi_0 = \frac{1}{\text{vol}(M)} \int_M \psi(M).dv, \]
and \( \Delta \psi = \lambda u \psi \). Moreover \( \lambda_u = n/r^2 \).

This results suggests us [46] to consider the following way to study spectral geometry of submanifolds in the complex projective space. First, it seems natural to try to find a good isometric immersion of \( \mathbb{C}P^m \) in a Euclidean space, say for instance
\[ \phi: \mathbb{C}P^m \rightarrow E^N \]
and so study submanifolds \( M \) in \( \mathbb{C}P^m (\tau:M^n \rightarrow \mathbb{C}P^m) \) by looking the spectral behaviour of the isometric immersion \( \psi = \phi \tau:M^n \rightarrow E^N \). Of course this idea can be extended to other symmetric spaces (see for instance [17]).

In this sense, we will use the isometric imbedding \( \phi: \mathbb{C}P^m \rightarrow HM(m+1,C) \) defined in the last section.

For example, one can put the following problem, which is formally like in the theorem of Takahashi: Assume that \( \tau:M^n \rightarrow \mathbb{C}P^m \) is an isometric immersion of a Riemannian manifold \( M^n \) in \( \mathbb{C}P^m \). Using \( \phi: \mathbb{C}P^m \rightarrow HM(m+1,C) \) one can see \( M^n \) like a submanifold in the Euclidean space \( HM(m+1,C) \)
\[ \psi:M^n \rightarrow HM(m+1,C). \]
The spectral behaviour of \( \psi \) must be closely related to the geometry of the immersion \( \tau:M^n \rightarrow \mathbb{C}P^m \). Is the minimality of \( \tau \) equivalent to the simplest spectral behaviour of \( \psi \) (to be mono-order)?

We will see in section 11 that this problem is not true because if \( M \) is a Kaehler submanifold of \( \mathbb{C}P^m \), then it is minimal. On the other hand, we will prove that the only complex submanifolds for which \( \psi \) is mono-order are totally geodesic and so linear varieties.

Therefore a more solid first problem is: To try to study the submanifolds \( \tau:M^n \rightarrow \mathbb{C}P^m \) such that the corresponding isometric immersion \( \psi:M^n \rightarrow HM(m+1,C) \) is mono-order (of order \( u \geq 1 \)). In section 11, we will classify all CR-minimal submanifolds in \( \mathbb{C}P^m \) satisfying last condition. As we will see the solution obtained indicates that complex submanifolds admit a more involved behaviour with respect to the spectral character of \( \psi \) and so we will study in a second problem complex submanifolds in \( \mathbb{C}P^m \) whose isometric immersion \( \psi \) is bi-order.
IQ. THE VOLUME OF A MINIMAL SUBMANIFOLD IN THE COMPLEX PROJECTIVE SPACE. We recall that a submanifold \( \tau: M \rightarrow \mathbb{CP}^m \) is called a CR-submanifold if it admits a pair of complementary distribution \((D, D^\perp)\) such that \(D\) is holomorphic and \(D^\perp\) is totally real. In general we will denote \( M \) by \( M^{2n+q} \) where \( 2n \) indicates the real dimension of \( D \) and \( q \) the dimension of \( D^\perp \).

Given a submanifold \( \tau: M^n \rightarrow \mathbb{CP}^m \), one consider \( \psi: M^n \rightarrow H\mathbb{M}(m+1, \mathbb{C}) \), where \( \psi = \phi \tau \). Let \( H^\perp \) be the normal component to \( \mathbb{CP}^m \) of the mean curvature vector of \( \psi \). We will start this section by giving the best possible estimation for the length of \( H^\perp \), [17], [46].

**Lemma 10.1.** (a) Let \( M^n \) be an \( n \)-dimensional submanifold of \( \mathbb{CP}^m \), then

\[
\frac{n+1}{2n} \leq g(H^\perp, H^\perp) \leq \frac{n+2}{2n}.
\]

Moreover, the first equality (respectively the second equality) holds if and only if \( M^1 \) is totally real (respectively \( n \) is even and \( M^n \) is Kaehlerian).

(b) In particular if \( M^{2n+q} \) is a CR-submanifold of \( \mathbb{CP}^m \), then

\[
g(H^\perp, H^\perp) = \frac{(2n+q)^2 + 4n + q}{2(2n+q)}.
\]

**Proof.** Take any point \( A \) in \( M \) and an orthonormal basis \( \{E_1, \ldots, E_n\} \) of \( T_A M \), then trivially one has

\[
H^\perp = \frac{1}{n} \sum_{i=1}^{n} \delta(E_i, E_i).
\]

Now we use lemma 8.9 to obtain

\[
g(H^\perp, H^\perp) = \frac{1}{n^2} \sum_{i,j} g(\delta(E_i, E_i), \delta(E_j, E_j)) = \frac{n+1}{2n} + \frac{1}{2n^2} \sum_{i,j} g(E_i, J E_j)^2.
\]

Define the endomorphism \( P \) on \( T_A M \) by \( \langle PX, Y \rangle = \langle JX, Y \rangle \) for all \( X, Y \) in \( T_A M \), then

\[
g(H^\perp, H^\perp) = \frac{n+1}{2n} + \frac{1}{2n^2} g(P, P).
\]

Because \( P \) is nothing but the tangential component of \( J \), one has

\[
0 \leq g(P, P) \leq n,
\]

where the first equality (respectively, the second equality) holds if and only if \( M \) is totally real at \( A \) (respectively, \( n \) is even and \( M^n \) is Kaehlerian at \( A \)). Now (10.1) follows from (10.3) and (10.4) and the equalities (10.1) are also characterized.
(b) If $M^{2n+q}$ is a CR-submanifold, at any point $A$ in $M$, one can choose an orthonormal basis of $T_A M$, say for instance $(E_1, \ldots, E_n, J_1, \ldots, J_n, F_1, \ldots, F_q)$ where $(E_1, \ldots, E_n, J_1, \ldots, J_n)$ span $D_A$ and $(F_1, \ldots, F_q)$ span $D_A^r$. Now we use lemma 8.9 and a direct computation like in the proof of (a) to obtain (10.2).

**Lemma 10.2.** Let $M^{2n+q}$ be a CR-submanifold of $\mathbb{C}P^m$ and $\hat{\gamma}_M$ the restriction of the second fundamental form of $\phi : \mathbb{C}P^m \to \text{HM}(m+1, \mathbb{C})$ to the tangent bundle of $M$. Then

\[(10.5) \quad g(\hat{\gamma}_M, \hat{\gamma}_M) = \frac{1}{2} ((2n+q)^2 + 4n + 3q).\]

The proof is again an immediate consequence of lemma 8.9.

**Corollary 10.3.** Let $M^{2n+q}$ be a CR-submanifold of $\mathbb{C}P^m$. Denote by $\rho$ the scalar curvature of $M$ and by $H$ (respectively by $\sigma$) the mean curvature vector (respectively the second fundamental form) of $M$ in $\mathbb{C}P^m$. Then

\[(10.6) \quad \rho = \frac{(2n+q)^2 + 4n + 3q}{4} + (2n+q)^2 g(H,H) - |\sigma|^2.\]

**Proof.** It is easy to see that

\[\rho = (2n+q)^2 g(H,H) + (2n+q)^2 g(H^\perp, H^\perp) - |\sigma|^2 - |\hat{\gamma}_M|^2.\]

Now (10.6) follows from last formula joint formulas (10.2) and (10.5).

In order to obtain an estimate for the volume of any minimal submanifold of $\mathbb{C}P^m$, we start by recalling the following proposition [14]

**Proposition 10.4.** Let $M^n$ be an $n$-dimensional submanifold in the Euclidean space $\mathbb{E}^m$, then

\[(10.7) \quad \int_M |H'|^n dv \geq C_n,\]

where $H'$ denotes the mean curvature vector of $M$ and $C_n$ the volume of the $n$-dimensional unit sphere. Moreover the equality in (10.7) holds if and only if $M$ is imbedded as the standard sphere in a $(n+1)$-dimensional affine subspace of $\mathbb{E}^m$.

Now we obtain the following theorem,[46]
THEOREM 10.5.- Let $M^n$ be an $n$-dimensional minimal submanifold of $\text{CP}^m$, then

(10.8) \[ \text{vol}(M) \geq c_n \left( \frac{2n}{n+2} \right)^{n} \]

Proof.- Because $M$ is minimal in $\text{CP}^m$, $H^1$ is essentially the mean curvature vector of $M$ in $\text{HM}(m+1, \mathbb{C})$. Therefore from (10.1) and (10.7) one get (10.8).

If $M = S^2 = \text{CP}^1$ it is obvious that the equality in (10.8) follows. Conversely, assume that equality in (10.8) holds, then one obtains:

(i) from the equality in (10.7), $M^n$ is imbedded as a standard sphere of certain radius $R$ in an $(n+1)$-dimensional Euclidean space, and

(ii) from equality in the second part of (10.11, n is even, say $n=2k$ and $M^{2k}$ is a complex submanifold of $\text{CP}^m$.

In particular $M^n = S^{2k}(R)$ must be a Kähler manifold and so $k=1$. But $\text{vol}(M^2) = R^2 c_2$ and so from the equality in (10.8) one get $R=1$.

Furthermore $M^2$ is already a CR-minimal submanifold of $\text{CP}^m$ and so from (10.6) one get $\rho = 2 - |\sigma|^2$, which implies that $\sigma = 0$ and the proof is finish.

Remark.- The estimate obtained in the last theorem has been improved by B. Y. Chen (see [17]).

11. THE FIRST EIGENVALUE OF A MINIMAL SUBMANIFOLD IN THE COMPLEX PROJECTIVE SPACE. The main purpose of this section is to obtain an upper bound for the first eigenvalue of a minimal submanifold of $\text{CP}^m$. The estimate obtained will be the best possible.

We start by solving the following problem : Let $M^{2n+q}$ be a CR-minimal submanifold of $\text{CP}^m$ ($\tau: M \rightarrow \text{CP}^m$) then $M$ can be looked as a submanifold of a certain Euclidean space ($\psi: M \rightarrow \text{HM}(m+1, \mathbb{C}), \psi = \phi \cdot \tau$). From Takahashi's Theorem it seems natural to ask: When $M$ is minimal in some hypersphere of $\text{HM}(m+1, \mathbb{C})$? The complete solution to this problem is given in the following theorem, [46]

THEOREM 11.1.- Let $M^{2n+q}$ be a CR-minimal submanifold of $\text{CP}^m$. Then $M$ is minimal in some sphere of $\text{HM}(m+1, \mathbb{C})$ if and only if either:

(a) $q=0$ and $M^{2n}$ is a totally geodesic complex submanifold of $\text{CP}^m$, or

(b) $n=0$ and $M^q$ is a totally totally real submanifold of $\text{CP}^m$ for which there exists a totally geodesic complex submanifold $\text{CP}^q$ of $\text{CP}^m$.
such that $M^q$ is a totally real submanifold of $\mathbb{C}P^q$.

Proof.- First we prove the sufficient condition. Let $M^{2n}$ be like in (a), then $M^{2n}$ must be some linear variety $\mathbb{C}P^n$ and so it admits an isometric imbedding $\psi: \mathbb{C}P^n \to \text{HM}(n+1,\mathbb{C})$ which is certainly minimal in a sphere (see Proposition 8.7). Suppose now that $M^q$ is in situation (b). For each point $A$ in $M^q$, one takes an orthonormal basis $\{E_1, \ldots, E_q\}$ in $T_A M$, then $\{E_1, \ldots, E_q, JE_1, \ldots, JE_q\}$ is an orthonormal basis of $T_A \mathbb{C}P^q$. If $H^\prime$ (respectively $\sigma'$) denotes the mean curvature vector (respectively the second fundamental form) of $\psi: M^q \to \text{HM}(q+1,\mathbb{C})$ and because $\psi$ is minimal, then

$$H^\prime = \frac{1}{q} \sum_{i=1}^{q} \sigma'(E_i, E_i) = \frac{1}{q} \sum_{i=1}^{q} \delta(E_i, E_i).$$

Now from proposition 8.7 and lemma 8.3, one obtains

$$H^\prime_A = \frac{1}{2q} \left( \sum_{i=1}^{q} \delta(E_i, E_i) + \frac{q}{q+1} \delta(JE_i, JE_i) \right) = H^\prime_A = \frac{1}{2q} \left( I - (q+1)A \right) = - \frac{q+1}{2q} (A - \frac{1}{q+1} I)$$

for all $A$ in $M$.

As $M^q$ is immersed through $\mathbb{C}P^q$ in a sphere of $\text{HM}(q+1,\mathbb{C})$, whose center is $\frac{1}{q+1} I$, last formula proves us that $M^q$ must be minimal in such a sphere.

Now we will prove the necessary condition. Assume that $M^{2n+q}$ is a CR-minimal submanifold of $\mathbb{C}P^m$ which is minimal in some sphere, say $S$ of $\text{HM}(m+1,\mathbb{C})$. Let $Q$ be the center of $S$, it is clear that one can assume that $Q$ is a diagonal matrix, otherwise one uses the repeated argument of the isometry of type $A \to \text{PAP}^{-1}$ for some $P$ in $\text{U}(m+1)$.

Let $H^\prime$ be the mean curvature vector of $\psi: M \to \text{HM}(m+1,\mathbb{C})$. Because $M$ is minimal in $S$, $H^\prime$ must be normal to $S$ at each point of $M$ and so $H^\prime = h(A - Q)$ where $h$ is a non-zero constant and $A$ denotes any point of $M$ in which is computed $H^\prime$.

On the other hand and because $M$ is minimal in $\mathbb{C}P^m$, $H^\prime$ is normal to $\mathbb{C}P^m$ at each point $A$ in $M$. Therefore one get $Q \in T_A^+(\mathbb{C}P^m)$ at each point $A \in M$. Consequently $M$ is contained in the linear subspace $L$ of $\text{HM}(m+1,\mathbb{C})$ defined by

$$L = \{ A \in \text{HM}(m+1,\mathbb{C}) / A Q = QA \}.$$

Suppose
Spectral geometry of submanifolds

\[ Q = \begin{pmatrix} a_1 & \cdots & a_r \\ \vdots & \ddots & \vdots \\ a_r & \cdots & a_1 \end{pmatrix} \]

Then

\[ \text{CP}^m \cap L = \left\{ \left( \frac{A_1}{I_{r \times r}} \right) \text{ such that } A_1^2 = A_1 \text{ and } \sum_{i=1}^{r} \text{Trace } A_i = 1 \right\}. \]

Because each \( A_1 \) is a Hermitian matrix with \( A_1^2 = A_1 \), then for all \( A \) in \( M \) there exists an index \( j \in \{1, \ldots, r\} \) such that \( \text{Trace } A_j = 1 \) and \( \text{Trace } A_i = 0 \quad i \neq j \). Therefore

\[ \text{CP}^m \cap L = \bigcup_{j=1}^{r} \left\{ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \right\} \text{ such that } A_j^2 = A_j \text{ and } \text{Trace } A_j = 1 \}

But \( M \) is connected and so it must be contained in some component of \( \text{CP}^m \), that is \( M \subset \text{CP}^k \), \( k \leq m \). On the other hand \( M \) is also minimal in the great sphere \( S \). Therefore the problem is reduced to study CR-minimal submanifolds of \( \text{CP}' \) which are minimal in a sphere of \( \text{HM}(r+1, \mathbb{C}) \) whose center is \( aI \), with \( a \in \mathbb{R} \) and \( 1 \) being the identity matrix of order \( r+1 \).

The mean curvature vector \( H' \) of \( M \) in \( \text{HM}(r+1, \mathbb{C}) \) is nothing but \( H^1 \) because \( M \) is minimal in \( \text{CP}' \), then

\[ H^1 = \frac{1}{2n+q} \sum_{i} \langle E_i, E_i \rangle, \]

which joint lemma 8.8 imply

**(11.1)** \[ g(H^1, A-aI) = -1 \]

Now by using (10.2), (11.1) and since \( H^1 = H' = h(A-aI) \), one get

\[ h = -g(H^1, H^1) = -\frac{(2n+q)^2 + 4n + q}{2(2n+q)^2}, \]

and so

**(11.2)** \[ g(A-aI, A-aI) = -\frac{1}{h} = -\frac{2(2n+q)^2}{(2n+q)^2 + 4n + q} \quad \text{for all } A \text{ in } M. \]

Notice that formula (11.2) gives us the radius of the sphere \( S \).

On the other hand

**(11.3)** \[ g(A-aI, A-aI) = 2 - 4a + 2a^2(r+1). \]

From (11.2) and (11.3) one get the following equation
\[(11.4) \quad (r+1)((2n+q)^2 + 4n + q)a^2 - 2((2n+q)^2 + 4n + q)a + 4n + q = 0,\]

and so one has
\[(11.5) \quad (2n+q)^2 \geq r(4n+q) \quad \text{and} \quad 4nq \geq 5nq,\]

which implies \(n=0\) and \(q=r\) (case (b)) or \(q=0\) and \(n=r\) (case (a)).

The following lemma can be looked as a special case of a formula due to R. C. Reilly [45]. Any way we obtain it by using a more simple method, [47].

**Lemma 11.2.** Let \(\psi: M^n \rightarrow \mathbb{E}^N\) be an isometric immersion of a Riemannian manifold in the Euclidean space. If \(H'\) denotes the mean curvature vector of \(\psi\) and \(\lambda_1\) the first eigenvalue of \(M\), then
\[(11.6) \quad n \int_M g(H',H') \, dv = \lambda_1 \text{vol}(M) \geq 0.\]

**Proof.** First one takes \(k=t=1\) in proposition 5.4 to obtain
\[(11.7) \quad \int_M g(\Delta^2 \psi, \psi) \, dv = \lambda_1 \int_M g(\Delta \psi, \psi) \, dv \geq 0,\]

the equality holding if and only if \(\psi\) is of order 1. Now it is not difficult to see that
\[(11.8) \quad \Delta g(\psi, \psi) = -2n + 2g(\psi, \Delta \psi) = -2n(1+g(H', \psi)),\]

where we used formula (5.7). From (5.7) and (11.7) one get
\[n^2 \int_M g(H',H') \, dv + \lambda_1 n \int_M g(H',\psi) \, dv \geq 0,\]

which combined with (11.8) gives us (11.6).

**Corollary 11.3.** [17]. Let \(M^n\) be a minimal submanifold of \(\mathbb{C}P^m\). Then
\[(11.9) \quad \lambda_1 \leq -\frac{n+2}{2}.\]

Moreover the equality holds if and only if \(n\) is even and \(M = \mathbb{C}P^{n/2}\) imbedded in \(\mathbb{C}P^m\) as a totally geodesic complex submanifold.

**Proof.** Since \(M^n\) is minimal in \(\mathbb{C}P^m\), the mean curvature vector of \(M^n\) in \(HM(m+1, \mathbb{C})\) is \(H'\). Furthermore, from (10.1),
\[g(H', H') \leq \frac{n+2}{2},\]

now (11.9) follows from (10.1) and (11.6).
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Assume now \( \lambda_1 = (n+2)/2 \), then one has the second equality in (10.1) which implies that \( n \) is even, say \( n = 2k \) and \( M^{2k} \) is a complex submanifold of \( \mathbb{CP}^m \). Moreover one has equality in (11.6) gives us that \( \psi: M^{2k} \rightarrow \text{HM}(m+1,C) \) is of order one, in particular it must be mono-order and so from Takahashi's theorem \( \psi \) is minimal at some sphere in \( \text{HM}(m+1,C) \), from theorem 11.1 it is totally geodesic.

Using the same method B. Y. Chen,[17], showed the following

PROPOSITION 11.4.- Let \( M^n \) be a minimal submanifold of \( \mathbb{RP}^m \) (the \( m \)-dimensional real projective space with sectional curvature \( 1 \)). Then

\[
\lambda_1 \leq 2(n+1).
\]

Moreover the equality holds if and only if \( M^n \) is a \( \mathbb{RP}^n \) imbedded as a totally geodesic submanifold in \( \mathbb{RP}^m \).

PROPOSITION 11.5.- Let \( M^n \) be a minimal submanifold of \( \mathbb{QP}^m \) (the \( m \)-dimensional quaternion projective space with quaternion sectional curvature \( 1 \)). Then

\[
\lambda_1 \leq \frac{n+4}{2}.
\]

Moreover the equality holds if and only if \( n \) is a multiple of 4 and \( M \) is a \( \mathbb{QP}^{n/4} \) imbedded in \( \mathbb{QP}^m \) as a totally geodesic submanifold.

COROLLARY 11.6.- Let \( M^{2n+q} \) be a CR-minimal submanifold of \( \mathbb{CP}^m \). Then

(11.10)

\[
\lambda_1 \leq \frac{(2n+q)^2 + 4n + q}{2(2n+q)}.
\]

Moreover if the equality holds, then either: (a) \( q=0 \) and \( M^{2n} \) is a totally geodesic complex submanifold of \( \mathbb{CP}^m \), or (b) \( n=0 \) and \( M^q \) is a minimal totally real submanifold of some \( \mathbb{CP}^q \) totally geodesic complex submanifold in \( \mathbb{CP}^m \).

Proof.- The formula (11.10) is an immediate consequence from (11.6) and (10.2). The equality in (11.10) holds if and only if the corresponding immersion \( \psi: M \rightarrow \text{HM}(m+1,C) \) is of order 1, in particular it is mono-order and so from theorem 11.1 one obtains the second part.

COROLLARY 11.7.- Let \( M^{2n+q} \) be a CR-minimal submanifold of \( \mathbb{CP}^m \) which is minimal at some sphere of \( \text{HM}(m+1,C) \). Then
\[
\frac{(2n+q)^2 + 4n + q}{2(2n+q)} \in \text{Spec}(M, g)
\]

Proof. If \( R \) denotes the radius of the sphere in which \( M \) is minimal then for Takahashi's theorem, one get
\[
\frac{(2n+q)}{R^2} \in \text{Spec}(M, g).
\]

Moreover formula (11.2) gives us that
\[
R^2 = \frac{2(2n+q)^2}{(2n+q)^2 + 4n + q}
\]

COROLLARY 11. 8,[46],[23].- Let \( M^{2n} \) be a complex submanifold of \( \mathbb{C}P^n \). Then
\[
\lambda_1 \leq n+1.
\]

Moreover the equality holds if and only if \( M^{2n} \) is totally geodesic.

COROLLARY 11.9,[46].- Let \( M^q \) be a minimal totally real submanifold of \( \mathbb{C}P^n \).

(i) If there exists a linear subvariety \( \mathbb{C}P^q \) such that \( M^q \) is a totally real submanifold of \( \mathbb{C}P^q \), then \( (q+1)/2 \in \text{Spec}(M, g) \).

(ii) If \( \lambda_1 = (q+1)/2 \), then there exists a linear subvariety \( \mathbb{C}P^q \) of \( \mathbb{C}P^m \) such that \( M \) is a totally real submanifold of \( \mathbb{C}P^q \).

Proof.-(i) It follows from corollary 11.7 because in this case \( n=0 \).
(ii) It follows from the second part of corollary 11.6.

12. PARALLEL COMPLEX SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE.- In last section we knew that the only complex submanifolds of \( \mathbb{C}P^m \) which are minimal in some sphere of \( \text{HM}(m+1, \mathbb{C}) \) are totally geodesic and therefore, under conditions we are assuming in this paper, they are linear subvarieties. Consequently these submanifolds are the unique complex submanifolds of \( \mathbb{C}P^m \) whose corresponding isometric immersions \( \psi: M \longrightarrow \text{HM}(m+1, \mathbb{C}) \) admit the simplest spectral behaviour (they are mono-order).

After totally geodesic complex submanifolds, parallel complex submanifolds (that is, complex submanifolds with parallel second fundamental form) are the more simple complex submanifolds. Therefore it seems natural to study the spectral behaviour of \( \psi: M \longrightarrow \text{HM}(m+1, \mathbb{C}) \) when \( M \) is a parallel complex submanifold of \( \mathbb{C}P^m \).
In this section we study parallel complex submanifolds of $\mathbb{C}P^m$ which are Einstein. First we will give some examples of such kind of submanifolds which are not totally geodesic.

12.1 The complex quadric.- Consider the imbedding $\phi : \mathbb{C}P^{n+1} \rightarrow \text{HM}(n+2, \mathbb{C})$ given in section 8. The standard projection of $\mathbb{C}P^{n+2} \rightarrow (0)$ over $\mathbb{C}P^{n+1}$ is given as follows: For all $z = (z_0, \ldots, z_{n+1}) \in \mathbb{C}^{n+2} \rightarrow (0)$

$$z \mapsto \phi(z) = (1/zz^t)z^t . z$$

On $\mathbb{C}P^{n+1}$ one consider the homogeneous coordinate system $(z_0, \ldots, z_{n+1})$ determined from the above projection. As we know the complex quadric is defined by $Q_n = \{(z_0, \ldots, z_{n+1}) \in \mathbb{C}P^{n+1} / \sum z_i^2 = 0 \}$. $Q_n$ is a complex hypersurface of $\mathbb{C}P^{n+1}$ which is holomorphically isometric to the Hermitian symmetric space $SO(n+2)/SO(2)xSO(n)$. In order to identify $Q_n$ in $HM(n+2, \mathbb{C})$ by means of the imbedding $\phi$ one notices that

$$\phi(A) = A = (1/zz^t)z^t z$$

and $A A^t = (1/zz^t)z^t (\sum z_i^2)z$ for all $A \in \mathbb{C}P^{n+1}$. Therefore

$$Q_n = \{ A \in \mathbb{C}P^{n+1} / AA^t = 0 \}.$$

We also recall that $Q_n$ has parallel second fundamental form. Moreover a well-known result due to Smyth, [52], affirms that the only Einstein complex hypersurfaces (of course complete) of the complex projective space are (up congruence) linear subvarieties and complex quadrics.

12.2 The Veronese imbedding.- Let $S_k(\mathbb{C}^{n+1})$ be the complex vector space whose elements are homogeneous polynomials of degree $k$ over $\mathbb{C}^{n+1}$. Denote by $S_k^*$ its dual space. If $d = \text{dim} S_k - 1$, one considers the complex projective space $\mathbb{C}P^d$ over $S_k^*$. For each point $z \in \mathbb{C}^{n+1}$, one can define $F(z) \in S_k^*$ by

$$F(z): S_k \rightarrow \mathbb{C} \quad F(z)(p) = \langle p(z) \rangle \quad \text{for all polynomials } p \in S_k.$$

It is clear that $F(\lambda z)(p) = \lambda^k F(z)(p)$ for all $\lambda \in \mathbb{C}$, It proves that $F: \mathbb{C}^{n+1} \rightarrow S_k^*$ defines a rational mapping $f: \mathbb{C}P^{n+1} \rightarrow \mathbb{C}P^d$ which gives a holomorphic imbedding. If one considers over $\mathbb{C}P^d$ the Fubini-Study metric with constant holomorphic sectional curvature 1, then the standard metric via $f$
induces over $\mathbb{CP}^n$ the Fubini-Study metric with constant holomorphic curvature $c = 1/k$. In particular when $k = 2$, the isometric imbedding $f: \mathbb{CP}^n(1/2) \rightarrow \mathbb{CP}^d$ has parallel second fundamental form and it is called the Veronese imbedding.

12.3. The Segre imbedding.- One starts by taking the tensorial product mapping

$$\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} = \mathbb{C}^{n+m+n+m+1},$$

defined by $(z, w) \mapsto z \otimes w$. If $\lambda, \mu \in \mathbb{C}$, then $(\lambda z, \mu w) \mapsto \lambda \cdot z \otimes w$ and so this mapping induces a holomorphic one which is already holomorphic imbedding from $\mathbb{CP}^n \times \mathbb{CP}^m$ in $\mathbb{CP}^{n+m+n+m}$. This imbedding is called the Segre imbedding. If one takes on $\mathbb{CP}^{n+m+n+m}$ the Fubini-Study metric with holomorphic sectional curvature one, the induced metric over $\mathbb{CP}^n \times \mathbb{CP}^m$ is the Riemannian product of the Fubini-Study metrics on $\mathbb{CP}^n$ and $\mathbb{CP}^m$ both with holomorphic sectional curvature one. This imbedding is also parallel. Moreover $\mathbb{CP}^n \times \mathbb{CP}^m$ is Einstein if and only if $n = m$.

12.4. The Plücker imbedding.- Consider the complex Grassmannian of 2-dimensional planes, $G(2,n,\mathbb{C}) = \{ \Pi \in \mathbb{C}^{n+2} \mid \Pi \text{ is a 2-dimensional complex plane} \}$. If $\Pi^2(\mathbb{C}^{n+2}) = \mathbb{C}^{n+2}$, one defines

$$\mathbb{C}^{n+2} \times \mathbb{C}^{n+2} \rightarrow \mathbb{C}^{n+2} \otimes \mathbb{C}^{n+2}$$

by $(z, w) \mapsto z \otimes w$. If $(z, w)$ and $(\tilde{z}, \tilde{w})$ are two basis of the same element $\Pi \in G(2,n,\mathbb{C})$ then

$$\tilde{z} \tilde{w} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \otimes w,$$

with $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$,

where $\tilde{z} = az + bw$, $\tilde{w} = cz + dw$. Consequently the above mapping induces a holomorphic imbedding from $G(2,n,\mathbb{C})$ in $\mathbb{CP}^{\binom{n+1}{2}}$. It is called the Plücker imbedding. The induced metric over $G(2,n,\mathbb{C})$ from the Fubini-Study metric with holomorphic sectional curvature one via the above imbedding is nothing but the standard metric over the Grassmannian and the imbedding is also parallel.

12.5. The rank two Hermitian symmetric spaces $M_6 = SO(10)/U(5)$ and $M_{26} = E_6/Spin(10) \times T$ admit holomorphic standard imbedding in $\mathbb{CP}^1$ and $\mathbb{CP}^{26}$ respectively which have parallel second fundamental form. They are obtained by using the representation theory of Lie groups (to see [37],[55]).
The complete classification for Einstein parallel complex submanifold in complex projective space is consequence of results in [37] and [55]. If \( M \) is an Einstein complex submanifold of a complex projective space then \( M \) is parallel if and only if \( M \) is totally geodesic or \( M \) is an imbedded submanifold which is congruent with the standard of some of the submanifolds described before, that is, with the standard imbedding of some \( M_i, 1 \leq i \leq 6 \), where

\[
M_1 = \mathbb{CP}^n(\frac{1}{2}); \quad M_2 = Q_n, n \geq 3; \quad M_3 = \mathbb{CP}^n \times \mathbb{CP}^n; \quad M_4 = G(2,s,C), s \geq 3;
\]

\[
M_5 = \text{SO}(10)/U(5); \quad M_6 = E_6/\text{Spin}(10)\times T.
\]

The first eigenvalue of \( M_i, 1 \leq i \leq 6 \), can be computed by using the following fact, [39]: If \( M \) is an Einstein homogeneous Kaehler manifold with positive scalar curvature \( \rho \), then \( x_1 = \rho/n \), \( n \) being the complex dimension of \( M \).

In [36], T. Nagano gave a theoretical method to compute the spectrum of classical symmetric spaces. In particular one obtains the second eigenvalue of \( M_i, 1 \leq i \leq 5 \).

Recently S. Udagawa, (["Spectral geometry of Kaehler submanifolds of a complex projective space" preprint]) computed the complete spectrum of \( M_6 = E_6/\text{Spin}(10)\times T \), namely

\[
\text{Spec}(M_6) = \{ 2m_1(m_1+m_2+8) + m_2(m_2+1)/ m_1, m_2 \in \mathbb{Z}^+ \} = \{ 0 < \lambda_1 = 12 < \lambda_2 = 18 \ldots \}.
\]

Table 1. Einstein parallel complex submanifolds

<table>
<thead>
<tr>
<th>Submanifold</th>
<th>( n )</th>
<th>( p )</th>
<th>( \rho )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 = \mathbb{CP}^n(\frac{1}{2}) )</td>
<td>( n )</td>
<td>( \frac{1}{2}n(n+1) )</td>
<td>( \frac{1}{2}n(n+1) )</td>
<td>( \frac{1}{2}(n+1) )</td>
<td>( n+2 )</td>
</tr>
<tr>
<td>( M_2 = Q_n, n \geq 3 )</td>
<td>( n )</td>
<td>1 ( n^2 )</td>
<td>( n )</td>
<td>( n+2 )</td>
<td></td>
</tr>
<tr>
<td>( M_3 = \mathbb{CP}^n \times \mathbb{CP}^n )</td>
<td>( 2n )</td>
<td>( n^2 )</td>
<td>( 2n(n+1) )</td>
<td>( n+1 )</td>
<td>( 2n+2 )</td>
</tr>
<tr>
<td>( M_4 = G(2,s,C), s \geq 3 )</td>
<td>( 2s )</td>
<td>( \frac{1}{2}s(s-1) )</td>
<td>( 2s(s+2) )</td>
<td>( s+2 )</td>
<td>( 2s+2 )</td>
</tr>
<tr>
<td>( M_5 = \text{SO}(10)/U(5) )</td>
<td>1 ( 0 )</td>
<td>5</td>
<td>80</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>( M_6 = E_6/\text{Spin}(10)\times T )</td>
<td>16 ( 10 )</td>
<td>192</td>
<td>12</td>
<td>18</td>
<td></td>
</tr>
</tbody>
</table>

In this table \( n \) denotes complex dimension, \( p \) full complex dimension, \( \rho \) is the scalar curvature and \( \lambda_i \) the \( i \)-th eigenvalue.
13. COMPLEX SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE
WHICH ARE BI-ORDER. To motivate the study of complex submanifolds
M in $\mathbb{CP}^n$ whose corresponding imbedding $\psi: M \rightarrow H\mathbb{M}(m+1, \mathbb{C})$ is bi-order, we shall see what is the spectral behaviour of $\psi$ when M is the complex quadric
$$Q_n = \{ A \in \mathbb{CP}^{n+1} / AA^t = 0 \}.$$ 

It is clear that the imbedding of $Q_n$ in $H\mathbb{M}(n+2, \mathbb{C})$ is $SO(n+2)$-equivariant. One takes the point $C \in Q_n$ given by
$$C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{where} \quad c = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

It is not difficult to see that $T_C(Q_n) = \{ X \in T_C(\mathbb{CP}^{n+1}) / XC^t + CX^t = 0 \}$. Moreover by choosing an appropiate basis of $T_C(Q_n)$ one obtains the mean curvature vector of $\psi: Q_n \rightarrow H\mathbb{M}(n+2, \mathbb{C})$ at C to be
$$\eta = \frac{1}{2n}(I - (n+1)C - C^t)$$

and since $\psi$ is $SO(n+2)$-equivariant, last formula is true at any point $A \in Q_n$, that is,
$$(13.1) \quad \eta = \frac{1}{2n}(I - (n+1)\psi - \psi^t).$$

Finally it is easy to see that $\psi = \psi_0 + \psi_1 + \psi_2$, where $\psi_0 = \frac{1}{n+2} I$, is the center of gravity of $\psi$.

$$\psi_1 = \frac{1}{2}(\psi - \psi^t) \quad \text{with} \quad \Delta \psi_1 = n \psi_1 = \lambda_1 \psi_1,$$
$$\psi_2 = \frac{1}{2}(\psi + \psi^t) - \frac{1}{n+2} I \quad \text{with} \quad \Delta \psi_2 = \lambda_2 \psi_2 = (n+2)\psi_2,$$

which proves us that $\psi: Q_n \rightarrow H\mathbb{M}(n+2, \mathbb{C})$ is bi-order of order $\{1, 2\}$.

Before give a complete classification for complex submanifolds of $\mathbb{CP}^n$ whose corresponding imbeddings in $H\mathbb{M}(m+1, \mathbb{C})$ are bi-order, we will give a few technical lemmas whose proofs can be obtained directly.

**Lemma 13.1.** Let $M^n$ be a complex submanifold of $\mathbb{CP}^{n+p}$. If $\eta$ denotes the mean curvature vector of $\psi: M^n \rightarrow H\mathbb{M}(n+p+1, \mathbb{C})$, then

$$(13.2) \quad \eta = \frac{1}{2n} \sum_{i=1}^n \gamma(E_i, E_i),$$

$$(13.3) \quad \Delta \eta = (n+1)\eta + \frac{1}{n} \sum_{i,j} \frac{1}{2} \dot{\partial} \left( \sigma(E_i, E_j, E_i, E_j) \right) - \frac{1}{n} \sum_{i,j} \dot{\partial} \left( \sigma(E_i, E_j, \sigma(E_i, E_j)) \right).$$


where \( \sigma \) and \( \Lambda \) denote respectively the second fundamental form and the Weingarten endomorphism of \( \tau: M^n \rightarrow \mathbb{C}P^{n+p} \) and \( \{E_i\} \) is a local orthonormal basis of vector fields over \( M \).

Remark.- One notices that the vector fields along \( M \) defined by \( \eta \) and \( \Delta_n \) are normal to \( \mathbb{C}P^{n+p} \). Moreover if one puts

\[
\sigma(E_i, E_j) = \Gamma^k_{ij} h^k
\]

then (13.3) can be written as

\[
(3.14) \quad \Delta_n = (n+1) \eta + \int h^k \delta_j \delta_k (E_j, E_k) - \frac{1}{n} \int h_{ij} h_{ij} \delta (\xi, \xi)
\]

**Lemma 13.2**.- Let \( M^n \) be a complex submanifold of \( \mathbb{C}P^{n+p} \) and \( \psi: M^n \rightarrow \mathbb{H}M(n+p+1, \mathbb{C}) \). Then

\[
(13.5) \quad g(\psi, \psi) = 2
\]

\[
(13.6) \quad g(\psi, \eta) = -1
\]

\[
(13.7) \quad g(\psi, \Delta_n) = -(n+1)
\]

\[
(13.8) \quad g(\eta, \eta) = \frac{(n+1)^2}{2n}
\]

\[
(13.9) \quad g(\eta, \Delta_n) = \frac{(n+1)^2}{n^2} + (1/2n^2) \| \sigma \|^2
\]

\[
(13.10) \quad g(\Delta_n, \Delta_n) = \frac{(n+1)^3}{2n^2} + \frac{(n+1)}{n} \| \sigma \|^2 + \frac{1}{n^2} \int \xi_{\lambda \mu} (\text{Tr} A_\lambda \xi_{\mu})^2 + \frac{1}{n} \| A_\lambda \|^2
\]

Let \( T^A_M \) the normal space of \( \tau: M^n \rightarrow \mathbb{C}P^{n+p} \) at \( A \) and define a tensor \( T: T^A_M \times T^A_M \rightarrow \mathbb{R} \) by \( T(\xi, \psi) = \text{Tr}(\xi \psi) \) for all \( \xi, \psi \in T^A_M \). The length of \( T \) is estimated in the following lemma.

**Lemma 13.3**.- Let \( M^n \) be a complex submanifold of \( \mathbb{C}P^{n+p} \), then

\[
(13.11) \quad \frac{1}{2p} \leq |T|^2 \leq \frac{1}{2} |\sigma|^4
\]

Moreover first equality holds if and only if \( T = k \text{g} \) (\( k \) being some constant and \( g \) the restricted metric over \( T^A_M \)).

Proof.- \( |T|^2 = \int \xi (\text{Tr}(\Lambda, \Lambda'))^2, |\sigma|^2 = \int \xi \text{Tr} \Lambda^2 \).

Let \( G \) be the metric over the space of symmetric bilinear forms on \( T^A_M \) with the metric \( g \) over \( T^A_M \). Since \( T \) and \( g \) belong to such space, from the Schwartz inequality one has

\[
G(T, g) \leq |T|^2 |g| \leq |T|^2 |\sigma|^4,
\]

which proves the first inequality. The equality holding if and only
\( T = kg \) for some constant \( k \). The second inequality is well-known, [41].

**Theorem 13.4**. Let \( M^n \) be a full complex submanifold of \( \mathbb{C}P^{n+p} \). Then \( \psi : M^n \rightarrow H\mathbb{M}(n+p+1,C) \) is of order \( \{u_1, u_2\} \) (for certain natural number \( u_1, u_2 \)) if and only if

1. \( M \) is Einstein, and
2. \( T = kg \) for some real number \( k \).

**Proof.** Assume \( \psi \) is of order \( \{u_1, u_2\} \), then

\[ \Delta \eta = a \eta + b(\psi - Q) \]

for some real number \( a, b \) where \( Q \) is the center of gravity of \( \psi \) and \( \eta \) its mean curvature vector.

If \( b = 0 \), \( \Delta \psi = a \psi \), that is \( \Delta \psi - a \psi = C \) (constant) which implies that \( \psi \) is monotonous and from Theorem 11.1 \( M^n \) is totally geodesic, satisfying (1) and (2).

We now consider \( b \neq 0 \), because \( \psi, \eta \) and \( \Delta \eta \) define vector fields along \( M \) which are normal to \( \mathbb{C}P^{n+p} \) at each point of \( M \), one obtains that \( Q \) is in \( T^A_{A}(\mathbb{C}P^{n+p}) \) for all \( A \in M \), which proves that \( M \) is contained in the linear subspace \( L \) of \( H\mathbb{M}(n+p+1,C) \) defined by

\[ L = \{ A \in H\mathbb{M}(n+p+1,C) / AQ = QA \}. \]

From a well-known argument one can consider \( Q \) to be diagonal. Moreover by using a similar argument as in the proof of Theorem 11.1 one get

\[ Q = \frac{1}{n+p+1} I, \]

\( I \) being the identity matrix of order \( n+p+1 \).

Therefore

\[ (13.12) \Delta \eta = a \eta + b(\psi - Q). \]

Now we use some technical arguments to see the behaviour of the second fundamental form \( \sigma \) of \( \psi : M^n \rightarrow \mathbb{C}P^{n+p} \). Take \( r \neq s, s^* \) and use (8.13) and (8.14) to obtain

\[ (13.13) g(\sigma(E_r, E_s), \Delta \eta) = a g(\sigma(E_r, E_s), \eta). \]

Now from (8.11), (13.4) and (13.13), one get

\[ (13.14) \sum_{I} h^I \delta_{h^I} = 0. \]

Since any complex submanifolds satisfies

\[ (13.15) \sum_{I} h^I \delta_{h^I} = 0, \]
from (13.14) and (13.15) one get that \( \sum_{\lambda} \lambda_{i}^{2} \) is diagonal. Now we apply 
\[ g(\delta(E_{r},E_{r}),-) \] to (13.12) to obtain
\[
\frac{(n+1)^2}{2n} + \frac{1}{n} \sum_{i=1}^{n} h_{i}^2 r_{i} = \frac{n+1}{2n} a - b,
\]
which proves
\[
\sum_{\lambda} \lambda_{i}^{2} = [\frac{n+1}{2} a - nb - \frac{(n+1)^2}{2}] I
\]
Finally from (6.15) and (13.16) one get \( M \) is Einstein.
In order to prove (2) it is enough to apply the same argument over the normal bundle. The sufficient condition is essentially technical and no difficult.

The following corollary, [47], gives us precisely the eigenvalues \( \lambda_{1} \) and \( \lambda_{2} \) corresponding to the order \( \{u_{1}, u_{2}\} \) in last theorem.

COROLLARY 13.5.- Let \( M^{n} \) be a full complex submanifold of \( CP^{n+p} \). If \( M \) is Einstein and \( T = kg \), then
\[
\frac{1}{2} \left\{ n+1 + \frac{p+n}{pn} |\sigma|^{2} + \sqrt{(n+1 - \frac{p+n}{pn} |\sigma|^{2})^{2} + \frac{4}{n} |\sigma|^{2}} \right\}
\]
are eigenvalues of \( M \).

Proof.- According last theorem, one has
\[
\Delta n = a n + b(\psi - \frac{1}{n+p+1} I) \quad \text{for certain} \quad a, b \in \mathbb{R}
\]
\[
g(\psi, \Delta \psi) = ag(A, \eta) + bg(\psi, \psi - \frac{1}{n+p+1} I)
\]
\[
g(n, \Delta n) = ag(n, n) + bg(n, \psi - \frac{1}{n+p+1} I)
\]
Now we use lemma 13.2 to obtain
\[
a = n+1 + \frac{p+n}{pn} |\sigma|^{2}
\]
\[
b = \frac{p+n+1}{2pn} |\sigma|^{2}
\]
Since \( \psi: M \to \mathbb{HM}(n+p+1, C) \) is of order \( \{u_{1}, u_{2}\} \), one has
\[
\psi - \frac{1}{n+p+1} I = \psi u_{1} + \psi u_{2},
\]
\[
-2n \eta = \lambda_{1} \psi u_{1} + \lambda_{2} \psi u_{2},
\]
\[
-2n \Delta \eta = \lambda_{1}^{2} \psi u_{1} + \lambda_{2}^{2} \psi u_{2},
\]
which joint (13.12) gives us
\[
\left( \frac{\lambda_2^2 u_1 - a \lambda_1 u_1 + b}{2n} \right) \psi_{u_1} + \left( \frac{\lambda_2^2 u_1 - a \lambda_2 u_2 + b}{2n} \right) \psi_{u_2} = 0,
\]
that is \( \lambda_1 \) and \( \lambda_2 \) are roots of the equation \( \lambda^2 - a \lambda + 2nb = 0 \).

14. PARALLEL COMPLEX SUBMANIFOLDS AND BI-ORDER COMPLEX SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE ARE VERY CLOSE. In last section we shown that complex submanifolds of the complex projective space which are Einstein with \( T = kg \) have a good spectral behaviour for their corresponding immersions in Euclidean space. They are just the second step in the study of the Spectral Geometry of complex submanifolds in the complex projective space.

Now we will see that these conditions are close to a simple behaviour of the second fundamental form.

**THEOREM 14.1, [47].** Let \( M^n \) be a complex submanifold of \( CP^{n+p} \) whose immersion is full.

(i) If \( M \) is Einstein with \( T = kg \), then

\[
(14.1) \quad |\sigma|^2 \geq \frac{np(n+2)}{2p+n}.
\]

The equality holds if and only if \( v_0 = 0 \), with \( \sigma \neq 0 \).

(ii) If \( |\sigma|^2 = \frac{np(n+2)}{2p+n} \), then \( v_A = 0 \) if and only if \( M \) is Einstein with \( T = kg \).

**Proof.** (i) If \( M \) is Einstein, from (6.15) one has \( Tr \left( \sum_{\lambda} K^2 \right) = \frac{|\sigma|^4}{2n} \).

From \( T = kg \) and lemma 13.3, one get \( |\mathcal{T}|^2 = \sum_{\lambda} (\lambda \Lambda^\mu \Lambda_{\lambda})^2 = \frac{1}{2p} |\sigma|^4 \).

Now one uses last formulas in (6.13) to obtain

\[
\frac{1}{2} |\sigma|^2 = |\nu \sigma|^2 + \frac{n+2}{2} |\sigma|^2 - \frac{2p+n}{2np} |\sigma|^4.
\]

Since \( |\sigma|^2 \) is constant and so \( \Delta |\sigma|^2 = 0 \), one get

\[
|\nu \sigma|^2 = \frac{2p+n}{2np} |\sigma|^2 - \frac{n+2}{2} |\sigma|^2,
\]

which proves (i).

(ii) It is clear that \( \Delta |\sigma|^2 = 0 \), now from (6.13), (6.15) and lemma 13.3 one get

\[
|\nu \sigma|^2 \leq 2Tr \left( \frac{1}{2} |\sigma|^4 + \frac{1}{2p} |\sigma|^2 - \frac{n+2}{2} |\sigma|^2 \right) \geq \frac{1}{n} |\sigma|^4 + \frac{n+2}{2p} |\sigma|^2 - \frac{n+2}{2} |\sigma|^2 = 0.
\]
Remark.- If $M$ is Einstein with $T=kg$ and we assume that $\omega = 0$, with $\omega \neq 0$, then from the above lemma one has

$$|\sigma|^2 = \frac{np(n+2)}{2p+n}$$

and so the eigenvalues in corollary 13.5 are

$$n\left(\frac{n+p+1}{2p+n}\right), n+2 \in \text{Spec}(M).$$

As consequence, we will obtain a nice characterization of the Veronese imbedding. First we shall see the Veronese imbedding is of order $(1,2)$. In fact, $\omega = 0$ and

$$|\sigma|^2 = n(n+1) - p = \frac{1}{2}n(n+1)$$

Since $p = \frac{1}{2}n(n+1)$, we can write

$$|\sigma|^2 = \frac{np(n+2)}{2p+n}.$$ 

Therefore from theorem 14.1, $M$ is Einstein (well-known), $T=kg$ and the corresponding eigenvalues are

$$\lambda_1 = \frac{n(n+p+1)}{2p+n}, \quad \lambda_2 = \frac{n+1}{2}, \quad \lambda_3 = \frac{n+2}{2},$$

which proves us that the Veronese imbedding $\mathbb{C}P^n(1) \rightarrow \mathbb{C}P^{n+\frac{1}{2}n(n+1)}$ is of order $(1,2)$ in $\text{HM}(n+\frac{1}{2}n(n+1)+1,\mathbb{C})$.

**LEMMA 14.2**, [47].- Let $M^n$ be a complex submanifold of $\mathbb{C}P^{n+p}$ whose immersion is full. Suppose that $M$ is Einstein with $T=kg$. Then

(14.2) \quad \frac{1}{2}n(n+1) \geq p.$

Moreover the equality holds if and only if $\tau: M^n \rightarrow \mathbb{C}P^{n+p}$ is the Veronese imbedding.

**Proof.**- Because $|T|^2 = \frac{1}{2p} |\sigma|^4$ one has

$$|R|^2 = 2n(n+1) - 4|\sigma|^2 + \frac{1}{p} |\sigma|^4.$$ 

On the other hand $\rho = n^2(n+1)^2 + |\sigma|^4 - 2n(n+1)|\sigma|^2$. Since $\frac{1}{2}n(n+1)|R|^2 \geq \rho^2$ and equality holds if and only if $M$ has constant holomorphic sectional curvature. One get

$$\left\{ \frac{n(n+1)}{2p} - 1 \right\} |\sigma|^4 \geq 0,$$

that is $p \geq \frac{1}{2}n(n+1)$.

Moreover equality holds if and only if $M$ has constant holomorphic sectional curvature. But with codimension $\frac{1}{2}n(n+1)$ we know that $\tau: M \rightarrow \mathbb{C}P^{n+p}$ must be the Veronese imbedding, [11].
15. THE FIRST AND THE SECOND EIGENVALUES OF COMPLEX SUBMANIFOLDS IN THE COMPLEX PROJECTIVE SPACE. We already know that the first eigenvalue of an n-dimensional complex submanifold $M$ of $\mathbb{CP}^n$ satisfies: 

$$\lambda_1 \leq n+1$$

and the equality holds if and only if $M$ is totally geodesic. In this section we will obtain a spectral inequality for complex submanifolds of the complex projective space in terms of the five simplest spectral invariants: dimension, volume, total scalar curvature, first eigenvalue and second eigenvalue.

We start by taking $k=2$ and $t=1$ in Proposition 5.4 to obtain

$$\int_M g(\Delta^2 \psi, \psi) \, dv - (\lambda_1 + \lambda_2)\int_M g(\Delta \psi, \psi) \, dv + \lambda_1 \lambda_2 \int_M g(\Delta \psi, \psi) \, dv \geq 0,$$

and the equality holds if and only if the immersion $\psi: M \rightarrow H\mathbb{M}(m+1, \mathbb{C})$ is of order (1.2). Because $\Delta \psi = -2n \eta$, one can write from (15.1)

$$4n^2 \int_M g(\Delta \eta, \eta) \, dv = 4n^2 (\lambda_1 + \lambda_2)\int_M g(\eta, \eta) \, dv - 2n \lambda_1 \lambda_2 \int_M g(\eta, \eta) \, dv \geq 0.$$

THEOREM 15.1, [47]. - Let $M^n$ be a compact Kaehler submanifold of $\mathbb{CP}^m$, then

$$n[n+1+(n+1-\lambda_1)(n+1-\lambda_2)] \vol(M) \geq \int_M \rho \, dv.$$ 

Moreover if the equality holds, then $M$ is Einstein with $T = kg$ (if the immersion is full).

Proof. - Inequality (15.3) follows from (15.2) joint lemma 13.2. If equality holds, then $\psi: M^n \rightarrow H\mathbb{M}(m+1, \mathbb{C})$ is of order (1,2) and so from theorem 13.4 one obtains the second part.

COROLLARY 15.2, [23], [47]. - Let $M^n$ be an n-dimensional complex submanifold of $\mathbb{CP}^m$, then $\lambda_1 \leq n+1$. Moreover equality holds if and only if $M$ is totally geodesic.

Proof. - In proposition 5.4, one takes $k=1$, $t=1$ and $k=1$, $t=2$ respectively to obtain

$$\int_M g(\Delta^3 \psi, \psi) \, dv - \lambda_1 \int_M g(\Delta \psi, \psi) \, dv \geq 0,$$

and

$$\int_M g(\Delta^3 \psi, \psi) \, dv - \lambda_1 \int_M g(\Delta \psi, \psi) \, dv \geq 0.$$
Both equalities occur if and only if \( \psi: M \rightarrow H^1(M+1, C) \) is of order one. Since \( \Delta \psi = -2n\eta \), \( \Delta \) is self-adjoint, using lemma 13.2 one get

\[
(15.4) \quad 2n(n+1-\lambda_1)\text{vol}(M) \geq 0,
\]

\[
(15.5) \quad 2n(n+1)(n+1-\lambda_1)\text{vol}(M) + 2\int_M |\sigma|^2 d\nu \geq 0.
\]

Now (15.4) implies \( \lambda_1 \geq n+1 \). If the equality holds in (15.4), one also has the equality in (15.5) which implies \( \sigma = 0 \).

COROLLARY 15.3, [47].- Let \( M^n \) be a complex submanifold of \( CP^m \). Suppose that

\[
(15.6) \quad \lambda_1 = \frac{1}{n\text{vol}(M)} \int_M \rho \, d\nu
\]

and \( M \) is not totally geodesic. Then

\[
(15.7) \quad \lambda_2 \leq n+2.
\]

Moreover if the equality holds, then \( M \) is Einstein with parallel second fundamental form.

Proof.- From (15.3) and (15.6) one has

\[
n+1 + (n+1-\lambda_1)(n+1-\lambda_2) \geq \frac{1}{n\text{vol}(M)} \int_M \rho \, d\nu = \lambda_1,
\]

and so \( (n+1-\lambda_1)(n+2-\lambda_2) \geq 0 \). Because \( M \) is not totally geodesic \( \lambda_1 < n+1 \), then \( \lambda_2 \leq n+2 \) which proves (15.7).

The second part is very easy.

Remark.- Notice that condition (15.6) is natural because, for instance, every Einstein homogeneous Kaehler with positive scalar curvature satisfies it.

16. APPLICATIONS.

16.1.- Complete intersections.- Let \( M^n \) be a complex submanifold of \( CP^{n+p} \) determined by \( p \) polynomials \( P_1, \ldots, P_p \). If the Jacobian matrix associated with \( P_1, \ldots, P_p \) over \( \mathbb{P}^n(M) \) ( \( \mathbb{P}^n \) being the standard projection) is always of maximum rank, then we say that \( M^n \) is a complete intersection. In this case one has the following

THEOREM 16.1, [47].- Let \( M^n \) be a complex submanifold imbedded in \( CP^{n+p} \). Assume that \( M \) is a complete intersection of \( p \) homogeneous polynomials of degrees \( a_1, \ldots, a_p \). Then
Moreover the equality holds if and only if \( M = \mathbb{C}P^n \) or \( M = Q_n \) imbedded canonically.

Proof.- For complete intersections, the total scalar curvature was computed by K. Ogiue, [41] to be

\[
\int_M pdv = n(n+p+1-\tilde{a}_a)\text{vol}(M).
\]

Now (16.1) follows from (15.3) and (16.2).

Assume equality in (16.1), then one has equality in (15.3) and so from theorem 15.1 one get \( M \) must be Einstein. In these conditions a result due to J. Hano, [26], proves us that \( M = \mathbb{C}P^n \) (totally geodesic) or \( M = Q_n \) (canonical imbedding). The converse is obvious.

16.2.- The inverse problem for \( M_i \).- Because all elements in formulas (15.6) and (15.7) are spectral invariants, one can prove the so called inverse problem for \( M_i \). The following theorem was proved in [47] except for \( M_6 \), which was recently proved by S. Udagawa ("Spectral geometry of Kaehler submanifolds of a complex projective space" preprint).

**THEOREM 16.2.** Let \( M^n \) be an \( n \)-dimensional complex submanifold of \( \mathbb{C}P^m \). Assume that

\[
\text{Spec}(M) = \text{Spec}(M_i) \quad \text{for some} \quad i=1,..,6.
\]

Then \( M \) is imbedded and it is congruent with the standard imbedding of the corresponding \( M_i \) (of course \( \text{Spec}(M) \) means the spectrum of the induced metric).

Proof.- First we notice from table 1 (by looking at the dimension and \( \lambda \), for instance) that \( \text{Spec}(M_1) = \text{Spec}(M_j) \), \( i,j = 1,..,6 \) implies \( i=j \).

On the other hand, submanifolds \( M_i \) satisfy (15.6) and the equality in (15.7). Since they are spectral equalities, if \( M \) has the same spectrum of some \( M_i \), it must satisfy also (15.6) and equality in (15.7). Therefore \( M \) is Einstein and parallel but it is not totally geodesic (\( \lambda_i \neq n+1 \)). Consequently \( M \) is congruent with the standard imbedding of some \( M_j \), \( j=1,..,6 \). Now first remark we did proves us the theorem.
REFERENCES


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