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FOULSER'S COVERING THEOREM

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In [1] Foulser proved the following fundamental theorem regarding partial spreads.

COVERING THEOREM-THE BAER FORM:

Let V *be a vector space of dimension* 4r *over* GF(q), q *a prime,* r *an integer* ≥ 1. *Let* $\mathcal{N}$ *be a partial spread of* $1+p^r$ 2r*-dimensional subspaces which is covered by mutually disjoint Baer subplanes (subplanes of* $\mathcal{N}$ *of order* p^r *which are* 2r*-dimensional subspaces over* GF(p)). *Then the Baer subplanes are Desarguesian and* $\mathcal{N}$ *is the vector form of a regulus in* PG(3,K) *for some field* K *isomorphic to* $GF(p^r)$.

Similar situations concerning nets covered by subplanes continuously appear and one wonders about the nature of the net. That is:

Let V be a vector space of dimension 2s over GF(p). Let $\mathcal{N}$ be a partial spread of degree $1+p^t$ for some integer t. If there exists a set of mutually disjoint subplanes of $\mathcal{N}$ of order p^t which are also subspaces over GF(p) whose union covers the net $\mathcal{N}$, we shall simply say $\mathcal{N}$ is covered by subplanes.

So the general question becomes: if an arbitrary net $\mathcal{N}$ of degree $1+p^t$ is covered by subplanes of order p^t, are the subplanes Desarguesian and is the net $\mathcal{N}$ the vector form a $(\frac{s}{t}-1)$-regulus in $PG(\frac{2s}{t}-1,L)$ where L is a field isomorphic to

$GF(p^t)$? It has become apparent to the author that Foulser realized his methods answered this general question affirmatively, but this has gone essentially unnoticed to the workers in the field.

The purpose of this note is to show how this result may be obtained from Liebler's work [2] on enveloping algebras and Foulser's general ideas.

That is:

FOULSER'S COVERING THEOREM:

If a partial spread $\mathcal{N}$ *of degree* $1+p^t$ *within a vector space* V *of dimension* 2s *over* GF(p) *is covered by subplanes of order* p^t, *then the subplanes are Desarguesian and* $\mathcal{N}$ *is the vector form of a* $(\frac{s}{t}-1)$*-regulus in* $PG(\frac{2s}{t}-1, L)$ *for some field* $L \simeq GF(p^t)$.

The main distinction between the work of Foulser [1] and Liebler [2] is that Foulser essentially works from an associated coordinate structure of the net.

We initially follow Liebler [2]. Let $X \neq Y \in \mathcal{N}$. If $v \in V$ there exist unique vectors $pXY(v) \in X$ and $pYX(v) \in Y$ such that $v = pXY(v) + pYX(v)$. Associated with the net $\mathcal{N}$ are the slope transformations from X onto Y (also see Foulser [1], Lemma 1, p. 33 where X and Y are identified): If $Z \in \mathcal{N}$ define $sZXY : X \rightarrow Y$ by mapping $pXY(z) \rightarrow pYX(z)$ for $z \in Z$. Extend to a linear transformation of V $eZXY: V \rightarrow V$ by the mapping:

$$v \rightarrow sZXY(pXY(v)).$$

The GF(p)-algebra of linear transformations of V generated by

$\{eZXY$ for all $Z, X \neq Y \in \mathcal{N}\} = \mathcal{E}(\mathcal{N})$ is called the *enveloping algebra* of $\mathcal{N}$. (Note this is the direct sum of the algebra considered by Foulser.)

We state the following results of Liebler for convenience.

LEMMA (1.2) [2]: *A subspace* $U \subseteq V$ *is* $\mathcal{E}(\mathcal{N})$*-invariant if and only if* $\mathcal{N}_U = \{s \cap U \mid s \in \mathcal{N}\}$ *is a partial spread of* U.

LEMMA (1.3) [2] : *(Note that in our situation we have the hypotheses given by Liebler and that we have restated Liebler's results in our terms.)*

(a) *Each subplane of* $\mathcal{N}$ *(of the assumed cover) forms an irriducible* $\mathcal{E}(\mathcal{N})$*-invariant subspace* U.

(b) *Any* $\mathcal{E}(\mathcal{N})$*-invariant subspace* U *of dimension* $2t$ *over* $GF(p)$ *is a subplane of the cover.*

(c) *Let* L *denote the kernel of a subplane* T *of the cover. Then* $\mathcal{E}|T = \mathcal{E}(\mathcal{N}_T) = End_L(T)$.

THEOREM (1.4) [2]: *Let* T *be a subplane of the cover of* $\mathcal{N}$*. If* $\mathcal{E}(\mathcal{N})$ *acts faithfully on* T *then*

(a) $r = \dim_{GF(p)} V / \dim_{GF(p)} T$ *is an integer.*

(b) *Let* L *denote the kernel of* T*. The lattice of* $\mathcal{E}(\mathcal{N})$*-invariant subspaces is lattice isomorphic to the lattice of subspaces of an* r*-dimensional* L*-vector space in* $PG(r-1, L)$.

Proof of the Theorem:

V is a direct sum of subplanes of the cover so that $\frac{s}{t}$ is an integer.

Let P_1, P_2 be any two subplanes of the cover and let $W=P_1 \oplus P_2$ and $\mathscr{E}(\mathscr{N})|W$ be denoted by $\mathscr{E}(W)$. if P_3 is any subplane of the cover which intersects W nontrivially then P_3 is contained in W as $\mathscr{E}(\mathscr{N})$ acts irreducibly on P_3 and leaves W invariant. Hence, by the Krull-Schmidt Theorem, each subplane of the cover of W is $\mathscr{E}(W)$ isomorphic and $\mathscr{E}(W)$ acts faithfully on each subplane. Hence, if g is an element of $\mathscr{E}(\mathscr{N})$ and fixes P_1 pointwise then g must fix P_2 pointwise and since P_2 is arbitrary, g must be the identity mapping. Moreover, it now follows that any two $\mathscr{E}(\mathscr{N})$-invariant subspaces of dimension 2t are isomorphic as $\mathscr{E}(\mathscr{N})$-modules and are subplanes of the cover. So, $\mathscr{E}(\mathscr{N})$ clearly acts faithfully on the cover and by (1.3)(c), $\mathscr{E}(\mathscr{N}) \simeq \mathscr{E}(\mathscr{N}_T) = \mathrm{End}_L(T)$ so that $\mathscr{E}(\mathscr{N})$ is a simple ring. So, by (1.4)(b) (as $r = \frac{s}{t}$), the set of subplanes of the cover are in 1-1 correspondence with the number of points of PG(r 1,L). Since this number is $\frac{p^s-1}{p^t-1} = \frac{p^{tr}-1}{p^t-1} = \frac{q^r-1}{q-1}$ where $q=p^t$ (the points $\neq \mathcal{O}$ on a component of $\mathscr{N}$ are covered by $(p^s-1)/(p^t-1)$ subplanes), we have $|L| = p^t$ so that T is a Desarguesian subplane .

It remains to show that $\mathscr{N}$ is the vector form of a $(\frac{s}{t}-1)$ -regulus in $PG(\frac{2s}{t}-1,L)$. Since $\mathscr{E}(\mathscr{N}) \simeq \mathscr{E}(\mathscr{N}_T)$ then each element of $\mathscr{E}(\mathscr{N}_T)$ is induced from an element of $\mathscr{E}(\mathscr{N})$ as T is $\mathscr{E}(\mathscr{N})$-invariant. If T,U are elements of the cover then T and U are isomorphic as $\mathscr{E}(\mathscr{N})$-modules so that if $\sigma \in \mathscr{E}(\mathscr{N})$ induces an element of $\mathrm{End}_{\mathscr{E}(\mathscr{N}_T)}T$ then σ also induces an element of $\mathrm{End}_{\mathscr{E}(\mathscr{N}_U)}U$ of the same order on each subplane. This says that we may allow

L to act on V as V is a direct sum of isomorphic copies of T. Let $V = \bigoplus_{i=1}^{\frac{s}{t}} U_i$ for U_i an element of the cover for $i=1,\ldots,\frac{s}{t}$. Further, decompose $U_i = U_{1i} \oplus U_{2i}$ and write $V = \bigoplus_{i=1}^{\frac{s}{t}} U_{1i} \oplus \bigoplus_{i=1}^{\frac{s}{t}} U_{2i}$ where U_{ji} are t-dimensional subspaces of U_i for $j=1,2$. Represent a vector v in the form $(x_1,x_2,\ldots,x_{\frac{s}{t}},y_1,\ldots,y_{\frac{s}{t}})$ with $x_i \epsilon U_{1i}$ and $y_i \epsilon U_{2i}$ for each i. Let $U_{11} \simeq U_{1i}$ by f_i and $U_{21} \simeq U_{2i}$ by g_i. Let $\beta_i \epsilon L$ and simultaneously be considered to act on U_i for $i=1,\ldots,\frac{s}{t}$. For a fixed set $(\beta_1,\ldots,\beta_{\frac{s}{t}}) \neq (0,0,0,\ldots,0)$, form

$$\pi(\beta_1,\ldots,\beta_{\frac{s}{t}}) = \{(x_1\beta_1, f_2(x_1)\beta_2, f_3(x_1)\beta_3, \ldots, f_{\frac{s}{t}}(x_1)\beta_{\frac{s}{t}}, y_1\beta_1, g_2(y_1)\beta_2, \ldots, g_{\frac{s}{t}}(y_1)\beta_{\frac{s}{t}}) \text{ for all } x_1 \epsilon U_{11}, y_1 \epsilon U_{21}\}.$$

Since $\beta_i \epsilon \mathrm{End}_{\mathscr{E}(\mathscr{N}_{U_i})} U_i$, it follows that $\pi(\beta_1,\ldots,\beta_{\frac{s}{t}})$ is $\mathscr{E}(\mathscr{N})$-invariant and of dimension 2t. Hence, it must be that $\pi(\beta_1,\ldots,\beta_{\frac{s}{t}})$ is a subplane of the cover. And, although there are duplications, there are enough subplanes of this type to cover $\mathscr{N}$. That is, the cover consists of subplanes $\pi(\beta_1,\ldots,\beta_{\frac{s}{t}})$. (Note that we see that L acts transitively on each entry so that $(x_1\beta_1, f_2(x_1)\beta_2, \ldots, f_{\frac{s}{t}}(x_1)\beta_{\frac{s}{t}})$ for x_1 fixed takes on all vectors of $\bigoplus_{i=1}^{\frac{s}{t}} U_{1i}$ as $(\beta_1,\ldots,\beta_{\frac{s}{t}})$ varies over $\mathop{\times}_{i=1}^{\frac{s}{t}} L$.)

Now consider the Desarguesian plane Σ coordinatized by $GF(p^s)$. We consider $\Sigma = \{(x,y) \mid x,y \in GF(p^s)\}$ and regard lines in the form $y=xm$, $x = \Theta$ for $m \in GF(p^s)$.

Consider the partial spread $\mathscr{R}$ of Σ $\{y=x\alpha, x = \Theta$ for $\alpha \in GF(p^t)\}$. $y=x\alpha$ is $(x_1, x_2, \ldots, x_{\frac{s}{t}}, x_1\alpha, x_2\alpha, \ldots, x_{\frac{s}{t}}\alpha)$ for $x_i \in GF(p^t)$, $i=1,\ldots,\frac{s}{t}$ if we decompose $GF(p^s)$ over $GF(p^t)$.

With the isomorphisms suppressed, the vectors of $\pi(\beta_1, \ldots, \beta_{\frac{s}{t}})$ have the form $(x_1\beta_1, x_1\beta_2, \ldots, x_1\beta_{\frac{s}{t}}, y_1\beta_2, y_1\beta_2, \ldots, y_1\beta_{\frac{s}{t}})$ and clearly cover an isomorphic copy of $\mathscr{R}$.

That is, if $\mathscr{R}$ and $\mathscr{N}$ are regarded on the same points, they are replacements for each other and thus must be equal.

Within $PG(\frac{2s}{t}-1, GF(p^t))$, the subplanes of $\mathscr{R}$ are lines and the lines $y = x\alpha$, $x = \Theta$ are $(\frac{s}{t}-1)$-dimension subspaces. So we have a set of p^t+1 subspaces of dimension $(\frac{s}{t}-1)$ which are covered by lines. That is, $\mathscr{R}$ and hence $\mathscr{N}$ is a $(\frac{s}{t}-1)$-regulus in $PG(\frac{2s}{t}-1, L)$. This proves the theorem.

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