INTRODUCTION. Given two locally convex spaces E,F, and a set $\mathcal{M}$ of bounded subsets of E, let $\mathcal{L}_{\mathcal{M}}(E,F)$ denote the space of all linear continuous maps from E into F, provided with the topology of uniform convergence on all sets in $\mathcal{M}$. This treatise is mainly devoted to the study of the following question:

If the dual space $E'_{\mathcal{M}}$ $(:=\mathcal{L}_{\mathcal{M}}(E,\mathbb{K}))$ and F both have some property $\mathbb{P}$, does then $\mathcal{L}_{\mathcal{M}}(E,F)$ also possess this property (at least under reasonable additional hypotheses)?

For cumpleteness properties $\mathbb{P}$,A.Grothendieck has given a positive answer (under mild restrictions) to this question [15;p.9], whereas for the property of being a Schwartz space, D.J.Randthe gave a positive answer (see [17;p.353,1.Thm]). - In this treatise we will mostly be interested in properties which are invariant under the formation of certain final locally convex topologies, in which case the above problem is closely related to the question whether the functors $\mathcal{L}_{\mathcal{M}}(E,.)$ and $\mathcal{L}_{\mathcal{M}(.)}(.,F)$ are compatible with the formation of certain inductive and projective limits, respectively. Therefore the search for such compatibility statements will be part of our investigations.

In section one we first recall some fundamental facts about the spaces $\mathcal{L}_{\mathcal{M}}(E,F)$ and then investigate under what hypotheses the canonical continuous linear injections

$$\Phi: \bigoplus_{\iota\in I} \mathcal{L}_{\mathcal{M}}(E,F_\iota) \to \mathcal{L}_{\mathcal{M}'}(E,\bigoplus_{\iota\in I} F_\iota) \quad \text{and}$$

$$\Psi: \bigoplus_{\iota\in I} \mathcal{L}_{\mathcal{M}}(E_\iota,F) \to \mathcal{L}_{\mathcal{M}}(\prod_{\iota\in I} E_\iota,F), \text{ respectively,}$$

are topological isomorphisms.

In the special case that I is countably infinite and that $\mathcal{M}$ coincides with the system $\mathcal{B}$ of all bounded subsets, these hypotheses are just the "countable boundedness condition" (cbc) for E and the "countable neighbourhood condition" (cnc) for F, respectively. Both these properties as well as the property (fsb) of having a fundamental sequence of bounded sets, are investigated in the first part of section two. By making use of these properties we find hypotheses for a pair (E,F) of locally convex spaces which yield that a subset $\mathcal{H}$ in $\mathcal{L}_b(E,F)$ $(:=\mathcal{L}_{\mathcal{B}}(E,F))$ is bounded if and only if there exists a bornivorous barrel U in E such that $\mathcal{H}(U)$ is bounded in F.
Furthermore, carrying on the compatibility investigations of section one we show:
A locally convex space E satisfies (cbc) if and only if the canonical map $\Phi : \bigoplus_{n\in\mathbb{N}} \mathcal{L}_b(E,F_n) \to \mathcal{L}_b(E,\bigoplus_{n\in\mathbb{N}} F_n)$ is a topological isomorphism for every sequence $(F_n)_{n\in\mathbb{N}}$ of Hausdorff locally convex spaces, and a Hausdorff locally convex space F satisfies (cnc) if and only if the canonical map $\Psi : \bigoplus_{n\in\mathbb{N}} \mathcal{L}_b(E_n,F) \to \mathcal{L}_b(\prod_{n\in\mathbb{N}} E_n,F)$ is a topological isomorphism for every sequence $(E_n)_{n\in\mathbb{N}}$ of locally convex spaces.

The methods used in the proofs give rise to examples of the following kind: Even for rather "nice" spaces E,F the spaces $\mathcal{L}_b(E,\varphi)$ and $\mathcal{L}_b(\omega,F)$ need not even be countably quasibarrelled.

These results show that a decent behaviour of $\mathcal{L}_b(E,F)$ can only be expected if E satisfies (cbc) and F satisfies (cnc), which leads us close to A. Grothendieck's famous question [14;p.120]:

(*) If E is a metrizable locally convex space and F a DF-space, is then $\mathscr{L}_b(E,F)$ again a DF-space?

Unfortunately, this problem remains unsolved; nevertheless the contents of the sections three to five of this treatise are meant to be contributions to its solution.

In contrast to problem (*) it is well known that for two DF-spaces G,F the projective tensor product $G \otimes_\pi F$ is again a DF-space. In section three we recall under what circumstances the natural continuous linear injection $E_b' \otimes_\pi F \to \mathscr{L}_b(E,F)$ carries the DF-space property of $E_b' \otimes_\pi F$ over to $\mathscr{L}_b(E,F)$.

Section four is devoted to the following special case of problem (*): Let E be a Banach space and $F=\operatorname{ind}_{n\to} F_n$ an LB-space. What can be said about $\mathscr{L}_b(E,F)$? In particular, is the canonical map

$$\Phi : \operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n) \to \mathscr{L}_b(E,\operatorname{ind}_{n\to} F_n)$$

a topological isomorphism?

We obtain - roughly speaking - the following results.

(1) For every LB-space F the space $\mathscr{L}_b(l^1(I),F)$ is a DF-space.

(2) If $F = \operatorname{ind}_{n\to} F_n$ is a retractive LB-space, then

$$\Phi : \operatorname{ind}_{n\to} \mathscr{L}_b(l^1(I),F_n) \to \mathscr{L}_b(l^1(I), \operatorname{ind}_{n\to} F_n)$$

is a topological isomorphism.

(3) For every $1 \leq p < \infty$ there exist reflexive LB-spaces $\operatorname{ind}_{n\to} F_n$ such that $\Phi : \operatorname{ind}_{n\to} \mathscr{L}_b(l^p,F_n) \to \mathscr{L}_b(l^p,\operatorname{ind}_{n\to} F_n)$ is not open.

In section five - which is to a large extent dual to section four - we investigate another special case of problem (*):

Let F be a Banach space and E be a Fréchet space, represented as the projective limit $\operatorname{proj}_{\leftarrow n} E_n$ of a projective sequence of Banach spaces E_n ($n\in\mathbb{N}$).

What can be said about $\mathscr{L}_b(E,F)$? In particular, is the canonical map

$$\Psi : \operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(\operatorname{proj}_{\leftarrow n} E_n, F)$$

a topological isomorphism?

We present the following results:

(1') For every Fréchet space E the space $\mathscr{L}_b(E,l^\infty(I))$ is a DF-space.

(2') If $E = \operatorname{proj}_{\leftarrow n} E_n$ is a quasinormable reduced projective limit of a sequence of Banach spaces E_n, then the map

$$\Psi : \operatorname{ind}_{n\to} \mathscr{L}_b(E_n,l^\infty(I)) \to \mathscr{L}_b(E,l^\infty(I))$$

is a topological isomorphism.

(2") Let $E = \operatorname{proj}_{\leftarrow n} E_n$ be the strict projective limit of a sequence of Banach spaces (i.e. the canonical projections $E \to E_n$ ($n\in\mathbb{N}$) are all surjective) and let F be a Banach space with the λ extension property for some $\lambda \geq 1$. Then

$$\Psi : \operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(E,F)$$

is a topological isomorphism.

(3') For every $1 < q \leq \infty$ there exist reflexive Fréchet spaces $E = \operatorname{proj}_{\leftarrow n} E_n$ such that $\Psi : \operatorname{ind}_{n\to} \mathscr{L}_b(E_n,l^q) \to \mathscr{L}_b(E,l^q)$ is not open.

Furthermore, a detailed description of strict projective limits of a sequence of Banach spaces is worked out.

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1 would like to thank my husband Prof.Dr.P.Dierolf, who pointed out to me the context of spaces $\mathscr{L}_b(E,F)$ as an interesting field of research. Finally 1 would like to thank Prof. Dr.K.Floret and Dr.A.Defant, who took active interest in my investigations, spent numerous hours of discussions with me on this subject, and made many helpful comments.

NOTATIONS AND TERMINOLOGY

Let E be a locally convex space. By $\mathcal{U}_0(E)$ we denote the filter of all zero-neighbourhoods in E and by $\mathcal{B}(E)$ the system of all bounded subsets of E. E' denotes the topological dual of E and E^* its algebraic dual. For a dual pair $\langle F,G\rangle$ we denote by $\sigma(F,G)$, $\beta(F,G)$, $\tau(F,G)$ the weak, the strong, and the Mackey topology on F, respectively, whereas by $\beta^*(F,G)$ we denote the topology on F of uniform convergence on all $\beta(G,F)$-bounded subsets of G. We will write E_s, E_b, and E_{b^*} instead of $(E,\sigma(E,E'))$, $(E,\beta(E,E'))$, $(E,\beta^*(E,E'))$, respectively, and use similar notations for E' instead of E. For a subset A in E we denote by ΓA its absolutely convex hull, by $[A]$ its linear span, and by $A^\circ := \{f\in E': |f(x)| \leq 1 \text{ for all } x\in A\}$ its polar. A subset $A \subset E$ is called total in E if its linear span $[A]$ is dense in E. ▪ Given a linear subspace $L \subset E$, we denote by E/L the corresponding quotient space provided with the quotient topology.

Locally convex spaces are not tacitly assumed to be Hausdorff. Thus we will speak of pseudometrizable locally convex spaces E (which means that $\mathcal{U}_0(E)$ has a countable basis) and of seminormable spaces E (which means that E has a bounded and absolutely convex zero-neighbourhood).

$\mathbb{K}$ stands for one of the scalar fields $\mathbb{R}$ or $\mathbb{C}$, $\mathbb{R}_+^*$ denotes the set of (strictly) positive real numbers, and $\mathbb{N} = \{1,2,3,...\}$ the set of positive integers.

A map $f: E \to F$ between two locally convex spaces E, F is called a topological isomorphism if f is a linear homeomorphism onto F. Thus a topological isomorphism is by definition always surjective. If a map $f: E \to F$ has the property that the induced map $E \to f(E)$, $x \mapsto f(x)$, is a topological isomorphism (where $f(E)$ carries the relative topology induced by F), then we will call f a topological isomorphism onto its range.

Given a family $(E_\iota)_{\iota\in I}$ of locally convex spaces, we denote by $\prod_{\iota\in I} E_\iota$ the corresponding product space provided with the product topology and by $\bigoplus_{\iota\in I} E_\iota$ its locally convex direct sum. $p_\kappa: \prod_{\iota\in I} E_\iota \to E_\kappa$, $(x_\iota)_{\iota\in I} \mapsto x_\kappa$, denotes the canonical projection and $J_\kappa: E_\kappa \to \bigoplus_{\iota\in I} E_\iota$, $x \mapsto (\delta_{\iota\kappa} x)_{\iota\in I}$ (where $\delta_{\iota\kappa} := \begin{matrix}1\\0\end{matrix}$ if $\begin{matrix}\iota=\kappa\\ \iota\neq\kappa\end{matrix}$) stands for the natural injection ($\kappa\in I$). If $E_\iota = E$ for all $\iota\in I$, we write E^I instead of $\prod_{\iota\in I} E$. We also use the notations ω for $\mathbb{K}^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathbb{K}$ and φ for $\bigoplus_{n\in\mathbb{N}} \mathbb{K}$.

Let E be a linear space. If $(x_\iota)_{\iota\in I}$ is a family in E such that $K:=\{\iota\in I: x_\iota \neq 0\}$ is finite, let $\sum_{\iota\in I} x_\iota := \sum_{\iota\in K} x_\iota$. If $(A_\iota)_{\iota\in I}$ is a family of subsets in E such that $0\in A_\iota$ ($\iota\in I$), let $\sum_{\iota\in I} A_\iota := \bigcup_{K\subset I,\, K \text{ finite}} \sum_{\iota\in K} A_\iota$. Finally, if $(E_\iota)_{\iota\in I}$ is a family of linear spaces and A_ι a subset of E_ι containing 0 ($\iota\in I$), we will sometimes use the notation $\sum_{\iota\in I} A_\iota$ instead of $\sum_{\iota\in I} J_\iota(A_\iota)$ $(= \prod_{\iota\in I} A_\iota \cap \bigoplus_{\iota\in I} E_\iota) \subset \bigoplus_{\iota\in I} E_\iota$.

As far as the general locally convex terminology is concerned, we follow J.Horvath [16] and G.Köthe [20,21].

§1. THE SPACE $\mathscr{L}_{\mathscr{M}}(E,F)$

Let E and F be locally convex spaces, and let $\mathscr{L}(E,F)$ denote the vector space of all continuous linear maps from E into F. Let $\mathscr{M}$ be a subset of the set $\mathscr{B}(E)$ of all bounded subsets of E. By $\mathscr{L}_{\mathscr{M}}(E,F)$ we denote the space $\mathscr{L}(E,F)$ provided with the topology of uniform convergence on all sets in $\mathscr{M}$, and we use the abbreviation $E'_{\mathscr{M}}$ instead of $\mathscr{L}_{\mathscr{M}}(E,\mathbb{K})$. Clearly, $\mathscr{L}_{\mathscr{M}}(E,F)$ is a locally convex space. The sets

$$\mathscr{W}(M,U):=\{T\in\mathscr{L}(E,F):T(M)\subset U\}\quad(M\in\mathscr{M},\,U\in\mathscr{U}_0(F))$$

form a subbasis of $\mathscr{U}_0(\mathscr{L}_{\mathscr{M}}(E,F))$; they form a basis of $\mathscr{U}_0(\mathscr{L}_{\mathscr{M}}(E,F))$, if $\mathscr{M}$ is stable under finite unions. It should be mentioned that the notation $\mathscr{W}(M,U)$ is not without ambiguity since neither E nor F occur in the symbol. Nevertheless we will sometimes use this notation, when it will be clear from the context which spaces E,F are under consideration.

Important examples of sets $\mathscr{M}$ are, for instance,

the set $\mathscr{Y}(E)$ of all finite subsets of E; we write $\mathscr{L}_s(E,F)$ and E'_s instead of $\mathscr{L}_{\mathscr{Y}(E)}(E,F)$ and $E'_{\mathscr{Y}(E)} = (E',\sigma(E';E))$, respectively;

the set $\mathscr{C}(E)$ of all compact subsets of E;

the set $\mathscr{B}(E)$ of all bounded subsets of E; we write $\mathscr{L}_b(E,F)$ and E'_b instead of $\mathscr{L}_{\mathscr{B}(E)}(E,F)$ and $E'_{\mathscr{B}(E)} = (E',\beta(E',E))$, respectively.

Since $\mathscr{W}(M,U)$ is closed in $\mathscr{L}_s(E,F)$ for every $M\subset E$ and every closed subset $U\subset F$, we obtain that $\mathscr{U}_0(\mathscr{L}_{\mathscr{M}}(E,F))$ always has a basis consisting of sets which are closed with respect to pointwise convergence. The identity map $\mathcal{L}_{\mathcal{M}}(E,F) \to \mathcal{L}_s(E,F)$ is continuous if $\mathcal{M}$ covers E.

First of all we will collect some elementary statements about the interrelations between the formation of $\mathcal{L}_{\mathcal{M}}(E,F)$ and initial or final locally convex topologies. Straightforward proofs will be omitted.

(1.1.) (See G.Köthe [21;p.151].)

Let E be a locally convex space and $\mathcal{M} \subset \mathcal{B}(E)$.

(a) For all locally convex spaces, F,G and all $S \in \mathcal{L}(F,G)$ the canonical linear map

$$\mathcal{L}_{\mathcal{M}}(E,F) \to \mathcal{L}_{\mathcal{M}}(E,G),\ T \mapsto S \circ T,$$

is continuous.

(b) Let G be a locally convex space and $F \subset G$ a linear subspace. Then the canonical injection

$$\mathcal{L}_{\mathcal{M}}(E,F) \to \mathcal{L}_{\mathcal{M}}(E,G),\ T \mapsto J \circ T,$$

induced by the inclusion map $J: F \hookrightarrow G$, is a topological isomorphism onto its range.

(c) For every family $(F_\iota)_{\iota \in I}$ of locally convex spaces, the map

$$\mathcal{L}_{\mathcal{M}}(E, \prod_{\iota \in I} F_\iota) \to \prod_{\iota \in I} \mathcal{L}_{\mathcal{M}}(E,F_\iota),\ T \mapsto (P_\iota \circ T)_{\iota \in I},$$

is a topological isomorphism.

(In fact, since the map is bijective, it suffices to compare subbases of $\mathcal{U}_0$ of the demain space and the range space; this can be done in the usual way: $\prod_{\kappa \in I}(P_\kappa \circ \mathcal{W}(M, \prod_{\iota \in I} U_\iota)) =$ $= \prod_{\kappa \in I} \mathcal{W}(M,U_\kappa)$ $(M \in \mathcal{M},\ U_\iota \in \mathcal{U}_0(F_\iota)\ (\iota \in I)$, where $\{\iota \in I : U_\iota \neq F_\iota\}$ is finite).)

(d) Whenever a locally convex space F carries the initial topology with respect to a family $(F_\iota)_{\iota\in I}$ of locally, convex spaces and linear maps $S_\iota : F \to F_\iota$, $(\iota\in I)$, then $\mathcal{L}_{\mathcal{M}}(E,F)$ carries the initial topology with respect to

$$\mathcal{L}(E,F) \to \mathcal{L}_{\mathcal{M}}(E,F_\iota),\ T \mapsto S_\iota \circ T, \qquad (\iota \in I).$$

(In fact, this statement follows easily from (b) and (c).

Remark: The question what happens if F carries a final locally convex topology (e.g., if F is a direct sum or an inductive limit) is more involved and will be dealt with later; see (1.11),(2.10), (2.11),(4.2),(4.7),(4.8),(4.12),(5.16)(d).

(1.2) (See G.Köthe [21;p.147].)

Let F be a locally convex space. Roughly speaking, we want to find a connection between spaces $\mathcal{L}_{\mathcal{M}_1}(E_1,F)$ and $\mathcal{L}_{\mathcal{M}_2}(E_2,F)$ with respect to linear continuous maps $E_1 \to E_2$. Now clearly, one cannot say anything unless the systems $\mathcal{M}_i$ behave well under continuous linear maps. Thus we will for the moment assume the following situation: Let $\mathcal{M}$ be a functor from the category of locally convex spaces into the category of sets, which assigns to every locally convex space E a subset $\mathcal{M}(E) \subset \mathcal{B}(E)$ such that $T(\mathcal{M}(E)) \subset \mathcal{M}(F)$ for all locally convex spaces E,F and all $T\in\mathcal{L}(E,F)$. (The systems $\mathcal{Y}(E)$, V(E), $\mathcal{B}(E)$ and many others arise from such a functorial concept.) It will cause no confusion with our previous notations if we write $\mathcal{L}_{\mathcal{M}}(E,F)$ instead of $\mathcal{L}_{\mathcal{M}(E)}(E,F)$.

(a) For all locally convex spaces E,H and every $S\in\mathcal{L}(E,H)$ the canonical linear map:

$\mathscr{L}_{\mathscr{M}}(H,F) \to \mathscr{L}_{\mathscr{M}}(E,F)$, $T \mapsto T \circ S$,

is continuous.

(In fact, let $M \in \mathscr{M}(E)$ and $U \in \mathscr{U}_0(F)$. Then $S(M) \in \mathscr{M}(H)$ and $\mathscr{W}(S(M),U) \circ S \subset \mathscr{W}(M,U)$.)

(b) Let E be a locally convex space, let $L \subset E$ be a linear subspace, and let $Q : E \to E/L$ denote the quotient map. Assume that for every $M \in \mathscr{M}(E/L)$ there exists a finite subset $\mathscr{N} \subset \mathscr{M}(E)$ such that $\overline{\Gamma Q(\bigcup \mathscr{N})} \supset M$. Then the canonical injection

$$\mathscr{L}_{\mathscr{M}}(E/L,F) \to \mathscr{L}_{\mathscr{M}}(E,F),\ T \mapsto T \circ Q,$$

is a topological isomorphism onto its range.

(In fact, by the injectivity and continuity of the map, it suffices to show that for every $M \in \mathscr{M}(E/L)$ and $U = \overline{\Gamma U} \in \mathscr{U}_0(F)$ the set $\mathscr{W}(M,U) \circ Q$ is a zero-neighbourhood in the range of the map. Choose $\mathscr{N} \subset \mathscr{M}(E)$ according to the hypothesis. Then $\mathscr{W} := \bigcap_{N \in \mathscr{N}} \mathscr{W}(N,U) \in \mathscr{U}_0(\mathscr{L}_{\mathscr{M}}(E,F))$ and whenever $T \circ Q \in \mathscr{W}$, then $T(M) \subset T(\overline{\Gamma Q(\bigcup \mathscr{N})}) \subset \overline{\Gamma \bigcup_{N \in \mathscr{N}} (T \circ Q)(N)} \subset \overline{\Gamma U} = U$.)

(c) For every family $(E_\iota)_{\iota \in I}$ of locally convex spaces, the map

$\mathscr{L}_{\mathscr{M}}(\bigoplus_{\iota \in I} E_\iota, F) \to \prod_{\iota \in I} \mathscr{L}_{\mathscr{M}}(E_\iota, F)$, $T \mapsto (T \circ J_\iota)_{\iota \in I}$,

is a topological isomorphism.

(In fact, it again suffices to show that for every $M \in \mathscr{M}(\bigoplus_{\iota \in I} E_\iota)$ and $U = \overline{U} \in \mathscr{U}_0(F)$, the set $\prod_{\iota \in I}(\mathscr{W}(M,U) \circ J_\iota)$ belongs to $\mathscr{U}_0(\prod_{\iota \in I} \mathscr{L}_{\mathscr{M}}(E_\iota, F))$. By the continuity of the projections $P_\iota : \bigoplus_{\kappa \in I} E_\kappa \to E_\iota$ $(\iota \in I)$, we have that $P_\iota(M) \in \mathscr{M}(E_\iota)$ $(\iota \in I)$. Moreover, as M is bounded in $\bigoplus_{\iota \in I} E_\iota$, the set $K := \{\iota \in I : P_\iota(M) \not\subset \{0\}\}$ is finite. Now put $\mathscr{W}_\iota := \mathscr{W}(P_\iota(M), \frac{1}{\text{card } K} U) \subset \mathscr{L}(E_\iota, F)$

$(\iota\in I)$. Then $\prod_{\iota\in I}\mathscr{W}_\iota \in \mathscr{U}_0(\prod_{\iota\in I}\mathscr{L}_{\mathscr{M}}(E_\iota,F))$ and whenever $(T_\iota)_{\iota\in I}\in\sum_{\iota\in I}\mathscr{W}_\iota$ and $(x_\iota)_{\iota\in I}\in M\subset\bigoplus_{\iota\in I}E_\iota$, then
$\sum_{\iota\in I}T_\iota(x_\iota) = \sum_{\iota\in K}T_\iota(x_\iota) + \sum_{\iota\in I\setminus K}T_\iota(x_\iota) \in \sum_{\iota\in K}\frac{1}{\text{card } K}U+\overline{\{0\}}\subset\bar{U} = U$;
hence $(T_\iota)_{\iota\in I}\in\prod_{\iota\in I}(\mathscr{W}(M,U)\circ J_\iota)$.)

(d) Let E be a locally convex space carrying the final locally convex topology with respect to a family $(E_\iota)_{\iota\in I}$ of locally convex spaces and linear maps $S_\iota : E_\iota \to E$. Suppose that for every $M\in\mathscr{M}(E)$ there exists a finite subset $K\subset I$ and for every $\iota\in K$ a finite subset $\mathscr{N}_\iota\subset\mathscr{M}(E_\iota)$ such that $M\subset\bar{\Gamma}\bigcup_{\iota\in K}S_\iota(\bigcup\mathscr{N}_\iota)$.

Then $\mathscr{L}_{\mathscr{M}}(E,F)$ carries the initial topology with respect to $\mathscr{L}(E,F)\to\mathscr{L}_{\mathscr{M}}(E_\iota,F)$, $T\mapsto T\circ S_\iota$, $(\iota\in I)$.

(In fact, if $\bigcup\mathscr{M}(E)\subset\overline{\{0\}}^E$, then $\mathscr{L}_{\mathscr{M}}(E,F)$ carries the coarsest topology and there is nothing to prove. Thus we may assume that $\bigcup\mathscr{M}(E)\not\subset\overline{\{0\}}^E$. Since for every $x\in E\setminus\overline{\{0\}}$ and every $y\in E$ there is $T\in\mathscr{L}(E,E)$ such that $T(x) = y$, we obtain by the functorial properties of $\mathscr{M}$ that $\mathscr{M}(E)$ covers E. Now one deduces from the above hypothesis that $\sum_{\iota\in I}S_\iota(E_\iota)$ is dense in E, whence $\sum_{\iota\in I}S_\iota(E_\iota) = E$ (as E carries the final locally convex topology); now the assertion follows from (b) and (c).)

Remark: The question what happens if E carries an initial topology (e.g., if E is a product space or a projective limit) is again more involved and will be dealt with later; see (1.12), (2.13). (2.14). (5.2). (5.3), (5.7), (5.8). (5.9); (5.15).

Whenever E,F are linear spaces, f a linear functional on E and $y\in F$, we denote by $f\otimes y$ the linear map

$$f \otimes y : E \to F, \quad x \mapsto f(x)y.$$

The following well-known proposition (see G.Köthe [21;p.132 (2')]) establishes a connection between $\mathscr{L}_{\mathscr{M}}(E,F)$ and the spaces $E'_{\mathscr{M}}$ and F.

(1.3.) Let E,F be a locally convex spaces and let $\mathscr{M} \subset \mathcal{O}(E)$.

(a) For every $y \in F\setminus\{0\}$ the map

$$\Phi : E'_{\mathscr{M}} \to \mathscr{L}_{\mathscr{M}}(E,F), f \mapsto f \otimes y ,$$

is a topological isomorphism onto a topologically complemented subspace of $\mathscr{L}_{\mathscr{M}}(E,F)$.

(b) Assume that $\cup\mathscr{M}$ is total in E. Then for every $f \in E'\setminus\{0\}$ the map

$$\Psi : F \to \mathscr{L}_{\mathscr{M}}(E,F), \quad y \mapsto f \otimes y ,$$

is a topological isomorphism onto a topologically complemented subspace of $\mathscr{L}_{\mathscr{M}}(E,F)$.

Proof. Clearly, Φ and Ψ are both linear, continuous, and injective.

(a) Let $P : F \to F$ be a continuous linear projector such that $P(F) = [y]$. For every $T \in \mathscr{L}(E,F)$ there exists a unique $f_T \in E'$ such that $P \circ T = f_T \otimes y$. The map

$$\check{\Phi} : \mathscr{L}_{\mathscr{M}}(E,F) \to E'_{\mathscr{M}} , \quad T \mapsto f_T$$

is linear, continuous, and $\check{\Phi} \circ \Phi$ equals the identity map; consequently, $\Phi \circ \check{\Phi} : \mathscr{L}_{\mathscr{M}}(E,F) \to \mathscr{L}_{\mathscr{M}}(E,F)$ is a linear continuous projector onto $\Phi(E')$, which finishes the proof of (a).

(b) Since $f \neq 0$ and $\cup\mathscr{M}$ is total in E, there is $x \in \cup\mathscr{M}$ such that $f(x) \neq 0$. The map

$$\check{\Psi} : \mathscr{L}_{\mathscr{M}}(E,F) \to F, \quad T \mapsto \frac{1}{f(x)}T(x) ,$$

is linear, continuous and $\psi \circ \psi$ equals the identity map: consequently, $\psi \circ \psi$ is a linear continuous projector onto y(F), which finishes the proof of (b).

Thus, as $E'_{\mathcal{M}}$ and F can be looked upon as complemented subspaces of $\mathscr{L}_{\mathcal{M}}(E,F)$, any reasonable property that $\mathscr{L}_{\mathcal{M}}(E,F)$ might have, belongs also to $E'_{\mathcal{M}}$ and to F. In this situation one may ask what can be said about the converse implication.

(1.4) Question: Given two locally convex spaces E and F, such that $E'_{\mathcal{M}}$ and F both have a certain property (IP), does then $\mathscr{L}_{\mathcal{M}}(E,F)$ also possess this property (at least under reasonable additional hypotheses)?

One has following classical result of this type due to A.Grothendieck [15;Intr.;p.9]:

THEOREM. ***Let** E **and** F **be two Hausdorff locally convex spaces such that** $\mathscr{L}(E,F) = \mathscr{L}(E,F_s)$. (This assumption is satisfied, if, for instance,* E *is a Mackey **space.)** Moreover let* $\mathcal{M} \subset \mathscr{B}(E)$ *satisfy* $\cup \mathcal{M} = E$.

If $E'_{\mathcal{M}}$ ***and*** F ***are complete, then also*** $\mathscr{L}_{\mathcal{M}}(E,F)$ ***is complete.***

Remark. The hypothesis that $\cup \mathcal{M} = E$ cannot be dropped as the following example shows.

Let E denote a product $\prod_{\iota \in I} E_\iota$ of Banach spaces which contains the space φ as linear subspace. Then φ is not complemented in E as E is a Baire space and φ is not. Let $\mathcal{M} := \{B \subset E : B$ is finite and $B \subset \varphi\}$ and put $F := \varphi$. F is clearly complete; moreover, the Hausdorff locally convex space associated with $E'_{\mathcal{M}}$ (i.e. ,the quotient $E'_{\mathcal{M}}/\overline{\{0\}}^{E'_{\mathcal{M}}}$) is topologically isomorphic to $\varphi'_s = \varphi'_b = \omega$, hence complete. E, being a Baire space, is certainly a Mackey space. We will show now that $\mathscr{L}_{\mathscr{M}}(E,F)$ is not even sequentially complete. In fact, choose an increasing sequence $(L_n)_{n\in\mathbb{N}}$ of finite dimensional linear subspaces of φ such that $\varphi = \bigcup_{n\in\mathbb{N}} L_n$. For every $n\in\mathbb{N}$ let $T_n \in \mathscr{L}(E,F)$ be such that $T_n(E) = L_n$ and $T_n(x)=x$ for all $x\in L_n$. The sequence $(T_n)_{n\in\mathbb{N}}$ is clearly a Cauchy sequence in $\mathscr{L}_{\mathscr{M}}(E,F)$. Assume that $(T_n)_{n\in\mathbb{N}}$ converges to a map T in $\mathscr{L}_{\mathscr{M}}(E,F)$. Then $T(x) = x$ for all $x\in \bigcup_{n\in\mathbb{N}} L_n = \varphi = F$, which is contradictory to the fact that φ is not complemented in E.

One would like to have the above theorem of A.Grothendieck also valid for other completeness properties such as quasicompleteness, sequential completeness, local completeness, instead of completeness itself (cf. G.Köthe [21; p.143]). In order to provide a general statement covering all the above cases, we will introduce the following abstract concept.

(1.5) Let $\mathscr{A}$ be a functor from the category of locally convex spaces and continuous linear maps into the category of sets and maps, which satisfies the following two conditions:

(a) For every locally convex space E the set $\mathscr{A}(E)$ is a subset of the power set $\mathscr{P}(E)$ of all subsets of E.

(b) Whenever E, F are locally convex spaces and $T\in\mathscr{L}(E,F)$, then $\mathscr{A}(T) : \mathscr{A}(E) \to \mathscr{A}(F)$ satisfies $\mathscr{A}(T)(A)=T(A)$ for all $A\in\mathscr{A}(E)$.

In other words: $\mathscr{A}$ assigns to every locally convex space E a set $\mathscr{A}(E)$ of subsets of E such that $T(\mathscr{A}(E)) \subset \mathscr{A}(F)$ for all $T\in\mathscr{L}(E,F)$. (Compare with the functor $\mathscr{M}$ introduced in (1.2) which is a special case of the above notion.)

(1.6) ***Definition.*** Let $\mathscr{A}$ be a functor as described in (1.5). and let E be a locally convex space.
A subset D c E is called an d-complete subset of E, if for every $A \in \mathscr{A}(E)$ the intersection $A \cap D$ is contained in a complete subset of D.
E is called an $\mathscr{A}$-complete locally convex space, if E is an &-complete subset of itself.
A subset D c E is called $\mathscr{A}$-closed in E, if $\overline{A \cap D}^E$ c D for all A e&(E).

(1.7) ***Remark.*** Let $\mathscr{A}$ be as in (1.5).

(a) A locally convex space E is &-complete if and only if $\bar{A}^E$ is complete for every $A \in \mathscr{A}(E)$.

(b) An $\mathscr{A}$-closed subset D of an $\mathscr{A}$-complete locally convex space E is d-complete in E.

(c) An d-complete subset L of a locally convex space E which is also a linear subspace, is an &-complete locally convex space.

(d) Arbitrary products of d-complete locally convex spaces are again &-complete.

Proof. (a) If A is contained in a complete subset D, then A^E is a closed subset of the complete subset D^E.

(b) Let $A \in \mathscr{A}(E)$. Since $\bar{A}^E$ is complete, also $\overline{A \cap D}^E$ is complete and clearly $A \cap D$ c $\overline{A \cap D}^E$ c D.

(c) holds since by (1.5) every $A \in \mathscr{A}(L)$ belongs to d(E).

(d) Let $(E_\iota)_{\iota \in I}$ be a family of &-complete locally convex spaces and let $A \in \mathscr{A}(\prod_{\iota \in I} E_\iota)$. Then, by (1.5). the set $P_\iota(A)$ belongs to $\mathscr{A}(E_\iota)$ for every $\iota \in I$. Now, by (a), $\overline{P_\iota(A)}$ is a complete subset of E_ι ($\iota\in I$). Thus $\prod_{\iota\in I} P_\iota(A)$ is a complete subset of $\prod_{\iota\in I} E_\iota$ which contains A.

(1.8) PROPOSITION. *Let $\mathscr{A}$ be functor a described in (1.5). Let E, F be locally convex spaces such that $\mathscr{L}(E,F) = \mathscr{L}(E,F_s)$ and let $\mathscr{M} \subset \mathscr{B}(E)$ satisfy $\cup\mathscr{M} = E$. If $E'_{\mathscr{M}}$ and F are $\mathscr{A}$-complete, then also $\mathscr{L}_{\mathscr{M}}(E,F)$ is $\mathscr{A}$-complete.*

Proof. If a locally convex space G carries the coarsest topology, then also $\mathscr{L}_{\mathscr{M}}(E,G)$ carries the coarsest topology. Thus, by (1.1) (c), we may assume that F is Hausdorff. - Let $A\in\mathscr{A}(\mathscr{L}_{\mathscr{M}}(E,F))$. We have to show that $\bar{A}$ is complete or - equivalently (see [26;0.45]) - that every Cauchy net $(T_\iota)_{\iota\in I}$ in A converges in $\mathscr{L}_{\mathscr{M}}(E,F)$.

Let $(T_\iota)_{\iota\in I}$ be a Cauchy net in A. The inclusion $\mathscr{L}_{\mathscr{M}}(E,F) \to F^E$, where F^E carries the product topology, is continuous since $\cup\mathscr{M} = E$. By (1.7) (d) the product space F^E is $\mathscr{A}$-complete. Therefore $(T_\iota)_{\iota\in I}$ converges in F^E to a map $T : E \to F$, which is linear as F is Hausdorff.

Next we will show that T is continuous. Because of $\mathscr{L}(E,F) = \mathscr{L}(E,F_s)$ it suffices to show that $f\circ T\in E'$ for all $f\in F'$. Let $f\in F'$. The map $\mathscr{L}_{\mathscr{M}}(E,F) \to E'_{\mathscr{M}}$, $S \mapsto f\circ S$, is linear and continous by (1.1)(a). Moreover, $E'_{\mathscr{M}}$ is $\mathscr{A}$-complete, whence $(f\circ T_\iota)_{\iota\in I}$ converges in $E'_{\mathscr{M}}$ to some $g\in E'$. In particular, $(f\circ T_\iota)_{\iota\in I}$ converges to g in $\mathbb{K}^E$ provided with the product topology. On the other hand, by the continuity of f, the net $(f\circ T_\iota)_{\iota\in I}$ converges to $f\circ T$ in $\mathbb{K}^E$. Therefore $f\circ T = g$, whence $f\circ T$ is continuous.

Finally, since $\mathscr{U}_0(\mathscr{L}_{\mathscr{M}}(E,F))$ has a basis consisting of $\mathscr{L}_s(E,F)$-

closed sets and $(T_\iota)_{\iota\in I}$ converges to T in $\mathscr{L}_s(E,F)$, we obtain that $(T_\iota)_{\iota\in I}$ converges to T in $\mathscr{L}_{\mathscr{M}}(E,F)$.

(1.9) In order to show that Proposition (1.8) has many applications, we give a list of completeness properties which are of type "$\mathscr{A}$-complete".

complete: put $d(E) := \mathscr{P}(E)$

quasicomplete: put $\mathscr{A}(E) := \mathscr{B}(E)$;

p-complete: put $\mathscr{A}(E) := \{ A\subset E;$ A is precompact $\}$;

sequentially complete: put $\mathscr{A}(E) := \{\{ x_n : n\in\mathbb{N}\} : (x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ is a Cauchy sequences in E$\}$;

convex compactness property (in the sense of A.Wilansky [30;p.134]): put $\mathscr{A}(E) := \{\Gamma K$: K is a compact subset of E $\}$;

metric convex compactness property: put $\mathscr{A}(E) := \{\Gamma K$: K is a compact and pseudometrizable subset of E$\}$;

locally complete (in the sense of "Mackey sequentially complete", see P.Dierolf [10]):

put $d(E) := \{\Gamma(\{x_n : n\in\mathbb{N}\}) : (x_n)_{n\in\mathbb{N}}\in E^{\mathbb{N}}$ converges to zero)

(or $d(E) := \{\{x_n : n \in \mathbb{N}\} : (x_n)_{n\in\mathbb{N}}\in E^{\mathbb{N}}$ is a local Cauchy sequence$\}$).

(1.10) ***Remark.*** Semireflexivity does not behave well in the situation of (1.4). In fact, the Banach space $\mathscr{L}_b(l^2,l^2)$ contains an isometric copy of l^∞ and is thus not semireflexive. This example shows that $\mathscr{L}_b(E,F)$ does not inherit separability from E_b' and F.

We will insert now two more compatibility statements of type (1.1)(c) and (1.2)(c) (cf. G.Köthe [21;p.151 (12);p.148 (5); p.149 (8)]), which are less obvious than (1.1) and (1.2).

CONVENTION. Let E be a locally convex space and $\mathscr{M}\subset\mathscr{B}(E)$. We introduce the notation $\check{\mathscr{M}} := \{\cup\mathscr{N} : \mathscr{N}\subset\mathscr{M}$ is finite$\}$. Then, for every locally convex space F one has $\mathscr{L}_{\mathscr{M}}(E,F) = \mathscr{L}_{\check{\mathscr{M}}}(E,F)$ and the sets $\mathscr{W}(M,U)$ $(M\in\check{\mathscr{M}},\ U\in\mathscr{N}_0(F))$ form a basis of $\mathscr{U}_0(\mathscr{L}_{\mathscr{M}}(E,F))$.

(1.11) PROPOSITION. *Let E be a locally convex space, let $\mathscr{M}\subset\mathscr{B}(E)$, and let $(F_\iota)_{\iota\in I}$ be a family of Hausdorff locally convex spaces.*

(a) *The canonical map*

$$\Phi : \bigoplus_{\iota\in I}\mathscr{L}_{\mathscr{M}}(E,F_\iota) \to \mathscr{L}_{\mathscr{M}}(E,\bigoplus_{\iota\in I}F_\iota),\quad (T_\iota)_{\iota\in I} \mapsto \sum_{\iota\in I} J_\iota \circ T_\iota ,$$

is linear, continuous, and injective.

(b) *Let E satisfy: There exists a total subset $X\subset E$ such that*

$$\forall (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \quad \exists (\rho_n)_{n\in\mathbb{N}}\in(\mathbb{R}_+^*)^{\mathbb{N}} \quad \{\rho_n x_n : n\in\mathbb{N}\} \in \mathscr{B}(E).$$

Then Φ is also surjective.

(c) Assume that I is countable and that $\cup\mathscr{M}$ is total in E. Furthermore let $\mathscr{N}\subset\mathscr{B}(E)$ be such that (compare Definition (2.1))

$$(*)\quad \forall (M_n)_{n\in\mathbb{N}}\in \check{\mathscr{M}}^{\mathbb{N}} \quad \exists (\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}} \quad \bigcup_{n\in\mathbb{N}} \rho_n M_n \in \mathscr{N}.$$

Then $\Phi : \bigoplus_{\iota\in I}\mathscr{L}_{\mathscr{M}}(E,F_\iota) \to \mathscr{L}_{\mathscr{N}}(E,\bigoplus_{\iota\in I}F_\iota)$ *is open.*

Proof. (a) Linearity and injectivity are obvious; continuity follows from (1.1)(a).

(b) Let $T\in\mathscr{L}(E,\bigoplus_{\iota\in I}F_\iota)$. Assume that for every finite subset $K\subset I$ we have that $T(E)\not\subset F_K := \sum_{\iota\in K}J_\iota(F_\iota)$. Then we inductively find an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of finite subsets of I and a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $T(x_n)\in F_{K_{n+1}}\setminus F_{K_n}$ (here we use that F_K is a sed linear subspace in $\bigoplus_{\iota\in I} F_\iota$ as all F_ι are Hausdorff f) . Let $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ be such that $B := \{\rho_n x_n : n\in\mathbb{N}\}$ is bounded in E. Then T(B) is a bounded subset of $F_{\bigcup_{\mathbb{N}} K_n}$, hence contained in some F_{K_m} (again using the fact that all F_ι are Hausdorff), which is a contrcdiction to $T(\rho_m x_m) \notin F_{K_m}$. Thus $T(E) \subset F_K$ for some finite subset $K \subset I$, whence $T = \Phi((P_\iota \circ T)_{\iota\in I}) \in \Phi(\bigoplus_{\iota\in I} \mathscr{L}(E, F_\iota))$.

(c) We may assume that $I = \mathbb{N}$. Since the hypotheses of (c) imply those of (b), the map Φ is surjective.

Let $U \in \mathscr{U}_0(\bigoplus_{n\in\mathbb{N}} \mathscr{L}_{\mathscr{M}}(E, F_n))$. There are sequences $(M_n)_{n\in\mathbb{N}} \in \check{\mathscr{M}}^{\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{U}_0(F_n)$ such that $\sum_{n\in\mathbb{N}} \mathscr{W}(M_n, V_n) \subset U$. By hypothesis there is $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that: $N := \bigcup_{n\in\mathbb{N}} \rho_n M_n \in \mathscr{N}$.

Now $\sum_{n\in\mathbb{N}} \mathscr{W}(M_n, V_n) \supset \sum_{n\in\mathbb{N}} \mathscr{W}(N, \rho_n V_n)$ and $\sum_{n\in\mathbb{N}} \rho_n V_n$ belongs to $\mathscr{U}_0(\bigoplus_{n\in\mathbb{N}} F_n)$. Thus it remains to prove that $\mathscr{W}(N, \sum_{n\in\mathbb{N}} \rho_n V_n) \subset \Phi(\sum_{n\in\mathbb{N}} \mathscr{W}(N, \rho_n V_n))$. Let $T \in \mathscr{W}(N, \sum_{n\in\mathbb{N}} \rho_n V_n)$. By the surjectivity of Φ, there is $(T_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \mathscr{L}(E, F_n)$ such that $\sum_{n\in\mathbb{N}} J_n \circ T_n = T$. For every $n\in\mathbb{N}$ and $x\in N$ one has $T_n(x) = P_n(T(x)) \in \rho_n V_n$, whence $T_n \in \mathscr{W}(N, \rho_n V_n)$.

Remark. The hypothesis that all spaces F_ι be Hausdorff is essential for (b) and (c). In fact, every linear map $T : E \to \bigoplus_{\iota\in I} \overline{\{0\}}^{F_\iota}$ is continuous; thus, if E is infinite dimensional and if for infinitely many $\iota\in I$ the space F_ι is not Hausdorff, we easily find $T \in \mathscr{L}(E, \bigoplus_{\iota\in I} F_\iota)$ which does not belong to the range of Φ.

Statements in (2.11),(2.12),(2.15) (a) will show that the other hypotheses of (b) and (c) are also indispensable (and even sharp). Cf. also (2.10).

Naturally, our next aim will be to prove a statement "dual" to (1.11), i.e., to investigate spaces of the form $\mathscr{L}_{\mathscr{M}}(\prod_{\iota\in I}E_\iota, F)$. For this purpose it will be convenient to start from a functor $\mathscr{M}$ as described in (1.2) rather than to consider systems $\mathscr{M}\subset\mathscr{B}(\prod_{\iota\in I}E_\iota)$, $\mathscr{M}_\iota\subset\mathscr{B}(E_\iota)$ $(\iota\in I)$.

(1.12) PROPOSITION. *Let $(E_\iota)_{\iota\in I}$ be a family of locally convex spaces and F be a Hausdorff locally convex space. Moreover, let $\mathscr{M}$ be a functor as described in (1.2), i.e., $\mathscr{M}$ assigns to every locally convex space E a set $\mathscr{M}(E)\subset\mathscr{B}(E)$ such that $T(\mathscr{M}(E))\subset\mathscr{M}(G)$ for all $T\in\mathscr{L}(E,G)$. (We will again write $\mathscr{L}_{\mathscr{M}}(E,G)$ instead of $\mathscr{L}_{\mathscr{M}(E)}(E,G)$.)*

(a) *The canonical map*

$$\Psi : \bigoplus_{\iota\in I}\mathscr{L}_{\mathscr{M}}(E_\iota,F)\to\mathscr{L}_{\mathscr{M}}(\prod_{\iota\in I}E_\iota,F),\quad (T_\iota)_{\iota\in I}\mapsto\sum_{\iota\in I}T_\iota\circ P_\iota ,$$

is linear, continuous, and injective.

(b) *Let F satisfy the following condition:*

There exists a coarser Hausdorff locally convex topology $\mathscr{Y}$ on F such that

$$\forall\,(U_n)_{n\in N}\in\mathscr{U}_0(F,\mathscr{Y})^{\mathbb{N}}\quad\exists\,(\sigma_n)_{n\in\mathbb{N}}\in(\mathbb{R}_+^*)^{\mathbb{N}}\quad\bigcap_{n\in\mathbb{N}}\sigma_nU_n\in\mathscr{U}_0(F).$$

Then Ψ is also surjective.

(c) *Assume that I is countable. Let $\mathscr{N}\subset\mathscr{B}(\prod_{\iota\in I}E_\iota)$ be such that $\prod_{\iota\in I}M_\iota\in\mathscr{N}$ for all $(M_\iota)_{\iota\in I}\in\prod_{\iota\in I}\mathscr{M}(E_\iota)$, and let F satisfy the following condition (compare with Definition (2.1))*

$$(*)\qquad\forall\,(U_n)_{n\in\mathbb{N}}\in\mathscr{U}_0(F)^{\mathbb{N}}\quad\exists\,(\sigma_n)_{n\in\mathbb{N}}\in(\mathbb{R}_+^*)^{\mathbb{N}}\quad\bigcap_{n\in\mathbb{N}}\sigma_nU_n\in\mathscr{U}_0(F).$$

Then $\Psi : \bigoplus_{\iota\in I}\mathscr{L}_{\mathscr{M}}(E_\iota,F)\to\mathscr{L}_{\mathscr{N}}(\prod_{\iota\in I}E_\iota,F)$ is open.

Proof. (a) Linearity and injectivity are obvious; continuity follows from (1.2)(a).

(b) Let $T \in \mathscr{L}(\prod_{\iota\in I} E_\iota, F)$. Assume that for every finite subset $K \subset I$ we have that $T(\prod_{\iota\in I\setminus K} E_\iota) \not\subset \{0\}$. Then we inductively find a sequence $(\iota(n))_{n\in\mathbb{N}}$ in I satisfying $\iota(n) \neq \iota(m)$ $(n\neq m)$ and a sequence $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} E_{\iota(n)}$ such that $T(J_{\iota(n)}(x_n))\neq 0$ $(n\in\mathbb{N})$ (where $J_\kappa : E_\kappa \to \prod_{\iota\in I} E_\iota$ denotes the canonical injection $(\kappa\in I)$). In fact, whenever $K \subset I$, the set $\bigcup_{\iota\in I\setminus K} J_\iota(E_\iota)$ is total in $\prod_{\iota\in I\setminus K} E_\iota$; consequently, if $T(\bigcup_{\iota\in I\setminus K} J_\iota(E_\iota)) = \{0\}$ then $T(\prod_{\iota\in I\setminus K} E_\iota) = \{0\}$ since F is Hausdorff.

For every $n\in\mathbb{N}$ there is $U_n \in \mathscr{U}_0(F,\mathscr{Y})$ such that $T(J_{\iota(n)}(x_n))\notin U_n$. Let $(\sigma_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ be such that $U := \bigcap_{n\in\mathbb{N}} \sigma_n U_n \in \mathscr{U}_0(F)$. The sequence $(J_{\iota(n)}(\sigma_n x_n))_{n\in\mathbb{N}}$ converges to zero in $\prod_{\iota\in I} E_\iota$, whence $T(J_{\iota(m)}(\sigma_m x_m)) \in U \subset \sigma_m U_m$ for large m which is a contradiction to $T(J_{\iota(m)}(x_m))\notin U_m$. Thus $T(\prod_{\iota\in I\setminus K} E_\iota) = \{0\}$ for some finite subset $K \subset I$, whence $T = \Psi((T\circ J_\iota)_{\iota\in I}) \in \Psi(\bigoplus_{\iota\in I} \mathscr{L}(E_\iota, F))$.

(c) We may assume that $I=\mathbb{N}$. Since the hypotheses of (c) imply those of (b), the map Ψ is surjective.

Let $U \in \mathscr{U}_0(\bigoplus_{n\in\mathbb{N}} \mathscr{L}_{\mathscr{M}}(E_n, F))$. There are sequences $(M_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \breve{\mathscr{M}}(E_n)$ and $(V_n)_{n\in\mathbb{N}} \in \mathscr{U}_0(F)^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \mathscr{W}(M_n, V_n) \subset U$.(*) By hypothesis there is $(\sigma_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $V := \bigcap_{n\in\mathbb{N}} \sigma_n V_n \in \mathscr{U}_0(F)$.

(*) By the functorial properties of $\mathscr{M}$ we have that $\{0\} \in \mathscr{M}(E)$ for all locally convex spaces E. We thus may assume that $0 \in M_n$ for all $n\in\mathbb{N}$.

Since homotheties are linear and continuous, the functorial properties of $\mathcal{M}$ imply that $\sigma_n M_n \in \mathcal{M}(E_n)$ $(n\in\mathbb{N})$, whence $N:=\prod_{n\in\mathbb{N}}\sigma_n M_n \in \mathcal{N}$. Now $\sum_{n\in\mathbb{N}} \mathcal{W}(M_n, V_n) \supset \sum_{n\in\mathbb{N}} \mathcal{W}(\sigma_n M_n, V)$, and it remains to prove that $\mathcal{W}(N,V) \subset \Psi(\sum_{n\in\mathbb{N}}\mathcal{W}(\sigma_n M_n, V))$. Let $T\in\mathcal{W}(N,V)$. By the surjectivity of Ψ there is $(T_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \mathcal{L}(E_n, F)$ such that $\sum_{n\in\mathbb{N}} T_n \circ P_n = T$. For every $n\in\mathbb{N}$ and $x\in \sigma_n M_n$ one has $T_n(x) = T(J_n(x))\in T(N)\subset V$, whence $T_n \in \mathcal{W}(\sigma_n M_n, V)$.

Remark. The hypothesis that F be Hausdorff is again essential for (b) and (c). In fact, if F carries the coarset topology, then every linear map $\prod_{\iota\in I} E_\iota \to F$ is continuous, not only those which vanish on $\prod_{\iota\in I\setminus K} E_\iota$ for a suitable finite subset $K \subset I$.

Statements in (2.14). (2.15), (2.12) (a) will show that the other hypotheses of (b) and (c) are also indispensable. Cf. also (2.13).

§2. THE PROPERTIES (fsb). (cbc), AND (cnc)

The aim of this section is to investigate the question whether $\mathscr{L}_b(E,F)$ inherits from E'_b and F such properties as being barrelled, bornological, or quasibarrelled. Here the following three properties of locally convex spaces will play an important rôle.

(2.1) DEFINITION. *A locally convex space* E *is said to*

(a) *have* (fsb) *(= fundamental sequence of bounded sets) if there exists an increasing sequence* $(B_n)_{n\in\mathbb{N}}$ *of absolutely convex bounded subsets of* E *such that*

$$\underset{B\in\mathscr{B}(E)}{\forall}\ \underset{n\in\mathbb{N}}{\exists}\quad B\subset B_n$$

(such a sequence $(B_n)_{n\in\mathbb{N}}$ *will be called a fundamental sequence of bounded sets in* E *);*

(b) *satisfy* (cbc) (= *countable boundedness condition) if*

$$\underset{(B_n)_{n\in\mathbb{N}}\in\mathscr{B}(E)^{\mathbb{N}}}{\forall}\quad \underset{(\rho_n)_{n\in\mathbb{N}}\in(\mathbb{R}_+^*)^{\mathbb{N}}}{\exists}\quad \bigcup_{n\in\mathbb{N}}\rho_n B_n\in\mathscr{B}(E).$$

(c) satisfy (cnc) *(= countable neighbourhood condition) if*

$$\underset{(U_n)_{n\in\mathbb{N}}\in\mathscr{U}_0(E)^{\mathbb{N}}}{\forall}\quad \underset{(\sigma_n)_{n\in\mathbb{N}}\in(\mathbb{R}_+^*)^{\mathbb{N}}}{\exists}\quad \bigcap_{n\in\mathbb{N}}\sigma_n U_n\in\mathscr{U}_0(F)..$$

(Cf.(1.11) and (1.12); *the countable neighbourhood condition has been defined by K.Floret (in* [13;p.222]) *and others.*

(2.2) Remarks.

(a) We recall that A.Grothendieck [14;p.63/64] has called a locally convex space E a DF-space, if E has (fsb) and if E is countably quasibarrelled (i.e., every bornivorous intersection of a sequence of absolutely convex zero-neighbourhoods in E belongs to $\mathcal{U}_0(E)$).

Every DE-space E satisfies (cnc). In fact, let $(U_n)_{n\in\mathbb{N}}\in\mathcal{U}_0(E)^{\mathbb{N}}$ and let $(B_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in E. We may assume that every U_n is absolutely convex. For $n\in\mathbb{N}$ let $\sigma_n > 0$ be such that $B_n \subset \sigma_n U_n$. Then $\bigcap_{n\in\mathbb{N}} \sigma_n U_n$ is bornivorous (as $(B_n)_{n\in\mathbb{N}}$ is increasing) hence a zero-neighbourhood in E. (See also K.Floret [13;p.223] .)

(b) Seminormable spaces clearly have (fsb) and satisfy (cbc) as well as (cnc).

Moreover, a locally convex space E with (fsb) satisfies (cbc) if and only if E_{b^*} is seminormable. In fact, if $(B_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in E and if $(\rho_n)_{n\in\mathbb{N}}\in(\mathbb{R}_+^*)^{\mathbb{N}}$ is such that $B:=\bigcup_{n\in\mathbb{N}}\rho_n B_n\in \mathcal{B}(E) = \mathcal{B}(E_{b^*})$, then the closed absolutely convex hull of B is a bounded zero-neighbourhood in E_{b^*}.

Let I be an uncountable set; consider the Hilbert space $l^2(I)$ and denote by B its closed unit ball. The sets

$$\varepsilon B + l^2(I \setminus K) \qquad (\varepsilon > 0,\ K \subset I \text{ countable})$$

form a basis of $\mathcal{U}_0(l^2(I),\mathcal{T})$ for some locally convex topology $\mathcal{T}$ on $l^2(I)$. One easily verifies that $\mathcal{T}$ is stronger than the weak topology $\sigma(l^2(I),l^2(I))$ and strictly coarser than the norm topology. Moreover, $(l^2(I), \mathcal{T})$ is a DF-space. (In fact, any bornivorous intersection V of a sequence of $\mathcal{T}$-zero-neighbourhoods contains εB for some $\varepsilon > 0$ and $l^2(I \setminus K)$ for some countable subset $K \subset I$.) Consequently, $(l^2(I),\mathcal{T})$ has (fsb) and satisfies both (cbc) and (cnc) without being seminormable.

(c) Pseudometrizable locally convex spaces satisfy (cbc) (J.Horváth [16;p.116,Prop.3]). Moreover, for a pseudometrizable locally convex space E on has:

E has (fsb) $\Longleftrightarrow$ E is seminormable $\Longleftrightarrow$ E satisfies (cnc).

(The first equivalence follows from G.Köthe [20,p.393 (2)], the second is easy to verify.)

(d) Let E be a locally convex space. Then the following statements hold.

(α) E_b' satisfies (cbc) $\Longleftrightarrow$ E_{b*} satisfies (cnc);

(β) E_b' satisfies (cnc) $\Longleftrightarrow$ E satisfies (cbc);

(γ) E_b' is metrizable $\Longleftrightarrow$ E has (fsb);

(6) E_b' has (fsb) $\Longleftrightarrow$ E_{b*} is pseudometrizable.

All four equivalences follow by elementary duality arguments.

From (γ) and (α) one obtains

(E) E has (fsb) $\Longleftrightarrow$ E_{b*} has (fsb) and satisfies (cnc).

We will insert now a brief study of the stability properties of (fsb),(cnc), and (cbc).

(2.3) Clearly, (fsb) is stable with respect to the formation of linear subspaces and countable direct sums.

On the other hand, (fsb) fails to be stable with respect to uncountable direct sums (consider a linear space of uncountable dimension provided with the strongest locally convex topology) and to countable products (consider ω). The completion of a space with (fsb) need not have (fsb) (the completion of (φ, $\sigma(\varphi,\omega)$)) is topologically isomorphic to the topological product $\mathbb{K}^{\mathbb{R}}$). A locally convex space E need not have (fsb) even if it contains a linear subspace L such that L and E/L have (fsb) (see [27;p.27,3.4 Example]). Furthermore, the following example shows that (fsb) is not inherited by quotients.

Let $(E,\mathcal{T})$ be a locally convex space of countably infinite dimension such that every bounded set in $(E,\mathcal{T})$ has finite dimensional linear span and such that $(E,\mathcal{T})$ contains a dense linear subspace L of infinite codimension. Such a space can be obtained as a suitable linear subspace of I.Amemyia's and Y.Kōmura's separable incomplete Montel space (see [1] and [18]). Let $\mathcal{Y}$ be a metrizable but not normable locally convex topology on the quotient space E/L, and let $\mathfrak{X}$ denote the initial topology on E with respect to the identity map $E \to (E,\mathcal{T})$ and the quotient map $E \to (E/L,\mathcal{Y})$. By [27;p.22,2.9 Lemma], the quotient topology $\mathcal{X}/L$ is equal to $\mathcal{Y}$. Thus $(E,\mathcal{X})$ is a locally convex space with (fsb) (the bounded sets in $(E,\mathcal{X})$ have finite dimensional linear span and the dimension of E is countable) admitting the quotient space $(E,\mathcal{X})/L=(E/L,\mathcal{Y})$ which clearly does not have (fsb).

(2.4) The property (cnc) is stable with respect to the formation of linear subspaces, quotients, completions, and countable direct sums (see K.Floret [13;p.223, Proposition]).

On the other hand, (cnc) is neither stable with respect to countable products (consider ω and use (2.2)(c)) nor with respect to uncountable direct sums (in fact, in order to show that $\bigoplus_{r\in\mathbb{R}} \mathbb{K}$ does not satisfy (cnc) it suffices by (2.2)(d)(β) to prove that $\mathbb{K}^{\mathbb{R}}$ does not satisfy (cbc); as (cbc) is clearly stable with respect to linear subspaces, it is sufficient to show that $(\varphi, \sigma(\varphi,\omega)) \subset \mathbb{K}^{\mathbb{R}}$ does not satisfy (cbc) which clearly holds).

We finally show: Let E be a locally convex space containing a linear subspace L such that L and E/L satisfy (cnc). Then also E satisfies (cnc).

In fact, let $Q : E \to E/L$ denote the quotient map and let $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_o(E)^{\mathbb{N}}$. We may assume that $U_n = \Gamma U_n$ $(n\in\mathbb{N})$. Then there are $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ and $V \in \mathcal{U}_o(E)$, $V = \Gamma V$, such that $2V \cap L \subset \bigcap_{n\in\mathbb{N}} \rho_n U_n$. Moreover, there are $(\sigma_n)_{n\in\mathbb{N}} \in [1,\infty)^{\mathbb{N}}$ and $W \in \mathcal{U}_o(E)$ such that $Q(W) \subset \bigcap_{n\in\mathbb{N}} \sigma_n Q(V\cap U_n)$.
Now $W \subset \bigcap_{n\in\mathbb{N}}(\sigma_n(V\cap U_n) + L)$ whence

$$W \cap V \subset \bigcap_{n\in\mathbb{N}}(\sigma_n(V\cap U_n)+L\cap(V+\sigma_n V)) \subset \bigcap_{n\in\mathbb{N}}(\sigma_n(V\cap U_n)+\sigma_n(L\cap(V+V))$$

$$\subset \bigcap_{n\in\mathbb{N}}(\sigma_n(V\cap U_n)+\sigma_n\rho_n U_n) \subset \bigcap_{n\in\mathbb{N}}(\sigma_n+\sigma_n\rho_n)U_n .$$

(2.5) The property (cbc) is stable under the formation of linear subspaces and countable products (the latter statement follows via (2.2)(d)(β) from the stability of (cnc) under countable direct sums and from the fact that the strong dual of a product coincides with the locally convex direct sum of the strong duals of the factor spaces).

On the other hand, (cbc) is neither stable under uncountable products (in (2.4) we proved that $\mathbb{K}^{\mathbb{R}}$ does not satisfy (cbc)), nor under countable direct sums (consider, for instance, φ), nor under completions ($(l^2, \sigma(l^2,l^2))$ satisfies (cbc) but its completion $((l^2)^*, \sigma((l^2)^*,l^2))$ is topologically isomorphic to $\mathbb{K}^{\dim l^2} = \mathbb{K}^{\mathbb{R}}$, thus does not satisfy (cbc).

The following example will show that a locally convex space E which does not satisfy (cbc) may contain a linear subspace such that L and E/L satisfy (cbc).

We first recall: $\mathbb{K}^{\mathbb{R}}$ is topologically isomorphic to the completion of $(\varphi, \sigma(\varphi, \omega))$ - a space of infinite dimension in which every bounded set has a finite dimensional linear span. Consequently, whenever $\mathscr{X}$ is a locally convex topology on $\mathbb{K}^{\mathbb{R}}$ stronger than the product topology, then $(\mathbb{K}^{\mathbb{R}}, \mathscr{X})$ does not satisfy (cbc).

- Let $(E, \mathscr{T})$ denote the space $\mathbb{K}^{\mathbb{R}}$ provided with the product topology. By what was said above $(E,\mathscr{T})$ contains a dense linear subspace. L which is topologically isomorphic to $(l^2, \sigma(l^2, l^2))$, hence satisfies (cbc). Let $\mathscr{Y}$ be a metrizable locally convex topology on the quotient space E/L. Then, according to [27;p.22,2.9 Lemma], the space E endowed with the initial topology $\mathscr{X}$ with respect to the identity map $E \to (E, \mathscr{T})$ and the quotient map $E \to (E/L, \mathscr{Y})$, provides the announced counterexample since - because of $\mathscr{X} \supset \mathscr{T}$ - it does not satisfy (cbc) where-as $(L, \mathscr{X}|L) = (L, \mathscr{T}|L)$ and $(E/L, \mathscr{X}/L) = (E/L, \mathscr{Y})$ both satisfy (cbc).

Finally, the following example shows that (cbc) is not stable with respect to quotients.

Let E be a linear subspace of l^∞ containing c_0 such that $\dim(E/c_0)$ is countably infinite, and let $\mathscr{T}$ denote the norm topology induced by $(l^\infty, \|\ \|_\infty)$ on E. Since $\dim(E/c_0)$ is countable, the quotient space $(E/c_0, \mathscr{T}/c_0)$ has a separable weak dual (it is contained in ω) Consequently, there exists a metrizable weak topology $\mathscr{Q}$ on E/c_0 which is coarser than the norm topology $\mathscr{T}/c_0$.

As $\mathscr{Q}$ is metrizable and clearly nonnormable, there exists an increasing sequence $(B_n)_{n\in\mathbb{N}}$ of absolutely convex closed bounded subsets of $(E/c_o,\mathscr{Q})$ such that B_n does not absorb B_{n+1} $(n\in\mathbb{N})$ and such that B_1 is a zero-neighbourhood in the normed space $(E/c_o,\mathscr{T}/c_o)$. Let $\mathscr{Y}$ denote the strongest locally convex topology on E/c_o satisfying $\mathscr{Y}|B_n = \mathscr{Q}|B_n$ for all $n\in\mathbb{N}$. By W.Roelcke [25;p.64 Thm.4] the sequence $(B_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in $(E/c_o,\mathscr{Y})$, which implies that $(E/c_o,\mathscr{Y})$ does not satisfy (cbc).
Let $\mathscr{X}$ denote the initial topology on E with respect to the identity map $E \to (E,\sigma(E,l^1)) = (E,\sigma(l^\infty,l^1)|E)$ and to the quotient map $E \to (E/c_o,\mathscr{Y})$. As c_o is dense in $(E,\sigma(E,l^1))$, [27;p.22,2.9 Lemma] yields that the quotient topology $\mathscr{X}/c_o$ is equal to $\mathscr{Y}$. Thus our example will be completed if we show that $(E,\mathscr{X})$ satisfies (cbc). In fact, since $(\mathscr{T}/c_o)|B_1 \supset \mathscr{Y}|B_1$, we obtain by W.Roelcke [25;p.74,Lemma 8] that $\mathscr{T}/c_o \supset \mathscr{Y}$. Consequently, $\mathscr{T} \supset \mathscr{X} \supset \sigma(E,l^1)$. As $\mathscr{B}(E,\sigma(E,l^1)) = \mathscr{B}(E,\mathscr{T})$ and $(E,\mathscr{T})$ is normed, we obtain that $(E,\mathscr{X})$ satisfies (cbc).

After this deviation we return to the investigation of spaces $\mathscr{L}_b(E,F)$. Our next aim is to say something about the system of all bounded sets in $\mathscr{L}_b(E,F)$.

Let E,F to locally convex spaces. As in section one we define $\mathscr{W}(A,C) := \{T\in\mathscr{L}(E,F) : T(A) \subset C\}$ whenever $A \subset E$, $C \subset F$. A subset $\mathscr{H} \subset \mathscr{L}(E,F)$ belong to $\mathscr{B}(\mathscr{L}_b(E,F))$ if and only if for every $B\in\mathscr{B}(E)$ the set $\mathscr{H}(B) := \bigcup_{T\in\mathscr{H}} T(B)$ belongs to $\mathscr{B}(F)$.
In fact, $\mathscr{H}\in\mathscr{B}(\mathscr{L}_b(E,F)) \Longleftrightarrow$

$$\overset{\forall}{B\in\mathcal{B}(E)}\quad \overset{\forall}{U\in\mathcal{U}_o(F)}\quad \overset{\exists}{\rho_{B,U}>0}\quad \mathcal{H}\subset\rho_{B,U}\,\mathcal{W}(B,U)\iff$$

$$\overset{\forall}{B\in\mathcal{B}(E)}\quad \overset{\forall}{U\in\mathcal{U}_o(F)}\quad \overset{\exists}{\rho_{B,U}>0}\quad \mathcal{H}(B)\subset\rho_{B,U}\,U\iff\bigcup_{B\in\mathcal{B}(E)}\mathcal{H}(B)\in\mathcal{B}(F).$$

Moreover, for every $U\in\mathcal{U}_o(E_{b*})$ and every $B\in\mathcal{B}(F)$ the set $\mathcal{W}(U,B)$ belong to $\mathcal{B}(\mathcal{L}_b(E,F))$, as U absorbs all the sets in $\mathcal{B}(E)$. The following proposition provides conditions on E and F under which every $\mathcal{H}\in\mathcal{B}(\mathcal{L}_b(E,F))$ is contained in $\mathcal{W}(U,B)$ for suitable $U\in\mathcal{U}_o(E_{b*})$ and $B\in\mathcal{B}(F)$. It is clear that such a statement cannot be valid without some hypotheses (consider $\mathcal{H}:=\{Id\}$, where Id is the identity map of a quasibarrelled but not seminormable locally convex space).

Cf. also the localization principle in A.Defant, K.Floret [8;p.6 ff].

(2.6) PROPOSITION. *Let* E *and* F *be locally convex spaces and let one of the following conditions be satisfied:*

(a) E *has* (fsb) and F *satisfies* (cbc);

(b) E_{b*} *satisfies* (cnc) *and* F *is pseudometrizable;*

(c) E *satisfies* (cbc) *and* F *has* (fsb);

(d) E_{b*} *is pseudometrizable and there exists a locally convex topology* $\mathcal{Y}$ *on* F *coarser than* $\beta^*(F,F')$ *such that* $\mathcal{B}(F,\mathcal{Y})=\mathcal{B}(F)$ *and such that* $(F,\mathcal{Y})$ **satisfies** (cnc).

Then

$$\overset{\forall}{\mathcal{H}\in\mathcal{B}(\mathcal{L}_b(E,F))}\quad \overset{\exists}{U\in\mathcal{U}_o(E_{b*})}\quad \mathcal{H}(U)\in\mathcal{B}(F).$$

Proof. Let $\mathcal{H}\in\mathcal{B}(\mathcal{L}_b(E,F))$.

(a) Let $(B_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in E. Then for every $n\in\mathbb{N}$ the set $A_n := \mathcal{H}(B_n)$ is bounded in F. As F satisfies (cbc) there is $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $A := \overline{\Gamma \bigcup_{n\in\mathbb{N}} \rho_n A_n}$ belongs to $\mathcal{B}(F)$. $U := \overline{\Gamma \bigcup_{n\in\mathbb{N}} \rho_n B_n}$ is a bornivorous barrel in E, hence a zero-neighbourhood in E_{b^*}. For every $T\in\mathcal{H}$ one has $T(U) \subset A$, whence $S(U) \in \mathcal{B}(F)$.

(b) Let $(V_n)_{n\in\mathbb{N}}$ be a basis of $\mathcal{U}_o(F)$, $V_n = \overline{\Gamma V_n}$ $(n\in\mathbb{N})$. For every $n\in\mathbb{N}$ the set $U_n := \bigcap_{T\in\mathcal{H}} T^{-1}(V_n)$ is a bornivorous barrel in E, hence belongs to $\mathcal{U}_o(E_{b^*})$. As E_{b^*} satisfies (cnc), there is $(\sigma_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $U := \bigcap_{n\in\mathbb{N}} \sigma_n U_n \in \mathcal{U}_o(E_{b^*})$. $A := \bigcap_{n\in\mathbb{N}} \sigma_n V_n$ clearly belongs to $\mathcal{B}(F)$ and $\mathcal{H}(U) \subset A$.

(c) Let $(B_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in F such that $B_n = \overline{\Gamma B_n}$ $(n\in\mathbb{N})$. For every $n\in\mathbb{N}$ the set $V_n := \bigcap_{T\in\mathcal{H}} T^{-1}(B_n)$ is closed in E and absolutely convex.

Assume that for every $n\in\mathbb{N}$ there is $A_n \in \mathcal{B}(E)$ such that V_n does not absorb A_n. As E satisfies (cbc), there is $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $A := \bigcup_{n\in\mathbb{N}} \rho_n A_n \in \mathcal{B}(E)$. Since $\mathcal{H}(A)$ is bounded in F, there is $m\in\mathbb{N}$ such that $\mathcal{H}(A) \subset B_m$; in particular $\mathcal{H}(\rho_m A_m) \subset B_m$, hence $\rho_m A_m \subset V_m$ which is a contradiction.

Consequently, there is $n\in\mathbb{N}$ such that V_n is bornivorous and hence belongs to $\mathcal{U}_o(E_{b^*})$. Clearly $\mathcal{H}(V_n)$ is contained in B_n and therefore bounded in F.

(d) Let $(U_n)_{n\in\mathbb{N}}$ be a basis of $\mathcal{U}_o(E_{b^*})$.

Assume that for every $n\in\mathbb{N}$ there is $V_n \in \mathcal{U}_o(F,\mathcal{Y})$ such that V_n does not absorb $\mathcal{H}(U_n)$. As $(F,\mathcal{Y})$ satisfies (cnc), there is $(\sigma_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $V := \bigcap_{n\in\mathbb{N}} \sigma_n V_n \in \mathcal{U}_o(F,\mathcal{Y}) \subset \mathcal{U}_o(F_{b^*})$. Thus $\bigcap_{T\in\mathcal{H}} T^{-1}(V)$ contains a bornivorous barrel in E, whence there is m such that $\mathscr{H}(U_m) \subset V$. In particular, $\mathscr{H}(U_m) \subset \sigma_m \setminus_m$, which is a contradiction.

Consequently, there is $n \in \mathbb{N}$ such that $\mathscr{H}(U_n)$ is bounded in $(F, \mathscr{T})$ and hence in F.

Remark. Comparing the four hypotheses of (2.6) with each other, one notices: the following condition

(α) E_b' and F satisfy (cbc);

is implied both by (a) and by (b) (use (2.2) (d), (ε) and (α)), and the following condition

(β) E_b' satisfies (cnc) and there exists a locally convex topology $\mathscr{Y}$ on F coarser than $\beta^*(F,F')$ such that $\mathscr{B}(F,\mathscr{Y}) = \mathscr{A}(F)$ and such that $(F,\mathscr{Y})$ satisfies (cnc);

is implied both by (c) and by (d) (use (2.2)(d)(c)).

Naturally one would like to know whether the conclusion of (2.6) remains valid if only (α) or (β) are satisfied. We do not know an answer to this question. In fact, we do not even know a locally convex space E satisfying (cbc) such that E_{b^*} is not pseudometrizable.

As a corollary of (2.6) we obtain the following statement (cf. (1.4)).

(2.7) PROPOSITION. *Let E and F be locally convex spaces such that E_b' and F both have (fsb). Then also $\mathscr{L}_b(E,F)$ has (fsb). Moreover, if $(A_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in E_b' and if $(B_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in F then the sets $\mathscr{W}(A_n^\circ, B_n)$ $(n\in\mathbb{N})$ form a fundamental sequence of bounded sets in $\mathscr{L}_b(E,F)$.*

Proof. It sufficesto prove the last assertion. And this one follows immediately from (2.2)(d)(6) and (2.6)(c).

The following statement can be regarded as another contribution to (1.4).

(2.8) PROPOSITION. *Let* E *and* F *be locally convex spaces such that* E'_b *and* F *both satisfy* (cnc). *Then also* $\mathscr{L}_b(E,F)$ *satisfies* (cnc).

Proof. Let $(\mathscr{W}_n)_{n\in\mathbb{N}} \in \mathscr{U}_0(\mathscr{L}_b(E,F))^{\mathbb{N}}$. For every $n\in\mathbb{N}$ there are $B_n \in \mathscr{B}(E)$ and $U_n \in \mathscr{U}_0(F)$ such that $\mathscr{W}(B_n,U_n) \subset \mathscr{W}_n$. As E satisfies (cbc) by (2.2)(d)(β), and as F satisfies (cnc), there are $(\rho_n)_{n\in\mathbb{N}}, (\sigma_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $B:=\bigcup_{n\in\mathbb{N}} \rho_n B_n \in \mathscr{B}(E)$ and such that $U:=\bigcap_{n\in\mathbb{N}} \sigma_n U_n \in \mathscr{U}_0(F)$.

Now $\bigcap_{n\in\mathbb{N}} \frac{\sigma_n}{\rho_n}\mathscr{W}_n \supset \bigcap_{n\in\mathbb{N}} \frac{\sigma_n}{\rho_n}\mathscr{W}(B_n,U_n) \supset \bigcap_{n\in\mathbb{N}} \mathscr{W}(\rho_n B_n,\sigma_n U_n) \supset \mathscr{W}(\bigcup_{n\in\mathbb{N}}\rho_n B_n, \bigcap_{n\in\mathbb{N}}\sigma_n U_n) = \mathscr{W}(B,U) \in \mathscr{U}_0(\mathscr{L}_b(E,F))$.

(2.9) PROPOSITION. *Let* E , F *be locally convex spaces and assume that one of the following conditions is satisfied:*

(a) E'_b *is metrizable and* F *satisfies* (cbc);

(b) E'_b *satisfies* (cbc) *and* F is *pseudometrizable.*

(Both hypotheses imply that E'_b *and* F *satisfy* (cbc).)

Then $\mathscr{L}_b(E,F)$ *satisfies* (cbc).

Proof. If (a) is satisfied. then (2.2)(d)(γ) and (2.6)(a) yield that the sets $\mathscr{W}(U,B)$ ($U\in \mathscr{U}_0(E_{b^*}), B\in\mathscr{B}(F)$) form a fundamental system of bounded sets in $\mathscr{L}_b(E,F)$. - If (b) is satisfied, then the same statement follows with the help of (2.2)(d)(a) and (2.6) (b).

Now let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{B}(\mathscr{L}_b(E,F))$. For every

$n\in\mathbb{N}$ there are $U_n \in \mathscr{U}_o(E_{b^*})$ and $B_n \in \mathscr{B}(F)$ such that $A_n \subset \mathscr{W}(U_n, B_n)$. Since E_{b^*} satisfies (cnc) and F satisfies (cbc), there are $(\sigma_n)_{n\in\mathbb{N}}$, $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $U := \bigcap_{n\in\mathbb{N}} \sigma_n U_n \in \mathscr{U}_o(E_{b^*})$ and $B := \bigcup_{n\in\mathbb{N}} \rho_n B_n \in \mathscr{B}(F)$.

Now $\bigcup_{n\in\mathbb{N}} \frac{\rho_n}{\sigma_n} A_n \subset \bigcup_{n\in\mathbb{N}} \frac{\rho_n}{\sigma_n} \mathscr{W}(U_n, B_n) \subset \bigcup_{n\in\mathbb{N}} \mathscr{W}(\sigma_n U_n, \rho_n U_n) \subset \mathscr{W}(U,B) \in \mathscr{B}(\mathscr{L}_b(E,F))$.

Remark. We do not know whether the implication

E_b' and F satisfy (cbc) $\Longrightarrow$ $\mathscr{L}_b(E,F)$ satisfies (cbc)

is always true?

We would like to mention that our structure theorem (1.11) has the following corollary (in the special case $I=\mathbb{N}$ and $\mathscr{M}=\mathscr{N}=\mathscr{B}$).

(2.10) PROPOSITION. *Let* E *be a locally convex space satisfying* (cbc). *Then for every sequence* $(F_n)_{n\in\mathbb{N}}$ *of Hausdorff locally convex spaces, then canonical map*

$$\Phi : \bigoplus_{n\in\mathbb{N}} \mathscr{L}_b(E, F_n) \to \mathscr{L}_b(E, \bigoplus_{n\in\mathbb{N}} F_n),\ (T_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} J_n \circ T_n,$$

is a topological isomorphism.

The following statement will show that the hypothesis for E to satisfy (cbc) is indeed essential.

(2.11) PROPOSITION. *Let* E *be a locally convex space containing a linear subspace* L *such that* L *does not satisfy* (cbc). *Then*

$$\Phi : \bigoplus_{n\in\mathbb{N}} E_b' \to \mathscr{L}_b(E, \varphi),\ (f_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} J_n \circ f_n,$$

is not relatively open (i.e., not an open map onto its range). Moreover, if L *has no quotient topologically isomorphic to*

φ, then $\mathscr{L}_b(E,\varphi)$ is not countably barrelled (*) *. If L contains a total bounde set, then $\mathscr{L}_b(E,\varphi)$ is not even countably quasibarrelled. (A locally convex space containing a total bounded set in particular does not admit φ as a quotient.)*

Proof. There exists a sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathscr{B}(L) \subset \mathscr{B}(E)$ such that $\bigcup_{n\in\mathbb{N}} \rho_n B_n$ is unbounded for every sequence $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$.

Let us first assume that $\Phi : \bigoplus_{n\in\mathbb{N}} E'_b \to \mathscr{B}_b(E,\varphi)$ is relatively open. Then there exist $B = \overline{\Gamma B} \in \mathscr{B}(E)$ and $U \in \mathscr{U}_o(\varphi)$, $U = \Gamma U$, such that $\mathscr{W} := \Phi(\bigoplus_{n\in\mathbb{N}} E'_b) \cap \{T \in \mathscr{L}(E,\varphi) : T(B) \subset U\} \subset \sum_{n\in\mathbb{N}} J_n \circ B_n^\circ$, where $J_n \circ B_n^\circ := \{J_n \circ f : f \in B_n^\circ\}$ $(n\in\mathbb{N})$.

Now there is $k\in\mathbb{N}$ such that B_k is not absorbed by B. Fix $\rho > 0$ such that $z := (\delta_{kn}\rho)_{n\in\mathbb{N}} \in U$. Since $\rho B_k \not\subset B = B^{\circ\circ}$, there is $f \in B^\circ$ and $x \in B_k$ such that $f(\rho x) > 1$.
The map $f \otimes z : E \to \varphi$, $y \to f(y)z$, clearly belongs to $\mathscr{W}$.
Consequently there is a sequence $(f_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} B_n^\circ \cap \bigoplus_{n\in\mathbb{N}} E'$ such that $f \otimes z = \sum_{n\in\mathbb{N}} J_n \circ f_n$, hence $\rho f = P_k \circ (f \otimes z) = P_k \circ (\sum_{n\in\mathbb{N}} J_n \circ f_n) =$

$= f_k$ (where $P_k : \varphi \to \mathbb{K}$ denotes the projection onto the k'th coordinate). This is a contradiction, as $x \in B_k$, $f_k \in B_k^\circ$, and $\rho f(x) = f(\rho x) > 1$.

Next choose an increasing sequence $(Y_n)_{n\in\mathbb{N}}$ of finite dimensional linear subspaces of φ such that $\varphi = \bigcup_{n\in\mathbb{N}} Y_n$. For every $n\in\mathbb{N}$ let $U_n = \overline{\Gamma U_n} \in \mathscr{U}_o(\varphi)$ be such that $Y_n \subset U_n$ and $U_n \neq \varphi$. Then for

(*) A locally convex space G is called countably (quasi)barrelled if every absorbent (bornivorous) intersection of a sequence of absolutely convex zero-neighbourhoods is itself a zero-neighbourhood.

every $n\in\mathbb{N}$ the set $\mathscr{W}_n := \mathscr{W}(B_n,U_n)$ is a closed and absolutely convex zero-neighbourhood in $\mathscr{L}_b(E,\varphi)$. Put $\mathscr{V} := \bigcap_{n\in\mathbb{N}} \mathscr{W}_n$.

If L has no quotient topologically isomorphic to φ, then for every $T\in\mathscr{L}(E,\varphi)$ the subspace T(L) is finite dimensional, hence contained in U_n for all $n \geq n(T)$; from this we clearly obtain that $\mathscr{V}$ is absorbent in $\mathscr{L}(E,\varphi)$ (recall that $B_n \subset L$ for all $n\in\mathbb{N}$).

If L contains a total bounded set A, then $\mathscr{V}$ is even bornivorous. In fact, let $\mathscr{H} \subset \mathscr{L}_b(E,F)$ be bounded. $\mathscr{H}(A)$ is bounded in φ, whence $\dim[\mathscr{H}(A)] < \infty$. Therefore there is $n(\mathscr{H}) \in \mathbb{N}$ such that $\mathscr{H}(L) \subset \overline{[\mathscr{H}(A)]} \subset Y_{n(\mathscr{H})} \subset Y_n \subset U_n$ $(n \geq n(\mathscr{H}))$. Since there is $\alpha \geq 1$ such that $\mathscr{H} \subset \alpha \bigcap_{n<n(\mathscr{H})} \mathscr{W}(B_n,U_n)$, we obtain that $\mathscr{H} \subset \alpha\mathscr{V}$.

Now the proof will be finished if we show that $\mathscr{V}\notin \mathscr{U}_o(\mathscr{L}_b(E,\varphi))$. Let us assume the contrary. Then there is $C\in\mathscr{B}(E)$ and $V = \Gamma V\in \mathscr{U}_o(\varphi)$ such that $\mathscr{W}(C,V) \subset \mathscr{V}$. Again there is $m\in\mathbb{N}$ such that B_m is not absorbed by $C^{\circ\circ}$. Choose $u\in\varphi \setminus U_m$ and $\sigma > 0$ such that $\sigma u\in V$. Since $\sigma B_m \not\subset C^{\circ\circ}$, there is $g\in C^\circ$ and $y\in B_m$ such that $g(\sigma y) > 1$. $S := \sigma g \otimes u \in \mathscr{L}(E,\varphi)$ belongs to $\mathscr{W}(C,V)$ as $g\in C^\circ$ and $\sigma u\in V$. Therefore $S\in\mathscr{V} \subset \mathscr{W}(B_m,U_m)$ which is not true as $y\in B_m$ and $S(y) = \sigma g(y)u = g(\sigma y)u \notin U_m$. Thus $\mathscr{V}\notin \mathscr{U}_o(\mathscr{L}_b(E,\varphi))$.

(2.12) ***Remarks***

(a) Let I be a set such that $\operatorname{card} I \geq \operatorname{card} \mathbb{R}$. We showed in (2.4) that $E := \mathbb{K}^I$ does not satisfy (cbc). Moreover $[0,1]^I$ is a total bounded set in $\mathbb{K}^I$. Thus we obtain from (2.11) that $\mathscr{L}_b(E,\varphi)$ is not countably quasibarrelled. hence neither bornological nor barrelled, though E'_b and φ both carry the strongest locally convex topology. It should be mentioned that this example also shows that in (1.12)(c) the countability of I is essential. In fact, φ satisfies (cnc), but the spaces $\bigoplus_{\iota\in I} \mathscr{L}_b(\mathbb{K}, \varphi) = \bigoplus_{\iota\in I} \varphi$ and $\mathscr{L}_b(\mathbb{K}^I, \varphi)$ are not topologically isomorphic (though the map Y from (1.12) is an algebraic isomorphism) since the first space is barrelled and the second is not.

In order to obtain some more applications of (2.11), let X be a reflexive nonnormable Fréchet space admitting continuous norms (e.g., $X=\mathscr{S}$ or $\mathscr{E}$). Then $E:=X'_b$ is a quasibarrelled nonnormable DF-space, hence does not satisfy (cbc) by (2.2)(b). As E has a total bounded set, (2.11) and (1.1)(c) yield that $\mathscr{L}_b(E,F)$ is not countably quasibarrelled whenever F is a locally convex space containing φ as a topologically complemented linear subspace.

(b) Let E be the linear space φ provided with the weak topology $\sigma(E, \omega)$. Then E does not satisfy (cbc), and E does not admit a quotient topologically isomorphic to φ $(= (\varphi, \beta(\varphi, \omega)))$, as all quotients of E carry a weak topology and $\beta(\varphi, \omega)$ is not a weak topology. Hence, by (2.11), $\mathscr{L}_b(E,\varphi)$ is not countably barrelled. On the other hand, we will show that $\mathscr{L}_b(E, \varphi)$ is bornological (hence quasibarrelled and hence countably quasibarrelled).

In fact, let $(e_n)_{n\in\mathbb{N}}$ be an algebraic basis in E. The map

$$\Theta : \mathscr{L}_b(E,\varphi) \to \varphi^{\mathbb{N}}, T \mapsto (T(e_n))_{n\in\mathbb{N}} .$$

is a topological isomorphism onto its range (where $\varphi^{\mathbb{N}}$ carries the product topology), since every bounded set in E has a finite dimensional linear span. Moreover, let $T : E \to \varphi$ be linear. Then T is continuous if and only if $\dim(E) < \infty$. In fact, if T is continuous, then T(E) carries a weak topology because E does so, and any linear subspace of φ carrying a weak topology is finite dimensional. - On the other hand, if $\dim T(E) < \infty$, then T has a representation of the form $\sum_{1\leq n\leq m} f_n \otimes z_n$, where $f_n \in E^* = E'$ and $z_n \in \varphi$ $(1\leq n\leq m)$, and is therefore continuous.

Consequently, $\Theta(\mathscr{L}(E, \varphi)) = \{(y_n)_{n\in\mathbb{N}} \in \varphi^{\mathbb{N}} : \dim[y_n : n\in\mathbb{N}] < \infty\} \supset \varphi^{(\mathbb{N})} := \{(y_n)_{n\in\mathbb{N}} \in \varphi^{\mathbb{N}} : \{n\in\mathbb{N} : y_n \neq 0\}$ is finite).

Consequently, $\Theta(\mathscr{L}(E, \varphi))$ (and hence $\mathscr{L}_b(E, \varphi)$) is bornological by a result of H.Pfister (in V.Eberhardt [11;1.6 Korollar b]).

(c) Let $E := \mathbb{K}^{[\mathbb{R}]} := \{(x_r)_{r\in\mathbb{R}} \in \mathbb{K}^{\mathbb{R}} : \{r\in\mathbb{R} : x_r \neq 0\}$ is countable $\}$ be provided with the relative topology induced by the product topology of $\mathbb{K}^{\mathbb{R}}$. Since for every countable subset $I \subset \mathbb{R}$ the space $\mathbb{K}^I$ is metrizable and hence satisfies (cbc), we obtain that

$$\forall_{(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}} \; \exists_{(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}} \; \{\rho_n x_n : n\in\mathbb{N}\} \in \mathscr{B}(E).$$

Consequently. by (1.11)(b), the canonical linear map $\Phi : \bigoplus_{n\in\mathbb{N}} E' \to \mathscr{L}(E.\varphi)$ is a bijection and by (1.11)(c), $\Phi : \bigoplus_{n\in\mathbb{N}} E'_s \to \mathscr{L}_b(E,\varphi)$ is open. - On the other hand, E does not satisfy (cbc) because $\mathbb{K}^{\mathbb{R}}$ does not satisfy (cbc) (see (2.4)) and as every $B \in \mathscr{B}(\mathbb{K}^{\mathbb{R}})$ is contained in the closure of a suitable bounded subset of $\mathbb{K}^{[\mathbb{R}]}$. Consequently, (2.11) implies that

$\Phi: \bigoplus_{n\in\mathbb{N}} E'_b \to \mathscr{L}_b(E,\varphi)$ is not open.

(b) The space φ does not satisfy (cbc), thus by (2.11) the map $\Phi: \bigoplus_{n\in\mathbb{N}} \omega \to \mathscr{L}_b(\varphi,\varphi)$ is not relatively open (recall that $\omega = \varphi'_b$). Furthermore, Φ is not surjective as the identity map $\varphi \to \varphi$ does not belong to the range of Φ. However, by (1.2)(c) $\mathscr{L}_b(\varphi,\varphi)$ is topologically isomorphic to $\prod_{n\in\mathbb{N}} \mathscr{L}_b(\mathbb{K},\varphi) = \prod_{n\in\mathbb{N}} \varphi = \omega\varphi$ (see G.Köthe [21,p.153;(14)]) and hence bornological and barrelled. Thus in (2.11) the additional hypotheses on L cannot be dropped.

Now we will give a similar treatment to (1.12) and begin with the following corollary of (1.12).

(2.13) PROPOSITION. *Let F be a locally convex Hausdorff space satisfying (cnc). Then for every sequence $(E_n)_{n\in\mathbb{N}}$ of locally convex spaces the canonical map*

$$\Psi : \bigoplus_{n\in\mathbb{N}} \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(\prod_{n\in\mathbb{N}} E_n,F),\ (T_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} T_n \circ P_n ,$$

is a topological isomorphism.

(2.14) PROPOSITION. *Let F be a locally convex space containing a linear subspace L such that the quotient space F/L does not satisfy (cnc). Then*

$$\Psi : \bigoplus_{\mathbb{N}} F \to \mathscr{L}_b(\omega,F),\ (y_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} P_n \otimes y_n ,$$

is not relatively open (i.e., not an open map onto its range).(*)

(*) This map Ψ is nothing else than the map Ψ from (2.13) applied to the case $E_n = \mathbb{K}$ $(n \in \mathbb{N})$, where $\mathscr{L}_b(\mathbb{K},F)$ and F are identified via the canonical topological isomorphism $\mathscr{L}_b(\mathbb{K},F) \to F$, $T \mapsto T(1)$.

Moreover, *if F/L contains no subspace topologically isomorphic to ω, then $\mathscr{L}_b(\omega, F)$ is not countably barrelled. If F/L admits continuous norms, then $\mathscr{L}_b(\omega,F)$ is not even countably quasibarrelled. (A locally convex space admitting continuous norms in particular does not contain ω as a subspace.)*

Proof. We may at once assume that L is closed in F.

There exists a sequence $(U_n)_{n\in\mathbb{N}}$ in $\mathscr{U}_o(F/L)$, $U_n = \Gamma U_n$ $(n\in\mathbb{N})$, such that $\bigcap_{n\in\mathbb{N}} \sigma_n U_n$ is not a zero-neighbourhood in F/L for every sequence $(\sigma_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$. Let $Q : F \to F/L$ denote the quotient map and put $V_n := Q^{-1}(U_n)$ $(n\in\mathbb{N})$. Furthermore, let $(e_n)_{n\in\mathbb{N}}$ denote the sequence of unit vectors in ω.

Let us first assume that $\Psi : \bigoplus_{n\in\mathbb{N}} F \to \mathscr{L}_b(\omega,F)$ is open onto its range. Then there exist $B\in\mathfrak{B}(\omega)$ and $U \in \mathscr{U}_o(F)$, $U = \Gamma U$, such that $\Psi(\bigoplus_{n\in\mathbb{N}} F) \cap \mathscr{W}(B,U) \subset \sum_{n\in\mathbb{N}} P_n \otimes V_n$ (where $P_n \otimes V_n := \{P_n \otimes y : y \in V_n\}$).

Now there is $k\in\mathbb{N}$ such that V_k does not absorb U. Fix $\rho > 0$ such that $|P_k(x)| \leq \rho$ for all $x\in B$. Since $\frac{1}{\rho} U \not\subset V_k$, there is $z \in \frac{1}{\rho} U \setminus V_k$. Then the map $P_k \otimes z : \omega \to F$ clearly belongs to $\mathscr{W}(B,U)$ and to the range of Ψ. Consequently, there is a sequence $(y_n)_{n\in\mathbb{N}} \in (\prod_{n\in\mathbb{N}} V_n) \cap \bigoplus_{n\in\mathbb{N}} F$ such that $P_k \otimes z = \sum_{n\in\mathbb{N}} P_n \otimes y_n$, whence $z = (P_k \otimes z)(e_k) = (\sum_{n\in\mathbb{N}} P_n \otimes y_n)(e_k) = y_k$. This is a contradiction as $z \notin V_k$ and $y_k \in V_k$.

Next let $\mathscr{W}_n := \mathscr{W}(\{e_n\}, V_n) = \{T\in \mathscr{L}(\omega,F) : T(e_n)\in V_n\}$ $(n\in\mathbb{N})$. Then for every $n\in\mathbb{N}$ the set $\mathscr{W}_n$ is a closed and absolutely convex zero-neighbourhood in $\mathscr{L}_s(\omega,F)$, hence in $\mathscr{L}_b(\omega,F)$. Put $\mathscr{W} := \bigcap_{n\in\mathbb{N}} \mathscr{W}_n$.

If F/L does not contain ω, then $\mathscr{W}$ is absorbent in $\mathscr{L}(\omega,F)$. In fact, for every $T\in \mathscr{L}(\omega,F)$ the space $Q(T(\omega))$ is then finite dimensional; consequently, there is $n(T)\in\mathbb{N}$ such that $T(e_n)\in L$ for $n \geq n(T)$. Therefore $T\in \mathscr{W}_n$ $(n\geq n(T))$ which implies that T is absorbed by $\mathscr{W}$.

If F/L admits a continuous norm, then $\mathscr{W}$ is even bornivorous. In fact, let $\mathscr{H} \subset \mathscr{L}_b(\omega,F)$ be bounded and let $V=\Gamma V \in \mathscr{U}_o(F/L)$ satisfy $\bigcap_{\varepsilon>0} \varepsilon V = \{0\}$. As ω is a barrelled space, the set $\mathscr{H}$ is equicontinuous, hence there is $n(\mathscr{H})\in \mathbb{N}$ such that $Q(T(\prod_{n\geq n(\mathscr{H})}\mathbb{K}))\subset V$ for all $T\in\mathscr{H}$. This implies that $Q(T(e_n)) = 0$ $(T\in \mathscr{H}$, $n\geq n(\mathscr{H}))$ whence $\mathscr{H}\subset \mathscr{W}_n$ for all $n\geq n(\mathscr{H})$. Since $\mathscr{H}\subset \alpha \bigcap_{n<n(\mathscr{H})} \mathscr{W}_n$ for some $\alpha>0$, we obtain that $\mathscr{W}$ absorbs $\mathscr{H}$.

Now the proof will be finished if we show that $\mathscr{W} \notin \mathscr{U}_o(\mathscr{L}_b(\omega,F))$. Let us assume the contrary. Then there are $C\in\mathscr{B}(\omega)$ and $W\in \mathscr{U}_o(F)$ such that $\mathscr{W}(C,W) \subset \mathscr{W}$. There is $m\in\mathbb{N}$ such that V_m does not absorb W. Fix $\sigma > 0$ such that $|P_m(x)| \leq \sigma$ for all $x\in C$. Since $\frac{1}{\sigma} W \not\subset V_m$, there is $w \in \frac{1}{\sigma} W\setminus V_m$. The map $P_m \otimes w : \omega \to F$ clearly belongs to $\mathscr{W}(C,W)$, hence to $\mathscr{W}(\{e_m\},V_m)$. But this cannot be true as $(P_m \otimes w)(e_m) = w \notin V_m$. Therefore $\mathscr{W} \notin \mathscr{U}_o(\mathscr{L}_b(\omega,F))$.

(2.15) Remarks.

(a) Let F be a linear space of dimension not less than the cardinality of $\mathbb{R}$ and let F be provided with the strongest locally convex topology. We showed in (2.4) that F does not satisfy (cnc). Moreover, F clearly admits continuous norms. Thus we obtain from (2.14) that $\mathscr{L}_b(\omega,F)$ is not countably quasibarrelled, hence neither bornological nor barrelled, although $\omega_b'(=\varphi)$ and F both carry the strongest locally convex

topology.
As in (2.12) (a) we would like to mention that the above example also shows that in (1.11)(c) the countability of I is essential. In fact, ω satisfies (cbc), but the spaces $\bigoplus_{r\in\mathbb{R}} \mathscr{L}_b(\omega,\mathbb{K}) = \bigoplus_{r\in\mathbb{R}}\varphi$ and $\mathscr{L}_b(\omega,\bigoplus_{r\in\mathbb{R}}\mathbb{K})$ are not topologically isomorphic (though the map Φ from (1.11) is an algebraic isomorphism) since the first space is barrelled and the second is not.

In order to obtain some more applications of (2.14), let F be a nonnormable Fréchet space admitting continuous norms. Then F does not satisfy (cnc), whence by (2.14) and (1.2)(c), $\mathscr{L}_b(E,F)$ is not countably quasibarrelled whenever E is a locally convex space containing ω (see also A.Grothendieck [15;Chap.II, p.92, Prop.14]).

(b) Let $F:=\{(x_n)_{n\in\mathbb{N}}\in\omega : \{n\in\mathbb{N}: x_n\neq 0\}$ is finite $\}$ be provided with the relative topology induced by ω on F. F does not satisfy (cnc) as it is metrizable and nonnormable. Consequently, by (2.14), the space $\mathscr{L}_b(\omega,F)$ is not countably barrelled. (*) - On the other hand, we will show that $\mathscr{L}_b(\omega,F)$ is bornological (hence quasibarrelled, hence countably quasibarrelled).
In fact, for every $n\in\mathbb{N}$ let $P_n : F\to\mathbb{K}$ denote the projection onto the n'th coordinate. Then by (1.1),(c) and (b), the map

$$\Theta : \mathscr{L}_b(\omega,F) \to \prod_{n\in\mathbb{N}} \mathscr{L}_b(\omega,\mathbb{K}) = (\omega_b')^{\mathbb{N}} = \varphi^{\mathbb{N}}, T\mapsto (P_n\circ T)_{n\in\mathbb{N}},$$

is a topological isomorphism onto its range. Thus it is sufficient to show that $\Theta(\mathscr{L}(\omega,F))$ is a bornological subspace of $\varphi^{\mathbb{N}}$. By a result of H.Pfister quoted already in (2.12)(b), we must

(*) As dim(F) is countable. F does not contain ω as a linear subspace.

only show that $\Theta(\mathscr{L}(\omega,F)) \supset \varphi^{(\mathbb{N})} := \{(y_n)_{n\in\mathbb{N}} \in \varphi^{\mathbb{N}} : \{n\in\mathbb{N} : y_n \neq 0\}$ is finite).
Let $(f_n)_{n\in\mathbb{N}} \in (\omega')^{\mathbb{N}}$ and let $n_o \in \mathbb{N}$ be such that $f_n = 0$ whenever $n > n_o$. Then $T := \sum_{n \leq n_o} f_n \otimes e_n : \omega \to F$ (where $e_n = (\delta_{nm})_{m\in\mathbb{N}} \in F$ $(n\in\mathbb{N})$ as usual) is linear and continuous and $(P_n \circ T)(x) = f_n(x)$ $(n\in\mathbb{N}, x\in\omega)$. - Indeed, one even has the equality $\Theta(\mathscr{L}(\omega,F)) = (\omega_b')^{(\mathbb{N})}$, as F does not contain w as a subspace.

(c) Let $F := \mathbb{K}^{(\mathbb{R})} = \{(a_r)_{r\in\mathbb{R}} \in \mathbb{K}^{\mathbb{R}} : \{r\in\mathbb{R} : a_r \neq 0\}$ is finite }; then $\mathscr{T} := \beta(\mathbb{K}^{(\mathbb{R})}, \mathbb{K}^{\mathbb{R}})$ is the strongest locally convex topology on F. Let $\mathscr{Y} := \sigma(\mathbb{K}^{(\mathbb{R})}, \mathbb{K}^{[\mathbb{R}]})$ (see (2.12)(c)).

We showed in (2.12)(c) that for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}^{[\mathbb{R}]}$ there is $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $\{\rho_n x_n : n\in\mathbb{N}\}$ is bounded in $(\mathbb{K}^{[\mathbb{R}]}, \sigma(\mathbb{K}^{[\mathbb{R}]}, \mathbb{K}^{(\mathbb{R})}))$.

From this we obtain that for every sequence $(U_n)_{n\in\mathbb{N}}$ in $\mathscr{U}_o(F,\mathscr{Y})$ there is $(\sigma_n)_{n\in\mathbb{N}}$ e (IR:)" such that $\bigcap_{n\in\mathbb{N}} \sigma_n U_n$ belongs to $\mathscr{U}_o(F,\mathscr{T})$.

Consequently. by (1.12)(b), the canonica1 map $\Psi : \bigoplus_{n\in\mathbb{N}} F \to \mathscr{L}(\omega,(F,\mathscr{T}))$ is a bijection. - On the other hand, $(F,\mathscr{T})$ does not satisfy (cnc). Therefore $\Psi : \bigoplus_{n\in\mathbb{N}} (F,\mathscr{T}) \to \mathscr{L}_b(\omega,(F,\mathscr{T}))$ is not open by (2.14).

(d) The space ω does not satisfy (cnc), thus by (2.14), the map $\Psi : \bigoplus_{n\in\mathbb{N}} \omega \to \mathscr{L}_b(\omega,\omega)$ is not relatively open. Furthermore, Ψ is not surjective as the identity map $\omega \to$ w does not belong to the range of Ψ. However, by (1.1) (c), the space $\mathscr{L}_b(\omega,\omega)$ is topologically isomorphic to $\prod_{n\in\mathbb{N}} \mathscr{L}_b(\omega,\mathbb{K}) = \prod_{n\in\mathbb{N}} \varphi = \omega\varphi$ and hence bornological and barrelled (see G.Köthe [21;p.153,(13)]). Thus in (2.14) the additional hypotheses on F/L cannot be dropped.

§3. CONNECTION BETWEEN $\mathscr{L}_b(E,F)$ AND THE PROJECTIVE TENSOR PRODUCT.

From the Proposition (2.11) and (2.14) we obtain that a generally decent behaviour of $\mathscr{L}_b(E,F)$ can only be expected if both spaces E_b' and F satisfy (cnc). So the abstract investigations contained in section two led us close to A.Grothendieck's famous question [14;p.120, quest'ion 7)]:

If E is a metrizable locally convex space and F a DF-space is then $\mathscr{L}_b(E,F)$ again a DF-space?

This problem - which 1 could not solve - mav rather naturallv be extended to the following question:

(3.1) Let E and F be locally convex spaces such that E_b' and F both have (fsb). If in addition, E_b' and F bcth have some property ($\mathbb{P}$), does then $\mathscr{L}_b(E,F)$ also posses ($\mathbb{P}$)?

(In order to verify that (3.1) is indeed an extension of Grothendieck's question, loca11 (2.7) and take ($\mathbb{P}$):="countably quasibarrelled").

The examples (4.7),(4.8),(4.9),(5.9),(5.11) will show that (3.1) has a negative answer for (P) $\in$ {barrelled, quasibarrelled, bornological, ultrabornological).

At this point we would like to reca11 the following well-known result (cf. G.Köthe [21;p.186,(7)] and H.Jarchow [17;p.335, 2.Thm]) and its consequences.

(3.2). Lemma. Let E and F be locally convex spaces .such that E_b' and F are both DF-spaces. Let $(\mathscr{W}_n)_{n\in\mathbb{N}}$ be a sequence of absolutely convex zero-neighbourhoods in $\mathscr{L}_b(E,F)$ such that $\mathscr{W}:=\bigcap_{n\in\mathbb{N}}\mathscr{W}_n$ absorbs all sets $D\otimes B:=\{f\otimes y:\ f\in D,\ y\in B\}$ $(D\in\mathscr{B}(E_b'),B\in\mathscr{B}(F))$.

Then there is $A\in W(E)$ and $V\in \mathcal{U}_o(F)$ such that $A^\circ \otimes V \subset \mathcal{W}$.

For the sake of completeness we will give a proof, which presents G.Köthe's methods of proof in a somewhat condensed form.

Proof. For every $n\in\mathbb{N}$ there are $P_n\in \mathcal{U}_o(E_b')$ and $Q_n\in\mathcal{U}_o(F)$ such that $P_n \otimes Q_n \subset \mathcal{W}_n$.

(a) For every $B\in\mathcal{B}(F)$ the set $\check{B}(\mathcal{W}):=\{f\in E':f\otimes B \subset\mathcal{W}\}$ belongs to $\mathcal{U}_o(E_b')$. In fact, let $B\in\mathcal{B}(F)$. For every $n\in\mathbb{N}$, the set $\check{B}(\mathcal{W}_n):=\{f\in E':f\otimes B \subset \mathcal{W}_n\}$ is absolutely convex and belongs to $\mathcal{U}_o(E_b')$, as $P_n\otimes Q_n \subset \mathcal{W}_n$ and Q_n absorbs B. Since $\check{B}(\mathcal{W})=\bigcap_{n\in\mathbb{N}} \check{B}(\mathcal{W}_n)$ and E_b' is a DF-space, it suffices to show that $\check{B}(\mathcal{W})$ is bornivorous. Let $D\in \mathcal{B}(E_b')$. Then $\mathcal{W}$ absorbs $D \otimes B$, whence $\check{B}(\mathcal{W})$ absorbs D.

(b) For every $D\in\mathcal{B}(E_b')$ the set $\hat{D}(\mathcal{W}):=\{y\in F:D\otimes y \subset\mathcal{W}\}$ belongs to $\mathcal{U}_o(F)$. In fact, let $D\in\mathcal{B}(E_b')$. For every $n\in\mathbb{N}$ the set $\hat{D}(\mathcal{W}_n):=\{y\in F: D\otimes y \subset \mathcal{W}_n\}$ is absolutely convex and belongs to $\mathcal{U}_o(F)$, as $P_n \otimes Q_n \subset \mathcal{W}_n$ and P_n absorbs D. Since $\hat{D}(\mathcal{W})=\bigcap_{n\in\mathbb{N}}\hat{D}(\mathcal{W}_n)$ and since F is a DF-space, we must only show that $\hat{D}(\mathcal{W})$ is bornivorous. But this follows from the fact that $\mathcal{W}$ absorbs $D \otimes B$ for every $B\in\mathcal{B}(F)$.

(c) Let $(B_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in F. Then according to (a), for every $n\in\mathbb{N}$ the set $\check{B}_n(\mathcal{W})= \{f\in E':f \otimes B_n \subset \mathcal{W}\}$ belongs to $\mathcal{U}_o(E_b')$. Since E_b' satisfies (cnc), there is $(\rho_n)_{n\in\mathbb{N}}\in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $U:=\bigcap_{n\in\mathbb{N}} \rho_n\check{B}_n(\mathcal{W})$ belongs to $\mathcal{U}_o(E_b')$.

$V:=\{y\in F:U\otimes y \subset \mathcal{W}\}$ is bornivorous as $U\otimes \frac{1}{\rho_n} B_n \subset \rho_n\check{B}_n(\mathcal{W}) \otimes \frac{1}{\rho_n} B_n \subset \check{B}_n(\mathcal{W}) \otimes B_n \subset \mathcal{W}$ for every $n\in\mathbb{N}$. Moreover, $U \otimes V \subset \mathcal{W}$. Thus the proof will be finished if we show that V is the intersection of a sequence of absolutely convex zero-neighbourhoods in

F.

Let $(D_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in E_b'. Then, for every $n\in\mathbb{N}$, the set $A_n := U_n \cap D_n$ is bounded in E_b' and, according to (b), the absolutely convex set $V_n := \hat{A}_n(\mathscr{W}) = \{y\in F : A_n \otimes y \subset \mathscr{W}\}$ belongs to $\mathscr{U}_0(F)$. Moreover $\bigcap_{n\in\mathbb{N}} V_n = \{y \in F : \forall_{n\in\mathbb{N}}\, A_n \otimes y \subset \mathscr{W}\} = \{y\in F: (\bigcup_{n\in\mathbb{N}} A_n) \otimes y \subset \mathscr{W}\} = V$.

(3.3) PROPOSITION. *Let E and F be locally convex spaces such that E_b' and F are both DF-spaces.*

If the canonical inclusion $E_b' \otimes_\pi F \to \mathscr{L}_b(E,F)$ is almost open (i.e., for all $A\in\mathscr{B}(E)$ and $V\in \mathscr{U}_0(F)$ the set $\overline{\Gamma(A^\circ \otimes V)}$ - where the closure is taken in $\mathscr{L}_b(E,F)$ - is a zero-neighbourhood in $\mathscr{L}_b(E,F)$, then $\mathscr{L}_b(E,F)$ is also a DF-space.

In particular, if E is a metrizable locally convex space and F a DF-space such that E or F are nuclear, then $\mathscr{L}_b(E,F)$ is a DF-space. (By H.Jarchow [17, p.491,3 Thm.] a metrizable locally convex space is nuclear if and only if its strong dual is nuclear.)

Proof. The first statement follows easily from (3.2). In fact, let $\mathscr{W}$ be a bornivorous intersection of a sequence $(\mathscr{W}_n)_{n\in\mathbb{N}}$ of absolutely convex zero-neighbourhoods in $\mathscr{L}_b(E,F)$. According to (3.2) there are $U \in \mathscr{U}_0(E_b')$ and $V \in \mathscr{U}_0(F)$ such that $U \otimes V \subset \mathscr{W}$, which implies that $\bar{\mathscr{W}} \in \mathscr{U}_0(\mathscr{L}_b(E,F))$. Because of $\overline{\mathscr{W}} \subset \bigcap_{n\in\mathbb{N}} \overline{\mathscr{W}}_n \subset \bigcap_{n\in\mathbb{N}} 2\mathscr{W}_n = 2\mathscr{W}$ we obtain that $\mathscr{W}$ is a zero-neighbourhood in $\mathscr{L}_b(E,F)$.

For the second statement let us assume that E is metrizable and that E or F are nuclear. We will show that then the natural inclusion map

$$J: E_b' \otimes_\pi F \to \mathscr{L}_b(E,F)$$

is a topological isomorphism onto a dense linear subspace of $\mathscr{L}_b(E,F)$. Then, in particular, the map J is almost open, and thus the second statement will follow from the first.

Let us mention first that, for all locally convex spaces X,Y, the ε-tensor product $X_b' \otimes_\varepsilon Y$ is by definition a topological subspace of the space Z of all those linear operators $T: X'' \to Y$ whose restriction to $B^{\circ\circ}$ is $\sigma(X'',X')$-continuous for all $B \in \mathscr{B}(X)$, provided with the topology of uniform convsrgence on $\{B^{\circ\circ}: B \in \mathscr{B}(X)\}$. A moment's reflection shows that the restriction map $Z \to \mathscr{L}_b(X,Y)$, $T \mapsto T|X$, is a topological isomorphism onto its range. Therefore. also the inclusion $X_b' \otimes_\varepsilon Y \to \mathscr{L}_b(X,Y)$ is a topological isomorphism onto its range. Now, since E or F are nuclear and. E is metrizable, we have that E_b' or F are nuclear, whence the identity map $E_b' \otimes_\pi F \to E_b' \otimes_\varepsilon F$ is a homeomorphism by [17;p.345,4.Remark] and the symmetry of $\otimes_\pi$ and $\otimes_\varepsilon$, respectively.

Taking all this together we obtain that $J : E_b' \otimes_\pi F \to \mathscr{L}_b(E,F)$ is a topological isomorphism onto its range. Thus it remains to show that $E' \otimes F$ is dense in $\mathscr{L}_b(E,F)$.

If E is nuclear, E has the approximation property [17;p.483,2.Cor.] whence $E' \otimes F$ is dense in $\mathscr{L}_{\mathscr{P}}(E,F)$ where $\mathscr{P} := \{P \subset E : P \text{ precompact}\}$ (see [17;p.398,1.Thm.]). By the nuclearity of E one has $\mathscr{P} = \mathscr{B}(E)$. whence $E' \otimes F$ is dense in $\mathscr{L}_b(E,F)$. - Let us finally assume that F is nuclear. Just as before one obtains that $F' \otimes F$ is dense in $\mathscr{L}_b(F,F)$. Let $T \in \mathscr{L}(E,F)$, $B \in \mathscr{B}(E)$, $U \in \mathscr{U}_0(F)$. Then there is $S \in F' \otimes F$ such that $(Id_F - S)(T(B)) \subset U$. $S \circ T$ belongs to $E' \otimes F$, as it has finite dimensional range, and $(T-S \circ T)(B) \subset U$. Thus again $E' \otimes F$ is dense in $\mathscr{L}_b(E,F)$.

Remarks. (a) In his note [2;Thm.4] the Usbekian mathematician V.A.Balakliets has stated (wittout a proof) the following theorem. Let E be a Fréchet space and H a DF-space such that

(i) $\mathscr{F}(E,H) := \{T \in \mathscr{L}(E,H) : \dim T(E) < \infty\}$ is dense in $\mathscr{L}_b(E,H)$;

(ii) The sets $Co\{f \otimes y : f \in B^\circ, y \in W\}$ $(B \in Q(E),\ W \in \mathscr{U}_0(F))$, where "Co" denotes the convex hull, form a basis of $\mathscr{U}_0(\mathscr{F}(E,H))$, when $\mathscr{F}(E,H)$ is provided with the relative topology induced by $\mathscr{L}_b(E,H)$.

Then $\mathscr{L}_b(E,H)$ is a DF-space.

SO. our Proposition (3.3) provides a proof for the result of Balakliets.

(b) If the canonical inclusion $E_b' \otimes_\pi F \to \mathscr{L}_b(E,F)$ is almost open and if $E_b' \otimes_\pi F$ is barrelled (resp. quasibarrelled), then also $E_b' \otimes_\varepsilon F$ and $\mathscr{L}_b(E,F)$ are barrelled (resp. quasibarrelled), as can easily be derived from the properties of a continuous and almost open linear map. Since the ε-tensor product of two Banach snaces is seldom barrelled (cf. the end of (4.9)), there are Banach spaces B and F such that the above inclusion is not almost open. Furthermore. since the projectfve tensor product of two barrelled (resp. quasibarrelled) DF-spaces is barrelled (resp. auasibarrelled) (see G.Köthe [21; p.186,(8)]), Example (4.7) and Proposition (4.10) will also show that the inclusion $E_b' \otimes_\pi F \to \mathscr{L}_b(E,F)$ need not be almost open even if E_b', F. and $\mathscr{L}_b(E,F)$ are DF-spaces.

We would like to add another statement about the relation between $\mathscr{L}_b(E,F)$ and the projective tensor product.

(1.4) ***Remark.*** Let E and F be locally convex spaces such that F is quasibarrelled and such that for every $T \in \mathscr{L}(E,F_b')$ there is $U \in \mathscr{U}_0(E)$ such that $T(U) \in \mathscr{B}(F_b')$. (These hypotheses are satisfied if, for instance, E and F are metrizable, see (2.6).)
Then the map $\Theta : (E \otimes_\pi F)_b' \to \mathscr{L}_b(E,F_b')$, $\Theta(f)(x)(y) := f(x \otimes y)$ $(x \in E,\ y \in F)$, is a continuous surjective isomorphism.
Let $R : (E \tilde{\otimes}_\pi F)_b' \to (E \otimes_\pi F)_b'$, $f \mapsto f|(E \otimes F)$, denote the canonical continuous surjective isomorphism.
Then the following statements hold.

(a) Θ is open if and only if the sets $\overline{\Gamma(A \otimes B)}$ $(A \in \mathscr{B}(E), B \in \mathscr{B}(F))$ (the closure being taken in $E \otimes_\pi F$) form a fundamental system of bounded sets in $E \otimes_\pi F$.

(b) $\Theta \circ R$ is open if and only if the sets $\overline{\Gamma(A \otimes B)}$ $(A \in \mathscr{B}(E), B \in \mathscr{B}(F))$ (the closure being taken in $E \tilde{\otimes}_\pi F$) form a fundamental system of bounded sets in $E \tilde{\otimes}_\pi F$.

Proof. Let $A \in \mathscr{B}(E)$ and $U \in \mathscr{U}_0(F_b')$. Then there is $B \in \mathscr{B}(F)$ such that $B^\circ \subset U$; now $D := A \otimes B$ is a bounded subset in $E \otimes_\pi F$. Whenever $f \in D^\circ$ $(\subset (E \otimes_\pi F)')$, then $\Theta(f)(x) \in B^\circ \subset U$ for all $x \in A$. This proves that Θ is (correctly defined and) continuous. Θ is clearly injective. - Let $T \in \mathscr{L}(E,F_b')$, and let $U \in \mathscr{U}_0(E)$ be such that $T(U) \in \mathscr{B}(F_b')$. F being quasibarrelled, there is $V \in \mathscr{U}_0(F)$, such that $V^\circ \supset T(U)$. $g : E \times F \to \mathbb{K}$, $g(x,y) := T(x)(y)$, is bilinear and continuous as $|g(x,y)| \leq 1$ whenever $(x,y) \in U \times V$. Thus g gives rise to a continuous linear functional $f \in (E \otimes_\pi F)'$, and clearly $\Theta(f) = T$. Therefore Θ is also surjective.

(a) Θ is open if and only if for every $D \in \mathscr{B}(E \otimes_\pi F)$ there are $A \in \mathscr{B}(E)$ and $B \in \mathscr{B}(F)$ such that $\Theta(D^\circ) \supset \{T \in \mathscr{L}(E, F_b') : T(A) \subset B^\circ\}$, or -equivalently - such that $D^\circ \supset (A \otimes B)^\circ$, i.e., $D^{\circ\circ} \subset \overline{\Gamma(A \otimes B)}$.

(b) is proved just in the same way.

(3.5) ***Corollary.*** Let E and F he metrizable locally convex spaces such that $\mathscr{L}_b(E, F_b')$ is bornological. Then the metrizable locally convex spaces $E \otimes_\pi F'$ and $E \tilde{\otimes}_\pi F$ are distinguished and the sets $\overline{\Gamma(A \otimes B)}$ $(A \in \mathscr{B}(E), B \in \mathscr{B}(F))$ (where the closure is taken in $E \tilde{\otimes}_\pi F$) form a fundamental system of bounded sets in $E \tilde{\otimes}_\pi F$. (Cf. G.Köthe [21;p.185, last line].)

Proof. $E \tilde{\otimes}_\pi F$ being metrizable, the strong dual $(E \tilde{\otimes}_\pi F)_b'$ admits a finer LB-space-topology $\mathscr{T}$, namely $\mathscr{T} = \beta((E \tilde{\otimes}_\pi F)', (E \tilde{\otimes}_\pi F)'')$.
The map $\Theta \circ R: ((E \tilde{\otimes}_\pi F', \mathscr{T}) \to \mathscr{L}_b(E, F_b')$ from (3.4) is a continuous isomorphism. $\mathscr{L}_b(E, F_b')$ is complete (see A.Grothendieck's Theorem after (1.4)) and bornological by hypothesis. Thus A.Grothendieck [15;p.17,Thm.B] implies that $\Theta \circ R: ((E \tilde{\otimes}_\pi F)', \mathscr{T}) \to \mathscr{L}_b(E, F_b')$ is a topological isomorphism. Cohsequently, $\Theta \circ R: (E \tilde{\otimes}_\pi F)_b' \to$ $\to \mathscr{L}_b(E, F_b')$ and $\Theta: (E \otimes_\pi F)_b' \to \mathscr{L}_b(E, F_b')$ are topological isomorphisms. Thus $E \tilde{\otimes}_\pi F$ and $E \otimes_\pi F$ have both a bornological strong dual and are consequently distinguished.
Furthermore, with the help of (3.4) we obtain the rest of the assertions.

§4. COUNTABLE INDUCTIVE LIMITS

In (1.11), (2.10), and (2.11) we investigated the question under which circumstances the canonical map

$$\bigoplus_{n\in\mathbb{N}} \mathscr{L}_b(E,F_n) \to \mathscr{L}_b(E,\bigoplus_{n\in\mathbb{N}} F_n)$$

is a topological isomorphism. Our next aim is to extend these investigations to the case of countable inductive limits instead of countable direct sums.

(4.1) Let F be a linear space and let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of linear subspaces of F covering F. Moreover, let each F_n be provided with a locally convex topology such that for every $n\in\mathbb{N}$ the inclusion $F_n \hookrightarrow F_{n+1}$ is continuous, and let F be endowed with the strongest locally convex topology such that all inclusion maps $I_n : F_n \to F$ are continuous $(n\in\mathbb{N})$. Then we will call F the locally convex inductive limit or simply the inductive limit of the increasing sequence of locally convex spaces $(F_n)_{n\in\mathbb{N}}$ and write $F=\operatorname{ind}_{n\to} F_n$.

Moreover, we will call an inductive limit $F = \operatorname{ind}_{n\to} F_n$

regular, if for every $B\in\mathscr{B}(F)$ there exists $n\in\mathbb{N}$ such that $B \subset F_n$ and $B \in\mathscr{B}(F_n)$;

retractive, if for every $B\in\mathscr{B}(F)$ there exists $n\in\mathbb{N}$ such that $B \subset F_n$ and such that F and F_n induce the same relative topology on B;

strict, if for every $n\in\mathbb{N}$ the inclusion $F_n \to F_{n+1}$ is a topological isomorphism onto its range, or - equivalently - if for every $n\in\mathbb{N}$ the inclusion $I_n : F_n \to F$ is a topological isomorphism onto its range. (Cf. G.Köthe [20;p.222].)

An inductive limit $F = \operatorname{ind}_{n\to} F_n$ of an increasing sequence of Banach spaces $(F_n)_{n\in\mathbb{N}}$ will be called an LB-space. Inductive limits $F=\operatorname{ind}_{n\to} F_n$ of DF-spaces F_n $(n\in\mathbb{N})$ are again DF-spaces, whence, in particular, every LB-space is a DF-space (cf.G.Köthe [20;p.402,(4)]).

(4.2) ***Remarks.*** Let E be a locally convex space and let $F=\operatorname{ind}_{n\to} F_n$ be the inductive limit of an increasing sequence of locally convex spaces $(F_n)_{n\in\mathbb{N}}$. The for every $n\in\mathbb{N}$, the canonical linear injections

$$\mathscr{L}_b(E,F_n) \to \mathscr{L}_b(E,F_{n+1}),\ T \mapsto I_{n+1,n} \circ T \quad \text{and}$$

$$\mathscr{L}_b(E,F_n) \to \mathscr{L}_b(E,F)\ ,\ T \mapsto I_n \circ T\ ,$$

$(I_{n+1,n}: F_n \to F_{n+1}$ and $I_n : F_n \to F$ denoting the inclusion maps) are continuous by (1.1)(a). Via these injections we may simultaneously identify $\mathscr{L}(E,F_n)$ $(n\in\mathbb{N})$ with linear subspaces of $\mathscr{L}(E,F)$, and then form the inductive limit $\operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n)$.

The inclusion map

$$\Phi : \operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n) \to \mathscr{L}_b(E,\operatorname{ind}_{n\to} F_n)$$

is continuous and satisfies $\Phi(T) = I_n \circ T$ whenever $T\in\mathscr{L}(E,F_n)$ $(n\in\mathbb{N})$.

(a) Assume that the inductive limit $F = \operatorname{ind}_{n\to} F_n$ is not regular. Then there exists a bounded sequence $(x_m)_{m\in\mathbb{N}}$ in F such that for every $n\in\mathbb{N}$, either $\{x_m : m\in\mathbb{N}\}$ is not contained in F_n or this set is contained but unbounded in F_n.

Let $\mathbb{E}^1$ denote the space $\mathbb{K}^{(\mathbb{N})} := \{(a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \{n\in\mathbb{N} : a_n \neq 0\}$ is finite $\}$, provided with the norm

$$\|\cdot\|_1 : \mathbb{E}^1 \to \mathbb{R},\ (a_n)_{n\in\mathbb{N}} \to \sum_{n\in\mathbb{N}} |a_n| .$$

$T : \mathbb{E}^1 \to F,\ (a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n x_n$, is linear and maps the closed unit ball B in $\mathbb{E}^1$ into $\Gamma\{x_m : m\in\mathbb{N}\}$ which is a bounded set in F. Thus T belongs to $\mathscr{L}(\mathbb{E}^1, F)$, but clearly T does not belong to the range of Φ.

(b) Assume that for every bounded subset $\mathscr{H}$ in $\mathscr{L}_b(E,F)$ there is $U\in\mathscr{U}_0(E)$ such that $X(U)$ is bounded in F. (By (2.6) this hypothesis is satisfied if, for instance, E is pseudometrizable and F is a DF-space.) Let the inductive limit $F = \operatorname{ind}_{n\to} F_n$ be regular. Then Φ is surjective (i.e., a continuous identity map), any subset $\mathscr{H} \subset \mathscr{L}(E,F)$ is bounded in $\mathscr{L}_b(E,F)$ if and only if it is bounded in $\operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n)$, and the inductive limit $\operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n)$ is regular.

Proof. All three statements will be proved if we show that every bounded subset of $\mathscr{L}_b(E,F)$ is contained in some $\mathscr{L}_b(E,F_n)$ and bounded there. If $\mathscr{H} \in \mathscr{B}(\mathscr{L}_b(E,F))$, then there is $U\in\mathscr{U}_0(E)$ such that $B := \mathscr{H}(U)$ belongs to $B(F)$; by the regularity of $\operatorname{ind}_{n\to} F_n$, there is $n\in\mathbb{N}$ such that B belongs to $\mathscr{B}(F_n)$; consequently every $T\in\mathscr{H}$ ($\subset \mathscr{W}(U,B)$) maps E continuously into F_n, and the set $\mathscr{H}$ is a bounded subset of $\mathscr{L}_b(E,F_n)$.

(c) Assume that for every bounded subset $\mathscr{H}$ in $\mathscr{L}_b(E,F)$ there is $U\in\mathscr{U}_0(E)$ such that $\mathscr{H}(U)$ is bounded in F. Suppose that the inductive limit $F = \operatorname{ind}_{n\to} F_n$ is retractive. Then for every bounded subset $\mathscr{H}$ in $\mathscr{L}_b(E,F)$ the spaces $\operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n)$ and $\mathscr{L}_b(E,F)$ induce the same relative topology on $\mathscr{H}$, and the inductive limit $\operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n)$ is retractive.

Proof. Since every retractive inductive limit is in narticular regular, the three conclusions of (b) are valid. Thus it suffices to show: for every $n \in \mathbb{N}$, every $U \in \mathscr{U}_0(E)$ and every $B \in \mathscr{B}(F_n)$ the spaces $\mathscr{L}_b(E,F_n)$ and $\mathscr{L}_b(E,F)$ induce the same relative topology on $\mathscr{W}(U,B)$ $(=\{T \in \mathscr{L}(E,F_n) : T(U) \subset B\})$. But this statement follows immediately from the subsequent Lemma (4.3) and (1.1)(b).

(d) If the inductive limit $F = \operatorname{ind}_{n\to} F_n$ is strict, then also $\operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n)$ is a strict inductive limit.

Proof. It suffices to prove that for every $n \in \mathbb{N}$ the relative topology induced by $\mathscr{L}_b(E,F)$ on $\mathscr{L}(E,F_n)$ coincides with the original topology of $\mathscr{L}_b(E,F_n)$. This statement again follows from Lemma (4.3) and (1.1)(b).

(4.3) ***Lemma***. Let F be a linear space and let $\mathscr{Y}$ and $\mathscr{T}$ be two locally convex topologies on F. Let $A \subset F$ be an absolutely convex subset such that $\mathscr{T}|A \subset \mathscr{Y}|A$. Then for every locally convex space E and every bornivorous subset $X \subset E$ the space $\mathscr{L}_b(E,(F,\mathscr{T}))$ induces a coarser topology on $\mathscr{H} := \{T \in \mathscr{L}(E,(F,\mathscr{Y})) \cap \mathscr{L}(E,(F,\mathscr{T})) : T(X) \subset A\}$ than $\mathscr{L}_b(E,(F,\mathscr{Y}))$.

Proof. Let $\mathscr{W} \in \mathscr{U}_0(\mathscr{L}_b(E,(F,\mathscr{T})))$ be given. Then there are $B \in \mathscr{B}(E)$ and $U \in \mathscr{U}_0(F,\mathscr{T})$ such that $\{T \in \mathscr{L}_b(E,(F,\mathscr{T})) : T(B) \subset U\} \subset \mathscr{W}$. Since X is bornivorous, we may assume that $B \subset X$. Because of $\mathscr{T}|A \subset \mathscr{Y}|A$, there is $V \in \mathscr{U}_0(F,\mathscr{Y})$ such that $V \cap A \subset U$. Now

$\mathscr{V} := \{T \in \mathscr{L}(E,(F,\mathscr{Y})) : T(B) \subset V\}$ belongs to $\mathscr{U}_0(\mathscr{L}_b(E,(F,\mathscr{Y})))$. If $T \in \mathscr{V} \cap \mathscr{H}$, then $T(B) = T(B) \cap T(X) \subset V \cap A \subset U$, whence $T \in \mathscr{W}$. Since $\mathscr{H}$ is absolutely convex. the assertion of the lemma follows with the help of G. Köthe [20;p.265,(5)].

From (4.2),(b) and (c), we obtain the following corollary.

(4.4) Let E be a pseudometrizable locally convex space; let $F = \operatorname{ind}_{n\to} F_n$ be the inductive limit of an increasing sequence of locally convex spaces $(F_n)_{n\in\mathbb{N}}$ and assume that F is a DF-space. Let one of the following two hypotheses be satisfied

(a) The inductive limit $\operatorname{ind}_{n\to} F_n$ is regular. and $\mathscr{L}_b(E,F)$ is bornological;

(b) The inductive limit $\operatorname{ind}_{n\to} F_n$ is retractive. and $\mathscr{L}_b(E,F)$ is a DF-space.

Then the .canonical map $\Phi : \operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n) \to \mathscr{L}_b(E,\operatorname{ind}_{n\to} F_n)$ is a topologica1 isomorphism.

(In the case (b) we use the fact that every DF-space is "lokaltopo logisch", i. e. carries the strongest locally convex topology agreeing with the original topology on each bounded set, see G.Köthe [20,p.398,(7)] .)

Even if E is a Banach space and $F = \operatorname{ind}_{n\to} F_n$ is a regular LB-space, the map $\Phi : \operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n) \to \mathscr{L}_b(E,\operatorname{ind}_{n\to} F_n)$ need not be a topologica1 isomorphism, as we will see with the help of the following proposition.

(4.5) PROPOSITION. *Let $\mathbb{E}^1 := \mathbb{K}^{(\mathbb{N})}$ be provided with the norm* $\|\cdot\|_1 : \mathbb{K}^{(\mathbb{N})} \to \mathbb{R}, (a_n)_{n\in\mathbb{N}} \to \sum_{n\in\mathbb{N}} |a_n|$ (cf.(4.2)(a).

Moreover let F be a locally convex space containing an increasing sequence $(B_n)_{n\in\mathbb{N}}$ of absolutely convex bounded subsets B_n $(n\in\mathbb{N})$ such that the following two conditions are satisfied.

(a) *The sequence $(nB_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in F;*

(b) $\forall U\in\mathscr{U}_0(F)\quad \exists B\in\mathscr{B}(F)\quad \forall n\in\mathbb{N}\quad U\cap B \not\subset B_n .$

Then $\mathscr{L}_b(\mathbb{E}^1,F)$ is not bornological.

If in addition, F is a DF-space and all B_n are closed in F $(n\in\mathbb{N})$, then $\mathscr{L}_b(\mathbb{E}^1,F)$ is not quasibarrelled.

Moreover, if F is supposed to be (*) locally complete (see (1.9)), then the same statement hold with $\mathbb{E}^1$ replaced by the Banach space l^1.

Proof. Let A denote the closed unit ball in $\mathbb{E}^1$. For every $n\in\mathbb{N}$ let $\mathscr{W}_n := \{T\in\mathscr{L}(\mathbb{E}^1,F): T(A)\subset B_n\}$. Since every bounded set in $\mathscr{L}_b(\mathbb{E}^1,F)$ is absorbed by some $\mathscr{W}_n$, the set $\mathscr{W} := \bigcup_{n\in\mathbb{N}} \mathscr{W}_n$ is bornivorous in $\mathscr{L}_b(\mathbb{E}^1,F)$. Moreover, $\mathscr{W}$ is absolutely convex as $(B_n)_{n\in\mathbb{N}}$ is increasing.
Suppose that $\mathscr{W}$ is a zero-neighbourhood in $\mathscr{L}_b(\mathbb{E}^1,F)$. Then there is $U = \Gamma U\in\mathscr{U}_0(F)$ such that $\mathscr{U} := \{T\in\mathscr{L}(\mathbb{E}^1,F): T(A)\subset U\}$ is contained in $\mathscr{W}$. On account of (b) there exists a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in U such that $x_n\notin B_n$ $(n\in\mathbb{N})$. The map $T:\mathbb{K}^{(\mathbb{N})}\to F$,

(*) in addition

$(a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n x_n$, is linear and $T(X) \subset \Gamma\{x_n : n\in\mathbb{N}\} \subset U$. In particular, T belongs to $\mathscr{L}(\mathbb{E}^1, F)$ and hence to $\mathscr{U}$. On the other hand, $T((\delta_{nm})_{m\in\mathbb{N}}) = x_n \notin B_n$, whence $T(A) \not\subset B_n$ and $T \notin \mathscr{W}_n$ $(n\in\mathbb{N})$. Consequently, $T \notin \mathscr{W}$, a contradiction. Therefore $\mathscr{W}$ does not belong to $\mathscr{U}_o(\mathscr{L}_b(\mathbb{E}^1, F))$ and $\mathscr{L}_b(\mathbb{E}^1, F)$ is not bornological.

Now we postulate that F is a DF-space and that B_n is closed in F for all $n\in\mathbb{N}$. Then the closure $\bar{\mathscr{W}}$ of $\mathscr{W}$ in $\mathscr{L}_b(\mathbb{E}^1, F)$ is *a* ***bornivorous*** *barrel* ***in*** *$\mathscr{L}_b(\mathbb{E}^1, F)$.*

Suppose that $\bar{\mathscr{W}}$ ***is a zero-neighbourhood in*** *$\mathscr{L}_b(\mathbb{E}^1, F)$.* ***Then*** *again* there is $U = \Gamma U \in \mathscr{U}_o(F)$ such that $\mathscr{U} := \{T\in \mathscr{L}(\mathbb{E}^1, F) : T(A) \subset U\}$ is contained in $\frac{1}{2}\bar{\mathscr{W}}$, and there is a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in U such that $x_n \notin B_n$ $(n\in\mathbb{N})$. For every $n\in\mathbb{N}$ there is $V_n = \Gamma V_n \in \mathscr{U}_o(F)$ such that $x_n \notin B_n + V_n$. $V := \bigcap_{n\in\mathbb{N}} (\frac{1}{2} B_n + V_n)$ is bornivorous in F on account of (a) and since the sequence $(B_n)_{n\in\mathbb{N}}$ is increasing. Consequently, V is a zero-neighbourhood in the DF-space F, and $\mathscr{V} := \{T\in \mathscr{L}(\mathbb{E}^1, F) : T(A) \subset V\}$ is a zero-neighbourhood in $\mathscr{L}_b(\mathbb{E}^1, F)$. Therefore $\mathscr{U} \subset \frac{1}{2}\mathscr{W} + \mathscr{V}$.

Now $T : \mathbb{K}^{(\mathbb{N})} \to F$, $(a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n x_n$, is linear, satisfies $T(A) \subset \Gamma\{x_n : n\in\mathbb{N}\} \subset U$, and thus belongs to $\mathscr{U}$. Because of $\mathscr{U} \subset \frac{1}{2}\mathscr{W} + \mathscr{V} = \bigcup_{n\in\mathbb{N}} (\frac{1}{2}\mathscr{W}_n + \mathscr{V})$, there is $n\in\mathbb{N}$ and $S\in \frac{1}{2}\mathscr{W}_n$ such that $T-S\in\mathscr{V}$. In particular, $(T-S)(A) \subset V \subset \frac{1}{2} B_n + V_n$, whence $x_n \in T(A) \subset S(A) + \frac{1}{2} B_n + V_n \subset \frac{1}{2}B_n + \frac{1}{2}B_n + V_n = B_n + V_n$, which is a contradiction to the choice of x_n and V_n. Therefore $\bar{\mathscr{W}}$ is not a zero-neighbourhood in $\mathscr{L}_b(\mathbb{E}^1, F)$ and $\mathscr{L}_b(\mathbb{E}, F)$ is not quasibarrelled.

Finally, if F is locally complete, then $\mathscr{L}_b(\mathbb{E}^1, F)$ and

$\mathscr{L}_b(l^1,F)$ are topologically isomorphic (cf. P.Dierolf [lo]).

The following statement is a variant of (4.5).

(4.6) PROPOSITION. *Let F be a locally convex space containing an increasing sequence* $(B_n)_{n\in\mathbb{N}}$ *of bounded Banach disks such that*

(a) *Every bounded Banach disk in* F *is absorbed by some* B_n

(b) *For every* $U \in \mathscr{U}_o(F)$ *there exists a bounded Banach disk* C *in* F *such that* $C \subset U$ *and such that* $C \not\subset B_n$ *for every* $n\in\mathbb{N}$

Then $\mathscr{L}_b(l^1,F)$ *is not ultrabornological.*

(Remark: Every LB-space F contains an increasing sequence of bounded Banach disks satisfying (a). cf. A.Grothendieck [15;p.16,Thm.A] .)

Proof. Let A denote the closed unit ball in l^1. For every $n\in\mathbb{N}$ let F_n denote the Banach space $([B_n],p_{B_n})$, where p_{B_n} denotes the Minkowski functional of B_n. Then we have the LB-space $\operatorname{ind}_{n\to} F_n$ and the continuous identity map $J:\operatorname{ind}_{n\to} F_n \to F$. Moreover, by (4.2). we have the continuous inclusion maps

$$\operatorname{ind}_{n\to} \mathscr{L}_b(l^1,F_n) \xrightarrow[\Phi_1]{} \mathscr{L}_b(l^1,\operatorname{ind}_{n\to} F_n) \xrightarrow[\Phi_2]{} \mathscr{L}_b(l^1,F).$$

Both maps Φ_1 and Φ_2 are surjective. In fact, if $T\in\mathscr{L}(l^1,F)$, then T(A) is a bounded Banach disk in F, hence contained in some nB_n. $S:l^1 \to F_n$, $x \mapsto T(x)$, belongs to $\mathscr{L}(l^1,F_n)$ and $\Phi_2(\Phi_1(S)) = T$. By A.Grothendieck's closed graph theorem [15; p.17,Thm.B], the space $\mathscr{L}_b(l^1,F)$ is ultrabornological if and only if the map $\Phi_2 \circ \Phi_1$ is open (reca11 that the spaces $\mathscr{L}_b(l^1,F_n)$ $(n\in\mathbb{N})$ are Banach spaces whence $\operatorname{ind}_{n\to} \mathscr{L}_b(l^1,F_n)$ is an LB-space). Thus it suffices to show that $\Phi_2\circ\Phi_1$ is not open.

For every $n\in\mathbb{N}$ let $\mathscr{W}_n := \{T\in\mathscr{L}(l^1,F) : T(A) \subset B_n\}$. Then $\mathscr{W} := \bigcup_{n\in\mathbb{N}} \mathscr{W}_n$ is an absolutely convex zero-neighbourhood in $\operatorname{ind}_{n\rightarrow} \mathscr{L}_b(l^1,F_n)$. Assume that $\mathscr{W}$ is a zero-neighbourhood in $\mathscr{L}_b(l^1,F)$. Then there is $U = \overline{\Gamma U} \in \mathscr{U}_0(F)$ such that $\mathscr{U} = \{T \in \mathscr{L}(l^1,F) : T(A) \subset U\}$ is contained in $\mathscr{W}$. By hypothesis there exists a bounded Banach disk $C \subset U$ such that $C \not\subset B_n$ for all $n\in\mathbb{N}$. Choose $x_n \in C \setminus B_n$ $(n\in\mathbb{N})$. Since C is a bounded Banach disk, there is $T\in\mathscr{L}(l^1,F)$ such that $T((\delta_{mn})_{m\in\mathbb{N}}) = x_n$ $(n\in\mathbb{N})$. T belongs to $\mathscr{U}$ as $T(A) = T(\overline{\Gamma\{(\delta_{mn})_{m\in\mathbb{N}} : n\in\mathbb{N}\}}) \subset \overline{\Gamma\{x_n : n\in\mathbb{N}\}} \subset \overline{\Gamma U} = U$. But T does not belong to $\mathscr{W}$ as $x_n \in T(A) \setminus B_n$ for every $n\in\mathbb{N}$. This contradiction shows that $\mathscr{W}$ is not a zero-neighbourhood in $\mathscr{L}_b(l^1,F)$ and that $\Phi_2 \circ \Phi_1$ is not open.

Next we will describe a class of LB-spaces F satisfying the hypotheses of (4.5) and thus obtain a negative answer to question (3.1) for $\mathbb{P} \in$ {barrelled, quasibarrelled, bornological, ultrabornological}.

(4.7) Example.

Let $(X_n,r_n)_{n\in\mathbb{N}}$ and $(Y_n,s_n)_{n\in\mathbb{N}}$ be two sequences of Banach spaces (*) such that for every $n\in\mathbb{N}$

(a) Y_n is a linear subspace of X_n and $s_n \geq r_n|Y_n$;

(b) $\{y \in Y_n : s_n(y) \leq 1\}$ is closed in (X_n,r_n).

Moreover, let $(Z,\|.\|)$ be a normal Banach sequence space, i.e.,

---

(*) r_n and s_n denoting the norms on X_n and Y_n, respectively.

$\mathbb{K}^{(\mathbb{N})} \subset Z \subset \mathbb{K}^{\mathbb{N}}$ algebraically, the inclusion $(Z, \|.\|) \hookrightarrow \mathbb{K}^{\mathbb{N}}$ is continuous, and whenever $(a_k)_{k\in\mathbb{N}} \in Z$, $(b_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ are such that $|b_k| \leq |a_k|$ $(k\in\mathbb{N})$, then $(b_k)_{k\in\mathbb{N}} \in Z$ and $\|(b_k)_{k\in\mathbb{N}}\| \leq \|(a_k)_{k\in\mathbb{N}}\|$. Thus for every $m\in\mathbb{N}$ the pro'jection

$$Pr_m : (Z,\|.\|) \to (Z, \|.\|), (a_k)_{k\in\mathbb{N}} \mapsto (b_k)_{k\in\mathbb{N}} \text{ where}$$

$$b_k:=a_k \quad (k\leq m) \text{ and } b_k:=0\,(k>m),$$

is norm-decreasing. Moreover. since every norm on the scalar field $\mathbb{K}$ is a multiple of the absolute value, there exists for every $k\in\mathbb{N}$ a positive number ρ_k such that

$$\|(\delta_{kl}a)_{l\in\mathbb{N}}\| = \rho_k|a| \text{ for all } a\in\mathbb{K}.$$

Let us suppose that - in addition - the following hypothesis is satisfied

(c) whenever $a = (a_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ satisfies $\|Pr_m(a)\| \leq 1$ for all $m\in\mathbb{N}$, then a belongs to Z and $\|a\| \leq 1$.

Remark: Condition (c) is satisfied for a normal Banach sequence space $(Z,\|.\|)$ if the norm $\|.\|$ has the Fatou property in the sense of A.C.Zaanen [31;p.446,Def.]. The spaces l^p $(1 \leq p \leq \infty)$ and their diagonal. transforms are normal Banach sequence spaces satisfying condition (c), whereas c_o does not satisfy condition (c).

Now. for every $n\in\mathbb{N}$ the linear space

$$F_n:=\{(x_k)_{k\in\mathbb{N}} \in \prod_{k<n} X_k \times \prod_{k\geq n} Y_k : ((r_k(x_k))_{k<n},(s_k(x_k))_{k\geq n})\in Z\}$$

is a normed space with respect to the norm

$$(x_k)_{k\in\mathbb{N}} \mapsto \|((r_k(x_k))_{k<n},(s_k(x_k))_{k\geq n})\|.$$

Let

$$B_n := \{(x_k)_{k\in N} \in F_n : \|((r_k(x_k))_{k<n},(s_k(x_k))_{k\geq n})\| \leq 1\}$$

denote its closed unit ball. Then $F_n \subset F_{n+1}$ and $B_n \subset B_{n+1}$ since $s_n \geq r_n|Y_n$ $(n\in\mathbb{N})$. Thus we may form the inductive limit $F := \operatorname{ind}_{n\to} F_n$. The canonical injection $F \to \prod_{k\in\mathbb{N}}(X_k,r_k)$ is continuous. In fact, given $n,k\in\mathbb{N}$, then the k'th projection $P_k : F_n \to X_k$, $(x_l)_{l\in\mathbb{N}} \mapsto x_k$, is continuous as $\|((r_l(x_l))_{l<n},(s_l(x_l))_{l\geq n})\| \geq \|(\delta_{kl}r_k(x_k))_{l\in\mathbb{N}}\| = \rho_k r_k(x_k)$ $((x_l)_{l\in\mathbb{N}} \in F_n)$.

We will prove now that every set B_n is closed in the product space $\prod_{k\in\mathbb{N}}(X_k,r_k)$, hence closed in $F = \operatorname{ind}_{n\to} F_n$.

Let $n\in\mathbb{N}$, and let $((x_k^{(l)})_{k\in\mathbb{N}})_{l\in\mathbb{N}}$ be a sequence in B_n converging to some element $(x_k)_{k\in\mathbb{N}}$ in $\prod_{k\in\mathbb{N}}(X_k,r_k)$. For every $k\geq n$ and $l\in\mathbb{N}$ we have that $1 \geq \|(\delta_{km}s_k(x_k^{(l)}))_{m\in\mathbb{N}}\| = \rho_k s_k(x_k^{(l)})$. Since the set $\{y\in Y_k : s_k(y) \leq \frac{1}{\rho_k}\}$ is closed in (X_k,r_k) on account of condition (b) and since $(x_k^{(l)})_{l\in\mathbb{N}}$ converges to x_k in (X_k,r_k), we obtain that $x_k \in Y_k$ and $s_k(x_k) \leq \frac{1}{\rho_k}$ for all $k \geq n$.

Thus - in view of condition (c) - it remains to prove that

$$\|((r_k(x_k))_{k<n},(s_k(x_k))_{n\leq k\leq m},(0)_{k>m})\| \leq 1 \quad \text{for all } m \geq n.$$

Assume that there $m \geq n$ such that

$$\|((r_k(x_k))_{k<n},(s_k(x_k))_{n\leq k\leq m},(0)_{k>m})\| > 1+\varepsilon \quad \text{for some } \varepsilon > 0.$$

Since for every $n \leq k \leq m$, the set $X_k \setminus \{y\in Y_k : s_k(y) \leq s_k(x_k) - \frac{\varepsilon}{m\rho_k}\}$ is open in (X_k,r_k) and contains x_k, there is $l_0\in\mathbb{N}$ such that $s_k(x_k^{(l)}) > s_k(x_k) - \frac{\varepsilon}{m\rho_k}$ for all $l > l_0$ and $n < k \leq m$.

Moreover we may assume that in addition

$r_k(x_k^{(l)} - x_k) \leq \frac{\varepsilon}{m\rho_k}$ for all $l \geq l_0$ and $k < n$. Now fix $l \geq l_0$. Then

$$1 \geq \|((r_k(x_k^{(l)}))_{k<n}, (s_k(x_k^{(l)}))_{n\leq k\leq m}, (0)_{k>m})\| \geq$$

$$\|((r_k(x_k) - r_k(x_k^{(l)} - x_k))_{k<n}, (\max\{0, s_k(x_k) - \frac{\varepsilon}{m\rho_k}\})_{n\leq k\leq m}, (0)_{k>m})\| \geq$$

$$\|((r_k(x_k))_{k<n}, (s_k(x_k))_{n\leq k\leq m}, (0)_{k>m})\| - \sum_{k<n}\|(\delta_{jk} r_k(x_k^{(l)} - x_k))_{j\in\mathbb{N}}\| -$$

$$- \sum_{n\leq k\leq m}\|(\delta_{jk}\frac{\varepsilon}{m\rho_k})_{j\in\mathbb{N}}\| > 1 + \varepsilon - \sum_{k<n}\rho_k r_k(x_k^{(l)} - x_k) - \sum_{n\leq k\leq m}\rho_k\frac{\varepsilon}{m\rho_k} \geq$$

$\geq 1 + \varepsilon - \sum_{k<n}\frac{\varepsilon}{m} - \sum_{n\leq k\leq m}\frac{\varepsilon}{m} = 1$, which is a contradiction.

Thus B_n is closed in the Fréchet space $\prod_{k\in\mathbb{N}}(X_k, r_k)$ for all $n\in\mathbb{N}$. Consequently each of the normed spaces F_n is a Banach space and the LB-space $F = \operatorname{ind}_{n\to} F_n$ is regular (since the closure $\overline{nB_n}^F$ ($n\in\mathbb{N}$) form a fundamental sequence of bounded sets in F, see G.Köthe [20; p.402,(4)]).

In the case that (Y_k, s_k) is a topological subspace of (X_k, r_k) (i.e., s_k and r_k induce the same topology on Y_k) for all $k\in\mathbb{N}$, we get in the above way strict LB-spaces $F = \operatorname{ind}_{n\to} F_n$ of the type which V.B.Moscatelli constructed in [23]. In contrast to this case we will suppose that - from now on - the following "opposite" condition is satisfied

(d) The norm s_k generates a strictly stronger topology on Y_k than the restriction $r_k|Y_k$ for every $k\in\mathbb{N}$.

We will show that the sequence $(B_n)_{n\in\mathbb{N}}$ in the DF-space F satisfies hypothesis (b) of (4.5).

In fact, by (d), every space Y_n contains a sequence $(x^{(n,m)})_{m\in\mathbb{N}}$ such that $s_n(x^{(n,m)}) = 2$ and $r_n(x^{(n,m)}) \leq \frac{1}{m}$ $(m\in\mathbb{N})$.
Let $z^{(n,m)} := (\delta_{jn} \frac{1}{\rho_n} x^{(n,m)})_{j\in\mathbb{N}}$ $(n,m\in\mathbb{N})$.
Then $\|(\delta_{jn} \frac{1}{\rho_n} s_n(x^{(n,m)}))_{j\in\mathbb{N}}\| = 2$ whence $z^{(n,m)} \in 2B_1 \setminus B_n$ and $\|(\delta_{jn} \frac{1}{\rho_n} r_n(x^{(n,m)}))_{j\in\mathbb{N}}\| \leq \frac{1}{m}$ whence $z^{(n,m)} \in \frac{1}{m} B_{n+1}$ for all $n,m\in\mathbb{N}$.
Let U be a zero-neighbourhood in F. Then for every $n\in\mathbb{N}$ there is $m(n) \in \mathbb{N}$ such that $\frac{1}{m(n)} B_{n+1} \subset U$. Thus the sequence $(z^{(n,m(n))})_{n\in\mathbb{N}}$ is contained in $2B_1 \cap U$, hence bounded, and $z^{(n,m(n))} \notin B_n$ for all $n\in\mathbb{N}$. We have proved that $(B_n)_{n\in\mathbb{N}}$ satisfies hypothesis (b) of (4.5). Now (4.5) implies that the space $\mathscr{L}_b(l^1,F)$ is not quasibarrelled, and that the canonical continuous linear bijection

$$\Phi : \operatorname*{ind}_{n\to} \mathscr{L}_b(l^1,F_n) \to \mathscr{L}_b(l^1, \operatorname*{ind}_{n\to} F_n) \qquad \text{(see (4.2) (b))}$$

is not a topological isomorphism (as $\operatorname*{ind}_{n\to} \mathscr{L}_b(l^1,F_n)$ is an LB-space and $\mathscr{L}_b(l^1, \operatorname*{ind}_{n\to} F_n)$ is not even quasibarrelled).

Remark.

(1) In (4.7) we menaged to realize the hvpotheses of (4.5) in a regular LB-space F; in fact, instead of (4.5)(b), the space F constructed in (4.7) satisfies the (formally) stronger condition

$$\exists B\in\mathscr{B}(F) \quad \forall U\in\mathscr{U}_0(F) \quad \forall n\in N \quad B\cap U \not\subset B_n .$$

Corollary (1.12) will show that retractive LB-spaces never satisfy the hypotheses of (4.5).

(2) In order to show that (4.6) has applications different from those of (4.5) we will consider the following variant of the above example.

Let $(X_n,r_n)_{n\in\mathbb{N}}$, $(Y_n,s_n)_{n\in\mathbb{N}}$ be given as before in (4.7) such that (a) and (d) are satisfied.

Then $G_n:=\{(x_k)_{k\in\mathbb{N}} \in \prod_{k<n} X_k \times \prod_{k\geq n} Y_k : ((r_k(x_k))_{k<n},(s_k(x_k))_{k\geq n})\in c_0\}$ is a Banach space with respect to the norm

$$\|\cdot\|_{(n)} : (x_k)_{k\in\mathbb{N}} \to \sup\{r_k(x_k):k<n\}\cup\{s_k(x_k):k\geq n\}.$$

The inclusion $G_n \to G_{n+1}$ is norm-decreasing ($n\in\mathbb{N}$) and one may form the LB-space $G:=\operatorname{ind}_{n\to} G_n$, which is in general not regular (we mention without proof that G.Köthe's incomplete LB-space [20;p.434/435] is of the above type). The Banach disks $B_n:=\{x\in G_n: \|x\|_{(n)}\leq 1\}$ ($n\in\mathbb{N}$) are increasing and satisfy (4.6) (a). Using the same double sequence $(z^{(n,m)})_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ as in (4.7) one easily shows (taking $C:=2B_1$) that (4.6)(b) is also satisfied. Thus by (4.6). the space $\mathscr{L}_b(l^1,G)$ is not ultrabornological.

(3) Given two locally convex spaces E,F, the canonical inclusion

$$E'_b \otimes_\varepsilon F \to \mathscr{L}_b(E,F)$$

is a topological isomorphism onto the subspace

$$\mathscr{F}_b(E,F):=\{T\in\mathscr{L}(E,F):\dim T(E)<\infty\} \text{ of } \mathscr{L}_b(E,F).\ (*)$$

(*) See H.Jarchow [17;p.330, 4. Prop., and p.344].

A.Defant and W.Govaerts [9;Thm.19 and Prop.4] have shown that - whenever E_b' is an $\mathscr{L}_\infty$-Banach space and F is a quasibarrelled DF-space - then $E_b' \otimes_\varepsilon F$ (and hence $\mathscr{F}_b(E,F)$) are quasibarrelled. Together with our Example (4.7) these results show that the spaces $\mathscr{L}_b(E,F)$ and $\mathscr{F}_b(E,F)$ may behave quite differently.

The following example complements (4.5) to (4.7).

(4.8) Example. For every $n \in \mathbb{N}$ let

$$a_{i,k}^{(n)} := \begin{cases} k & \text{whenever } (i,k) \in \mathbb{N} \times \mathbb{N},\ i \le n \\ 1 & \phantom{\text{whenever }} (i,k) \in \mathbb{N} \times \mathbb{N},\ i > n. \end{cases}$$

Moreover, let $1 \le q \le p \le \infty$. For every $n \in \mathbb{N}$ let

$$F_n := \{(x_{ik})_{(i,k)\in\mathbb{N}\times\mathbb{N}} \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}} : (\frac{1}{a_{i,k}^{(n)}} x_{ik})_{(i,k)\in\mathbb{N}\times\mathbb{N}} \in l^p(\mathbb{N}\times\mathbb{N})\}$$

be provided with the norm

$$(x_{ik})_{(i,k)\in\mathbb{N}\times\mathbb{N}} \mapsto \|(\frac{1}{a_{i,k}^{(n)}} x_{ik})_{(i,k)\in\mathbb{N}\times\mathbb{N}}\|_p ,$$

where $\|.\|_p$ denotes the usual norm on $l^p(\mathbb{N}\times\mathbb{N})$. Then F_n is a Banach space topologically isomorphic to l^p, one has $F_n \subset F_{n+1}$, and the inclusions $F_n \to F_{n+1} \to \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ are continuous for all $n \in \mathbb{N}$. The inductive limit $F := \operatorname{ind}_{n\to} F_n$ is a co-echelon space of order p (in the terminology of K.D.Bierstedt. R.G.Meise, W.H. Summers [5;1.2] the space F is a space of type k_p). For every $n \in \mathbb{N}$ the set

$$B_n := \{(x_{ik})_{(i,k)\in\mathbb{N}\times\mathbb{N}} \in F_n : \|(\frac{1}{a^{(n)}_{i,k}} x_{ik})_{(i,k)\in\mathbb{N}\times\mathbb{N}}\|_p \leq 1\}$$

is closed in $\mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ (as it was proved, for instance, in (4.7)); consequently the LB-space $F = \operatorname{ind}_{n\to} F_n$ is regular. Since $B_n \subset B_{n+1}$ $(n\in\mathbb{N})$, the sequence $(nB_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in F. Moreover, if $1 < p < \infty$, then all F_n and hence F are reflexive.

Let $E^q := l^q$ if $q < \infty$ and $E^q := c_o$ if $q = \infty$.

We will prove that the space $\mathscr{L}_b(E^q,F)$ is not quasibarrelled. From (4.5) and (4.7) we got such a statement only for the case q = 1.)

Let A denote the closed unit ball in E^q. For every $n\in\mathbb{N}$ let $\mathbb{B}_n := \{T\in\mathscr{L}(E^q,F) : T(A) \subset B_n\}$. Then $\mathscr{W} := \overline{\bigcup_{n\in\mathbb{N}} \mathbb{B}_n}$ is a bornivorous barrel in $\mathscr{L}_b(E^q,F)$ (the closure being taken in $\mathscr{L}_b(E^q,F)$). Let us assume that $\mathscr{W} \in \mathscr{U}_o(\mathscr{L}_b(E^q,F))$. Then there is $U = \Gamma U \in \mathscr{U}_o(F)$ such that $\mathscr{U} := \{T\in\mathscr{L}(E^q,F) : T(A) \subset U\} \subset \frac{1}{2}\mathscr{W}$, and there is a sequence $(m(n))_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \frac{1}{m(n)} B_n \subset U$.

$x^{(n)} := (\delta_{i,n+1}\,\delta_{k,m(n+1)+1}\,\frac{m(n+1)+1}{m(n+1)})_{(i,k)\in\mathbb{N}\times\mathbb{N}}$ belongs to $2B_1 \cap \frac{1}{m(n+1)} B_{n+1}$ but not to B_n. In fact, for all $n, m\in\mathbb{N}$

$$\|(\frac{1}{a^{(1)}_{i,k}}\delta_{i,n+1}\delta_{k,m+1}\frac{m+1}{m})_{(i,k)\in\mathbb{N}\times\mathbb{N}}\|_p =$$

$$= \|(\frac{1}{a^{(n)}_{i,k}}\delta_{i,n+1}\delta_{k,m+1}\frac{m+1}{m})_{(i,k)\in\mathbb{N}\times\mathbb{N}}\|_p = \frac{m+1}{m} \in]1,2] \quad \text{and}$$

$$\|(\frac{1}{a^{(n+1)}_{i,k}}\delta_{i,n+1}\delta_{k,m+1}\frac{m+1}{m})_{(i,k)\in\mathbb{N}\times\mathbb{N}}\|_p = \frac{1}{m}.$$

For every sequence $(a_n)_{n\in\mathbb{N}}\in E^q$ the sum $\sum_{n\in\mathbb{N}} a_n x^{(n)}$ is well-defined in $\mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ and belongs to $l^q(\mathbb{N}\times\mathbb{N})$ since $|x_{ik}^{(n)}| \leq 2\delta_{i,n+1}\delta_{k,m(n+1)+1}$. Because of $l^q(\mathbb{N}\times\mathbb{N}) \subset l^p(\mathbb{N}\times\mathbb{N}) \subset F_1$, the map $T:E^q \to F$, $(a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n x^{(n)}$, is well-defined and linear. If $a=(a_n)_{n\in\mathbb{N}}\in A$, then $\|T(a)\|_p \leq \|T(a)\|_q \leq 2$, whence $T(a) \in 2B_1$. Thus $T(A) \subset 2B_1$ and hence $T\in\mathscr{L}(E^q,F)$. Moreover, $T(A\cap\mathbb{K}^{(\mathbb{N})}) \subset \sum_{n\in\mathbb{N}}\{\alpha x^{(n)}: |\alpha| \leq 1\} \subset$

$\subset \sum_{n\in\mathbb{N}} \frac{1}{m(n+1)} B_{n+1} \subset U$, whence $T(A) = \overline{T(A\cap\mathbb{K}^{(\mathbb{N})})} \subset \bar{U} = U$. Thus $T \in \mathscr{U} \subset \frac{1}{2}\mathscr{W}$. Since $x^{(n)}\notin B_n = \overline{B_n}^F$, there is $V_n=\Gamma V_n\in\mathscr{U}_o(F)$ such that $x^{(n)} \notin B_n + V_n$ $(n\in\mathbb{N})$. $V:=\bigcap_{n\in\mathbb{N}}(\frac{1}{2} B_n + V_n)$ is bornivorous and therefore a zero-neighbourhood in the DF-space F.

$\mathscr{V} := \{S\in\mathscr{L}(E^q,F):S(A) \subset V\}$ belongs to $\mathscr{U}_o(\mathscr{L}_b(E^q,F))$, thus $T\in \frac{1}{2}\mathscr{W} \subset (\frac{1}{2} \bigcup_{n\in\mathbb{N}} \mathbb{B}_n) + \mathscr{V}$. Consequently, there are $n\in\mathbb{N}$ and $S\in\frac{1}{2}\mathbb{B}_n$ such that $(T-S)(A) \subset V \subset \frac{1}{2}B_n + V_n$, whence $T(A) \subset S(A) + \frac{1}{2}B_n+V_n \subset B_n + V_n$. - On the other hand, $x^{(n)} \in T(A) \setminus (B_n+V_n)$ which is a contradiction. Thus $\mathscr{L}_b(E^q,F)$ is not quasibarrelled, and the canonical map $\Phi : \operatorname{ind}_{n\to} \mathscr{L}_b(E^q,F_n) \to \mathscr{L}_b(E^q,F)$ is not a topological isomorphism.

If $p=q=2$, we get an example of an LB-space $F = \operatorname{ind}_{n\to} F_n$ such that every F_n is topologically isomorphic to l^2 and such that $\mathscr{L}_b(l^2,F)$ is not quasibarrelled.

In view of (4.5) to (4.8) one may ask whether a (negative) solution to the problem

If E is a Fréchet space and F a barrelled DF-space, is then $\mathscr{L}_b(E,F)$ barrelled?

can be obtained in an easier way than via (4.5) and (4.7). Indeed, one can find Banach spaces E and normed barrelled spaces F such that the normed space $\mathscr{L}_b(E,F)$ is not barrelled as the following example shows.

(4.9) Example.

Given two locally convex spaces E, F, let $\mathscr{F}_b(E,F)$ denote the space of all $T\in\mathscr{L}(E,F)$ such that $\dim T(E)<\infty$, provided with the relative topology induced by $\mathscr{L}_b(E,F)$. By H. Jarchow [17;p.330,4.Prop., and p.344] the space $\mathscr{F}_b(E,F)$ is topologically isomorphic to $E'_b \otimes_\varepsilon F$. (Cf. Remark (3) after (4.7):)

If E,F,G are locally convex spaces such that F is a dense linear subspace of G, then the canonical injection $\mathscr{F}_b(E,F)\to\mathscr{F}_b(E,G)$ identifies $\mathscr{F}_b(E,F)$ topologically with a dense linear subspace of $\mathscr{F}_b(E,G)$. In fact, the above map is a topological isomorphism onto its range by (1.1)(b). For the density it obviously suffices to show: whenever $B\in\mathscr{B}(E)$, $U\in\mathscr{U}_o(G)$, $x\in G$, and $f\in B^\circ\subset E'$ are given, then there is $y\in F$ such that $(f\otimes x - f\otimes y)(B)\subset U$; but this statement is certainly true - just choose $y\in F\cap(x-U)$.

Now let E be an infinite dimensional Banach space, and let F be a barrelled normed space such that every bounded Banach disk in F has finite dimensional linear span (or - equivalently - such that for every Banach space Z and every $T\in\mathscr{L}(Z,F)$ the dimension of T(Z) is finite). An example of such a space F is the space $m_o:=\{(x_n)_{n\in N}\in l^\infty:\{x_n:n\in N\}$ is finite $\}$ provided with the supnorm (see J.Batt, P.Dierolf, J.Voigt [3]).-

Furthermore, M.Valdivia [29] showed that every separable Banach space contains a dense hyperplane in which every bounded Banach disk has finite dimensional linear span. Since barrelledness is inherited by hyperplanes, every such hyperplane is another example.

Assume that $\mathscr{L}_b(E,F)$ is barrelled. Let $\tilde{F}$ denote a completion of F. Then $\mathscr{L}_b(E,F) = \mathscr{F}_b(E,F)$ is a dense linear subspace of $\mathscr{F}_b(E,\tilde{F})$, consequently, $\mathscr{F}_b(E,\tilde{F})$ and hence $E_b' \otimes_\varepsilon \tilde{F}$ are barrelled. From H.Jarchow [17;p.486,3.Prop.] we now obtain that for every $1 \leq p \leq \infty$ the Banach space $\tilde{F}$ fails to be an S_p-space with approximation property. Furthermore, by H.Jarchow [17;p.430] every infinite dimensional Banach space with an unconditional basis is an S_p-space for some $p \in [1,\infty]$.

Taking all these observations together, we get many examples of the announced kind.

In view of (4.5) to (4.8) one would like to know whether $\mathscr{L}_b(l^1,F)$ is a DF-space whenever F is a DF-space. In fact, one has the following result.

(4.10) PROPOSITION. *Let I be a set and let* $\mathbb{E}^1 := \mathbb{K}^{(I)} := \{(x_\iota)_{\iota\in I} \in \mathbb{K}^I : \{\iota\in I : x_\iota \neq 0\}$ *is finite) be provided with the norm*

$$\|.\|_1 : \mathbb{K}^{(I)} \to \mathbb{R},\ (x_\iota)_{\iota\in I} \to \sum_{\iota\in I} |x_\iota|.$$

Then for every DF-*space* F *the space* $\mathscr{L}_b(\mathbb{E}^1,F)$ *is again a* DF-*space.*

Proof. Let F be a DF-space and let $(B_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in F, $B_n = \Gamma B_n \subset B_{n+1}$ $(n\in\mathbb{N})$. Moreover, let

A denote the closed unit ball in $\mathbb{E}^1$. Then the sets $\mathbb{B}_n := \{T \in \mathscr{L}(\mathbb{E}^1, F) : T(A) \subset B_n\}$ $(n \in \mathbb{N})$ form a fundamental sequence of bounded sets in $\mathscr{L}_b(\mathbb{E}^1, F)$. Thus it remains to prove that $\mathscr{L}_b(\mathbb{E}^1, F)$ is countably quasibarrelled.

Let $(\mathscr{W}_n)_{n \in \mathbb{N}}$ be a sequence of absolutely convex zero-neighbourhoods in $\mathscr{L}_b(\mathbb{E}^1, F)$ such that $\mathscr{W} := \bigcap_{n \in \mathbb{N}} \mathscr{W}_n$ is bornivorous. For every $n \in \mathbb{N}$ choose $\rho_n > 0$ such that $\rho_n \mathbb{B}_n \subset \frac{1}{2^{n+1}} \mathscr{W}$. Then $\sum_{n \in \mathbb{N}} \rho_n \mathbb{B}_n$ is contained in $\frac{1}{2} \mathscr{W}$. Moreover, for every $n \in \mathbb{N}$, there is an absolutely convex zero-neighbourhood V_n in F such that

$$\mathscr{V}_n := \{T \in \mathscr{L}(\mathbb{E}^1, F) : T(A) \subset V_n\} \subset \frac{1}{2} \mathscr{W}_n .$$

$U_n := (\sum_{m \leq n} \rho_m B_m) + V_n$ is absolutely convex and belongs to $\mathscr{U}_0(F)$ $(n \in \mathbb{N})$, and $U := \bigcap_{n \in \mathbb{N}} U_n$ is bornivorous in F. In fact, if $m \in \mathbb{N}$, then $\rho_m B_m \subset U_n$ for all $n \geq m$, whence B_m is absorbed by U. Consequently, U is a zero-neighbourhood in the DF-space F.

We will show that $\mathscr{U} := \{T \in \mathscr{L}(\mathbb{E}^1, F) : T(A) \subset U\} \subset \bigcap_{n \in \mathbb{N}} \mathscr{W}_n$.

Let $T \in \mathscr{U}$ and let $n \in \mathbb{N}$. By $(e_\iota)_{\iota \in I}$ we denote the family of unit vectors $e_\iota = (\delta_{\iota\kappa})_{\kappa \in I}$ in $\mathbb{E}^1$ $(\iota \in I)$. For every $\iota \in I$ the element $T(e_\iota)$ belongs to U and hence to $U_n = \sum_{m \leq n} \rho_m B_m + V_n$. Consequently, there exist $b_{\iota, m} \in B_m$ $(m \leq n)$ and $v_\iota \in V_n$ such that

$$T(e_\iota) = \sum_{m \leq n} \rho_m b_{\iota, m} + v_\iota .$$

For every $m \leq n$ the linear map

$$T_m : \mathbb{K}^{(I)} \to F, \ (a_\iota)_{\iota \in I} \mapsto \sum_{\iota \in I} a_\iota b_{\iota, m} ,$$

satisfies $T_m(A) \subset \Gamma B_m = B_m$, whence $T_m \in \mathscr{L}(\mathbb{E}^1, F)$ and $T_m \in \mathbb{B}_m$.

Moreover $T - \sum_{m\leq n} \rho_m T_m$ belongs to $\mathscr{V}_n$ since $(T - \sum_{m\leq n} \rho_m T_m)(A) =$ $= \Gamma\{(T- \sum_{m\leq n} \rho_m T_m)(e_\iota): \iota \in I\} = \Gamma\{y_\iota : \iota \in I\} \subset V_n$. Consequently, $T \in \sum_{m\leq n} \rho_m \mathbb{B}_m + \mathscr{V}_n \subset \frac{1}{2}\mathscr{W} + \frac{1}{2}\mathscr{W}_n \subset \mathscr{W}_n$. Thus $\mathscr{U} \subset \bigcap_{n\in\mathbb{N}} \mathscr{W}_n$, whence $\mathscr{W}$ is a zero-neighbourhood in $\mathscr{L}_b(\mathbb{E}^1, F))$.

Remark: A.Defant observed that - using essentially the same methods as in the proof (4.10) - one can show the following stament:

If F is a quasinormable (*) locally convex space, then $\mathscr{L}_b(\mathbb{E}^1, F)$ is also quasinormable (wheve $\mathbb{E}^1$ has the same meaning as in (4.10)).

Proof. Let A denote the closed unit ball in $\mathbb{E}^1$ and let $\mathscr{W}$ be a zero-neighbourhood in $\mathscr{L}_b(\mathbb{E}^1, F)$. Then there is $U \in \mathscr{U}_o(E)$ such that $\mathscr{U} := \{T \in \mathscr{L}(\mathbb{E}^1, F): T(A) \subset U\} \subset \mathscr{W}$.

Since F is quasinormable, there is $V = \Gamma V \in \mathscr{U}_o(F)$ such that for every $\varepsilon > 0$ one finds a bounded subset $B(\varepsilon)$ in F such that $V \subset \varepsilon U + B(\varepsilon)$.

$\mathscr{V} := \{T \in \mathscr{L}(\mathbb{E}^1, F): T(A) \subset V\}$ belongs to $\mathscr{U}_o(\mathscr{L}_b(\mathbb{E}^1, F))$.

Let $\varepsilon > 0$ be given. Choose $B = \Gamma B \in \mathscr{B}(F)$ such that $V \subset \varepsilon U + B$. Then $\mathbb{B} := \{T \in \mathscr{L}(\mathbb{E}^1, F): T(A) \subset B\}$ is bounded in $\mathscr{L}_b(\mathbb{E}^1, F)$ and $\mathscr{V} \subset \varepsilon \mathscr{U} + \mathbb{B}$.

In fact, let $T \in \mathscr{V}$ and put $e_\iota := (\delta_{\iota\kappa})_{\kappa \in I} \in \mathbb{E}^1$ $(\iota \in I)$. Then for every $\iota \in I$ one has $T(e_\iota) \in V$ whence there are $u_\iota \in U$, $b_\iota \in B$ such

(*) A locally convex space F is called quasinormable if

$$\forall U \in \mathscr{U}_o(F)\ \exists V \in \mathscr{U}_o(F)\ \forall \varepsilon > 0\ \exists B \in \mathscr{B}(F)\quad V \subset \varepsilon U + B.$$

that $T(e_\iota) = \varepsilon u_\iota + b_\iota$.
$S: \mathbb{E}^1 \to F$, $(a_\iota)_{\iota\in I} \mapsto \sum_{\iota\in I} a_\iota b_\iota$, belongs to $\mathscr{L}(\mathbb{E}^1, F)$ and hence to $\mathbb{B}$, since $\{b_\iota : \iota\in I\}$ is a bounded set. Moreover,

$\frac{1}{\varepsilon}(T-S)(A) \subset \Gamma\{u_\iota : \iota\in I\} \subset U$, whence $T = \varepsilon(\frac{1}{\varepsilon}(T-S)) + S \in \varepsilon\mathscr{U} + \mathbb{B}$.

A variant of (4.10) is the following

(4.11) PROPOSITION. *Let* F *be a* DF-*space containing an increasing sequence* $(C_n)_{n\in\mathbb{N}}$ *of bounded Banach disks such that the closures* $\overline{C}_n$ $(n\in\mathbb{N})$ *for a fundamental sequence of bounded sets in* F. *(This hypothesis is satisfied if* F *is a locally complete* DF-*space or if* F *is an* LB-*space* (cf. G. Köthe [20; p.402, (4)]).) *Let* I *be a set. Then* $\mathscr{L}_b(l^1(I), F)$ *is a* DF-space.

Proof. We must show that $\mathscr{L}_b(l^1(I), F)$ is countably quasibarrelled. Let $(\mathscr{W}_n)_{n\in\mathbb{N}}$ be a sequence of absolutely convex zero-neighbourhoods in $\mathscr{L}_b(l^1(I), F)$ such that $\mathscr{W} := \bigcap_{n\in\mathbb{N}} \mathscr{W}_n$ is bornivorous. Let A denote the closed unit ball in $l^1(I)$, and for every $n\in\mathbb{N}$ let $\mathscr{C}_n := \{T \in \mathscr{L}(l^1(I), F) : T(A) \subset C_n\}$. Then there exists a sequence $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \rho_n \mathscr{C}_n \subset \frac{1}{2}\mathscr{W}$. Moreover. choose $V_n = \overline{\Gamma V_n} \in \mathscr{U}_0(F)$ such that $\mathscr{V}_n := \{T \in \mathscr{L}(l^1(I), F) : T(A) \subset V_n\} \subset \frac{1}{2}\mathscr{W}_n$ $(n\in\mathbb{N})$. Then $U_n := (\sum_{m\leq n} \rho_m C_m) + V_n$ is absolutely convex, belongs to $\mathscr{U}_0(F)$ $(n\in\mathbb{N})$, and $U := \bigcap_{n\in\mathbb{N}} U_n$ is bornivorous (since $\rho_m \overline{C}_m \subset U_n$ for all $n \geq m$), hence a zero-neighbourhood in the DF-space F. The proof will be finished if we show that

$$\mathscr{U} := \{T \in \mathscr{L}(l^1(I), F) : T(A) \subset \tfrac{1}{2} U\} \subset \bigcap_{n\in\mathbb{N}} \mathscr{W}_n.$$

Let $T\in\mathscr{U}$ and $n\in\mathbb{N}$. By $(e_\iota)_{\iota\in I}$ we denote the family of unit vectors in $l^1(I)$. For every $\iota\in I$ there exist $(c_{\iota,m})_{m\leq n} \in \prod_{m\leq n} C_m$ and $v_\iota \in V_n$ such that $T(e_\iota) = \frac{1}{2}(\sum_{m\leq n} \rho_m c_{\iota,m} + v_\iota)$.
As C_m is a bounded Banach disk, there is $T_m \in \mathscr{L}(l^1(I), F)$ such

that $T_m(e_\iota) = c_\iota m$ $(\iota \in I)$. Moreover, $T_m(A) \subset \bigcap_{\varepsilon>0}(1+\varepsilon)C_m \subset 2C_m$ $(m \leq n)$. $S:=2T-\sum_{m\leq n}\rho_m T_m$ satisfies $S(A) \subset \overline{\Gamma V_n}=V_n$. Consequently,

$$T=\frac{1}{2}\Big(\sum_{m\leq n}\rho_m T_m+S\Big) \in \frac{1}{2}\Big(\Big(\sum_{m\leq n}2\rho_m\mathscr{C}_m\Big)+\mathscr{V}_n\Big) \subset \sum_{m\leq n}\rho_m\mathscr{C}_m+\mathscr{V}_n \subset \frac{1}{2}\mathscr{W}+$$

$$+\frac{1}{2}\mathscr{W}_n \subset \mathscr{W}_n.$$

(4.12) ***Corollary.*** Let I be a set and let $\mathbb{E}^1:=\mathbb{K}^{(I)}$ be provided with the norm $\|.\|_1 : \mathbb{K}^{(I)} \to \mathbb{R}$, $(a_\iota)_{\iota\in I} \mapsto \sum_{\iota\in I}|a_\iota|$ as in (4.10). Moreover, let F be a DF-space which is the retractive inductive limit $\operatorname{ind}_{n\to} F_n$ of an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of locally convex spaces.
Then the canonical injection $\operatorname{ind}_{n\to}\mathscr{L}_b(\mathbb{E}^1,F_n) \to \mathscr{L}_b(\mathbb{E}^1,F)$ is a topological isomorphism.
If in addition, F satisfies the hypotheses of (4.11), then also the canonical injection $\operatorname{ind}_{n\to}\mathscr{L}_b(l^1(I),F_n) \to \mathscr{L}_b(l^1(I),F)$ is a topological isomorphism.

In particular, if $F = \operatorname{ind}_{n\to} F_n$ is a retractive LB-space, then also $\mathscr{L}_b(l^1(I),F)$ is a retractive LB-space, hence barrelled and bornological.

Proof. All the statements follow at once from (4.10) resp. (4.11) and (4.4),(4.2)(c).

Another rather curious consequence of (4.10) is the following

(4.13) PROPOSITION. *Let* F *be a* DF*-space and let* $(B_n)_{n\in\mathbb{N}}$ *be an increasing sequence of bounded, absolutely convex, and closed subsets of* F *such that* $(nB_n)_{n\in\mathbb{N}}$ *is a fundamental sequence of bounded sets in* F.
Then every null-sequence in F *is residually contained in some* B_m.

Proof. Assume that the assertion does not hold. Then there exists a null sequence $(x_n)_{n\in\mathbb{N}}$ in F such that for every $m\in\mathbb{N}$ the sequence $(x_n)_{n\in\mathbb{N}}$ does not residually lie in Bm. Then we find an increasing sequence $(k(n))_{n\in\mathbb{N}}$ in IN such that $x_{k(n)} \notin B_n$ for all $n\in\mathbb{N}$. Put $y_n := x_{k(n)}$ $(n\in\mathbb{N})$ and choose $V_n = \Gamma V_n \in \mathscr{U}_o(F)$ such that $y_n \notin B_n + V_n$ $(n\in\mathbb{N})$. Moreover. let $\mathbb{E}^1$ denote the space $\mathbb{K}^{(\mathbb{N})}$ provided with the norm $\|.\|_1 : (a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} |a_n|$, and let A denote the closed unit ball in $\mathbb{E}^1$, as usual.

Let $\mathbb{B}_n := \{T\in \mathscr{L}(\mathbb{E}^1,F) : T(A) \subset B_n\}$ $(n\in\mathbb{N})$. Then $\mathscr{W} := \overline{\frac{1}{\Gamma} \bigcup_{n\in\mathbb{N}} \mathbb{B}_n}$ is a bornivorous barrel in $\mathscr{L}_b(\mathbb{E}^1,F)$ (cf. the proof of (4.5)).

For every $m\in\mathbb{N}$ the linear map $T_m : \mathbb{E}^1 \to F$, $(a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\geq m} a_n y_n$, maps A into $\Gamma\{y_n : n\in\mathbb{N}\}$ and is thus continuous. Moreover, the sequence $(T_m)_{m\in\mathbb{N}}$ converges to zero in $\mathscr{L}_b(\mathbb{E}^1,F)$. In fact, the sets $\mathscr{V} := \{T\in \mathscr{L}(\mathbb{E}^1,F) : T(A) \subset V\}$ $(V=\Gamma V \in \mathscr{U}_o(F))$ form a basis of $\mathscr{U}_o(\mathscr{L}_b(\mathbb{E}^1,F))$ and $T_m\in\mathscr{V}$ whenever $\{y_n : n\geq m\} \subset V$.

Now, $\mathscr{L}_b(\mathbb{E}^1,F)$ is a DF-space by (4.10). $(T_m)_{m\in\mathbb{N}}$ is a null sequence, and $\mathscr{W}$ is a bornivorous barrel in $\mathscr{L}_b(\mathbb{E}^1,F)$. Consequently, by G.Köthe [20;p.398,(8)], there is $m_o\in\mathbb{N}$ such that $T_m\in\mathscr{W}$ for all $m \geq m_o$. As $U := \bigcap_{n\in\mathbb{N}}(\frac{1}{2} B_n + V_n)$ belongs to $\mathscr{U}_o(F)$ (same proof as in (4.5)), $\mathscr{U} := \{T\in \mathscr{L}(\mathbb{E}^1,F) : T(A) \subset U\}$ belongs to $\mathscr{U}_o(\mathscr{L}_b(\mathbb{E}^1,F))$, whence $T_{m_0} \in \frac{1}{2} \bigcup_{n\in\mathbb{N}} \mathbb{B}_n + \mathscr{U}$, i.e. there is $n_o\in\mathbb{N}$, $n_o \geq m_0$, such that $T_{m_0} \in \frac{1}{2} \mathbb{B}_{n_o} + \mathscr{U}$: Therefore $T_{m_0}(A) \subset \frac{1}{2} B_{n_0} + (\frac{1}{2} B_{n_o} + V_{n_0}) = B_{n_0} + V_{n_0}$, which is a contradiction to $y_{n_0} \in T_{m_0}(A) \setminus (B_{n_0} + V_{n_0})$.

Remark: We do not know, whether in (4.13) the closedness of the sets B_n ($n \in \mathbb{N}$) can be dropped.

15. A CLASS OF FRÉCHET SPACES

In (1.12),(2.13), and (2.14) we investigated the question, under which circumstances the canonical map

$$\bigoplus_{n\in\mathbb{N}} \mathscr{L}_b(E_n,F) \rightarrow \mathscr{L}_b(\prod_{n\in\mathbb{N}} E_n,F)$$

is a topological isomorphism. One of our aims in this section will be to extend these investigations to the case of countable projective limits instead of countable products, a concept which is in some sense dual to that studied in section four.

(5.1) Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of locally convex spaces; for all $m,n\in\mathbb{N}$, $m \geq n$, let $P_{nm} : E_m \rightarrow E_n$ be a continuous linear map such that P_{nn} equals the identity map and $P_{nm} \circ P_{ml} = P_{nl}$ $(l \geq m \geq n)$. Then we call the pair $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m\geq n})$ a projective sequence, and the space $E := \{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} E_n : \forall_{m\geq n} P_{nm}(x_m) = x_n \}$, provided with the relative topology induced by the product topology of $\prod_{n\in\mathbb{N}} E_n$, is called its projective limit and denoted $E = \text{proj}_{\leftarrow n}\, E_n$.

The canonical projections $E \rightarrow E_n, (x_m)_{m\in\mathbb{N}} \mapsto x_n$, will be denoted by P_n $(n\in\mathbb{N})$.

Furthermore. we will call a projective limit $E = \text{proj}_{\leftarrow n}\, E_n$ of a projective sequence $((E_n)_{n\in\mathbb{N}}, (P_{nm})_{m>n})$

reduced, if P,(E) is dense in E_n for all $n\in\mathbb{N}$, and

strict, if $P_n : E \rightarrow E_n$ is surjective and open for all $n\in\mathbb{N}$.

(Recall that a projective limit $E = \text{proj}_{\leftarrow n}\, E_n$ of a projective sequence $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m>n})$ is strict if and only if all the maps $P_{nm} : E_m \rightarrow E_n$ are surjective and open $(m\geq n)$.)

(5.2) Remarks. Let $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m>n})$ be a projective sequence of locally convex spaces, let E be its projective limit, and let F be a locally convex space. For all $m,n\in\mathbb{N}$, $m\geq n$, the linear maps

$$J_{mn} : \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(E_m,F),\ T \mapsto T \circ P_{nm}, \text{ and}$$
$$J_n : \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(E,F),\ T \mapsto T \circ P_n,$$

are continuous by (1.2)(a). Thus we may form the inductive limit $\lim\limits_{\rightarrow} J_{mn}(\mathscr{L}_b(E_n,F))$ (in the sense of G.Köthe [20;p.220]) and obtain the canonical linear and continuous map

$$\Psi : \lim_{\rightarrow} J_{mn}(\mathscr{L}_b(E_n,F)) \to \mathscr{L}_b(\operatorname{proj}_{\leftarrow n} E_n,F),\ \Psi\circ I_m = J_m (m\in\mathbb{N}),$$

where $I_m : \mathscr{L}_b(E_m,F) \to \lim\limits_{\rightarrow} J_{mn}(\mathscr{L}_b(E_n,F))$ denotes the canonical map.

(a) Assume that the projective limit $E = \operatorname{proj}_{\leftarrow n} E_n$ is reduced and that F is Hausdorff. Then the maps J_n and J_{mn} are injective for all $n\in\mathbb{N}$, $m \geq n$. Via these injections we may simultaneously identity $\mathscr{L}(E_n,F)$ $(n\in\mathbb{N})$ with linear subspaces of $\mathscr{L}(E,F)$, and the inductive limit $\lim\limits_{\rightarrow} J_{mn}(\mathscr{L}_b(E_n,F))$ may be considered as the inductive limit of the increasing sequence of locally convex spaces $(\mathscr{L}_b(E_n,F))_{n\in\mathbb{N}}$ in the sense of (4.1) ; the canonical map from above

$$\Psi : \operatorname{ind}_{n\rightarrow} \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(\operatorname{proj}_{\leftarrow n} E_n,F)$$

then becomes a continuous inclusion map.

(b) Let E be the reduced projective limit of a projective sequence $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m\geq n})$ of locally convex spaces and let F be a locally complete Hausdorff locally convex space such that for every bounded subset $\mathcal{H}$ in $\mathcal{L}_b(E,F)$ there is $U \in \mathcal{U}_o(E)$ such that $\mathcal{H}(U)$ is bounded in F (cf. (2.6)). Then Υ is surjective, any subset $\mathcal{H} \subset \mathcal{L}(E,F)$ is bounded in $\mathcal{L}_b(E,F)$ if and only if it is bounded in $\operatorname{ind}_{n\to} \mathcal{L}_b(E_n,F)$, and the inductive limit $\operatorname{ind}_{n\to} \mathcal{L}_b(E_n,F)$ is regular.

Proof. What we have to show is the following: every $\mathcal{H} \in \mathcal{B}(\mathcal{L}_b(E,F))$ is contained in some $\mathcal{L}_b(E_n F)$ and bounded there. By hypothesis, gives $\mathcal{H} \in \mathcal{B}(\mathcal{L}_b(E,F))$, there is $U \in \mathcal{U}_o(E)$ such that $B := \overline{\Gamma \mathcal{H}(U)}$ is bounded in F. Now there is $n \in \mathbb{N}$ and an open zero-neighbourhood V in E_n such that $U \supset P_n^{-1}(V)$. The set $\mathbb{B} := \{T \in \mathcal{L}(E_n,F) : T(V) \subset B\}$ is a bounded subset of $\mathcal{L}_b(E_n,F)$; thus it remains to show that $\mathcal{H} \subset J_n(\mathbb{B})$.

Let $T \in \mathcal{H}$. Because of $T(\ker P_n) \subset T(U) \subset B$ and since B is bounded in the Hausdorff space F, the map T vanishes on $\ker P_n$. Consequently. there exists a linear map $S : P_n(E) \to F$ such that $S \circ P_n = T$. Clearly, $S(V \cap P_n)(E)) \subset T(U) \subset B$. Since $P_n(E)$ is dense in E_n and since B is a closed, bounded Banach disk in F, the linear map S admits a linear extension $\check{T} : E_n \to F$ such that $\check{T}(\overline{V \cap P_n(E)}) \subset B$, whence $\check{T}$ is continuous and $\check{T}(V) \subset \check{T}(\overline{V \cap P_n(E)}) \subset B$. Thus $\check{T} \in \mathbb{B}$ and $J_n(\check{T}) = \check{T} \circ P_n = S \circ P_n = T$.

(c) Let E be the reduced projective limit of a projective sequence $((E_n)_{n\in\mathbb{N}}, (P_{nm})_{m>n})$ of locally convex spaces and let F be a locally complete Hausdorff locally convex space such that for every bounded subset $\mathcal{H}$ in $\mathcal{L}_b(E,F)$ there is $U \in \mathcal{U}_o(E)$ such that $\mathcal{H}(U)$ is bounded in F. Moreover assume that the following condition is satisfied

$$\underset{n\in\mathbb{N}}{\forall}\ \underset{U\in\mathcal{U}_o(E_n)}{\forall}\ \underset{m\geq n}{\exists}\ \underset{B\in\mathcal{B}(E_m)}{\forall}\ \underset{\check{B}\in\mathcal{B}(E)}{\exists}\ P_{nm}(B)\subset U+P_n(\check{B}).$$

Then for every boundcd subset $\mathcal{H}$ in $\mathcal{L}_b(E,F)$ the spaces $\operatorname{ind}_{n\to}\mathcal{L}_b(E_n,F)$ and $\mathcal{L}_b(E,F)$ induce the same relative topology on $\mathcal{H}$, and the inductive limit $\operatorname{ind}_{n\to}\mathcal{L}_b(E_n,F)$ is retractive.

Proof. Let $\mathcal{H}\in\mathcal{B}(\mathcal{L}_b(E,F))$ be given. Because of (b) and its proof there are $n\in\mathbb{N}$, $U\in\mathcal{U}_o(E_n)$, and $A=\Gamma A\in\mathcal{B}(F)$ such that $\mathcal{H}\subset J_n(\mathbb{B})$ where $\mathbb{B}:=\{T\in\mathcal{L}(E_n,F):T(U)\subset A\}$.

Now, by hypothesis, there is $m\geq n$ such that

$$\underset{B\in\mathcal{B}(E_m)}{\forall}\ \underset{\check{B}\in\mathcal{B}(E)}{\exists}\ P_{nm}(B)\subset U+P_n(\check{B}).$$

The proof will be finished if we show that $\mathcal{L}_b(E_m,F)$ and $\mathcal{L}_b(E,F)$ induce the same topology on $\mathbb{B}$.

Let $\mathcal{W}$ be a zero-neighbourhood in $\mathcal{L}_b(E,F)$. Then there are $B\in\mathcal{B}(E_m)$, $V=\Gamma V\in\mathcal{U}_o(F)$ such that $\mathcal{W}\supset\{T\in\mathcal{L}(E_m,F):T(B)\subset V\}$, and we may assume that $A\subset\frac{1}{2}V$. By hypothesis there is $\check{B}\in\mathcal{B}(E)$ such that $P_{nm}(B)\subset U+P_n(\check{B})$.

$\mathcal{V}:=\{T\in\mathcal{L}(E,F):T(\check{B})\subset\frac{1}{2}V\}$ is a zero-neighbourhood in $\mathcal{L}_b(E,F)$, and it remains to show that $\mathcal{V}\cap(\mathbb{B}\circ P_n)\subset\mathcal{W}\circ P_m$.

Let $T\in\mathcal{V}\cap(\mathbb{B}\circ P_n)$. Then there is $S\in\mathbb{B}$ such that $T=S\circ P_n=S\circ P_{nm}\circ P_m$. Because of $(S\circ P_{nm})(B)\subset S(U)+S(P_n(\check{B}))\subset A+T(\check{B})\subset$ $\subset\frac{1}{2}V+\frac{1}{2}V\subset V$, we obtain that $S\circ P_{nm}\in\mathcal{W}$ whence $T\in\mathcal{W}\circ P_m$.

(d) Let E be the reduced projective limit of a projective sequence $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m\geq n})$ of locally convex spaces E_n $(n\in\mathbb{N})$, and let F be a Hausdorff locally convex space. Assume that for every $n\in\mathbb{N}$ and every $B\in\mathcal{B}(E_n)$ there is $A\in O(E)$ such that

$P_n(A) \supset B$. Then the inductive limit $\operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F)$ is strict and the spaces $\mathscr{L}_b(E_m,F)$ and $\mathscr{L}_b(E,F)$ induce the same topology on $\mathscr{L}(E_m,F)$ for every $m\in\mathbb{N}$.

Proof. Let $m\in\mathbb{N}$, $B\in\mathscr{B}(E_m)$, and $\bar{U} = U\in\mathscr{U}_0(F)$ be given. Then there is $A\in\mathscr{B}(E)$ such that $P_m(A) \supset B$.

Now $\mathscr{W} := \{T\in\mathscr{L}(E,F) : T(A) \subset U\}$ belongs to $\mathscr{U}_0(\mathscr{L}_b(E,F))$ and $\mathscr{W}\cap(\mathscr{L}(E_m,F)\circ P_m \subset \{T\circ P_m : T\in\mathscr{L}(E_m,F), T(B)\subset U\}$.

In fact, if $S\in\mathscr{L}(E_m,F)$ satisfies $S\circ P_m \in \mathscr{W}$, then

$$S(B) \subset S(P_m(A) \subset \overline{(S\circ P_m)(A)} \subset \bar{U} = U.$$

Consequently, $\mathscr{L}_b(E_m,F)$ and. $\mathscr{L}_b(E,F)$ induce the same topology on $\mathscr{L}(E_m,F)$, which finishes the proof.

(5.3) ***Remark.*** Let E be a Fréchet space and let $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m\geq n})$ be a projective sequence of Banach spaces E_n $(n\in\mathbb{N})$ such that E is the reduced projective limit of the sequence $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m\geq n})$. Let F be a locally complete Hausdorff DF-space. Then by (5.2)(b), the canonical map $\Psi : \operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(E,F)$ is bijective. Moreover, it follows from (5.2) (b) that Ψ is a topological isomorphism if $\mathscr{L}_b(E,F)$ is bornological.

Next we will assume that in addition E is quasinormable (see the remark after (4.10)). Then for every $n\in\mathbb{N}$ and $U = \Gamma U \in \mathscr{U}_0(E_n)$ there is $m\geq n$ and an open $V \in \mathscr{U}_0(E_m)$ such that for every $\varepsilon > 0$ there exists $B\in\mathscr{B}(E)$ such that $P_m^{-1}(V) \subset \varepsilon P_n^{-1}(U)+B$, which implies that

$$P_{nm}(\tfrac{1}{\varepsilon}V) \subset \frac{1}{\varepsilon} P_{nm}(\overline{V\cap P_m(E)}^{E_m}) \subset \frac{1}{\varepsilon} \overline{P_{nm}(V\cap P_m(E))}^{E_n} \subset \frac{1}{\varepsilon} \overline{P_n(P_m^{-1}(V))}^{E_n} \subset$$

$$\subset \overline{P_n(P_n^{-1}(U) + \frac{1}{\varepsilon} B)}^{E_n} \subset \overline{U+P_n(\frac{1}{\varepsilon}B)}^{E_n} \subset 2U+P_n(\frac{1}{\varepsilon}B).$$

Thus the hypotheses of (5.2) (c) are satisfied, and we obtain that the inductive limit $\operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F)$ is retractive; furthermore (5.2)(c) implies that the canonical continuous linear bijection

$$\Psi : \operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(E,F)$$

is a topological isomorphism whenever $\mathscr{L}_b(E,F)$ is a DF-space (here we use that DF-spaces are "lokaltopologisch", see (4.4)).

Our next aim is to describe a certain class of Fréchet spaces which are closely related to (5.2)(d). We will do this with the help of the following two proposition, whose final formulation has been suggested by K.Floret and whose proofs can be found in [35].

(5.4) PROPOSITION. *Let* $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m\geq n})$ *be a projective sequence of Banach space* E_n $(n\in\mathbb{N})$ *such that its projective limit* $E = \operatorname{proj}_{\leftarrow n} E_n$ *is reduced. Then the following statements are equivalent.*

(a) *The maps* $P_{nm} : E_m \to E_n$ *are surjective for all* $m \geq n$;

(b) *For every* $n\in\mathbb{N}$ *and every* $B\in\mathscr{B}(E_n)$ *there exists* $A \in \mathscr{B}(E)$ *such that* $P_n(A)\supset B$.

(c) *For every* $n\in\mathbb{N}$ *and every* $B\in\mathscr{B}(E_n)$ *there exists* $A \in \mathscr{B}(E)$ *such that* $\overline{P_n(A)}^{E_n} \supset B$.

(d) *The inductive limit* $\operatorname{ind}_{n\to}(E_n)'_b$ *is a strict inductive limit.*

Furthermore, if one of these conditions is satisfied, then *the canonical map* $\psi: \operatorname{ind}_{n\to}(E_n)'_b \to E'_b$ *is a topological isomorphism.*

(5.5) PROPOSITION. *Let* E *be a Fréchet space. Then the following statements are equivalent.*

(a) *There exists a projective sequence* $((E_n))_{n\in\mathbb{N}},(P_{nm})_{m\geq n})$ *of Banach spaces* E_n $(n\in\mathbb{N})$ *such that*

(i) $E = \operatorname{proj}_{\leftarrow n} E_n$ *and the projective limit* $\operatorname{proj}_{\to n} E_n$ is *reduced;*

(ii) $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m\geq n})$ *satisfiss the conditions* **(a)** *to* (d) *in* **(5.4).**

(b) *There exist a decreasinq sequence* $(L_n)_{n\in\mathbb{N}}$ *of linear subspaces of* E *and* $B = \Gamma B \in \mathcal{B}(E)$ *such that the sets* $L_n + \frac{1}{n} + B$ $(n\in\mathbb{N})$ *form a basis of* $\mathcal{U}_o(E)$.

(c) *There exist a* decreasinq *sequence* $(L_n)_{n\in\mathbb{N}}$ *of linear subspaces of* E *and a sequence* $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(E)$, $B_n = \Gamma B_n$ $(n\in\mathbb{N})$, *such that the sets* $L_n + B_n$ $(n\in\mathbb{N})$ *form a basis of* $\mathcal{U}_o(E)$.

(d) *There exist a decreasinq sequence* $(L_n)_{n\in\mathbb{N}}$ *of linear subspaces of* E *and a sequence* $(B_n)_{n\in\mathbb{N}}$ in **I(E)**. $B_n = \Gamma B_n$ $(n\in\mathbb{N})$. *such that the sets* $\overline{L_n+B_n}^E$ $(n\in\mathbb{N})$ *form a basis of* $\mathcal{U}_o(E)$.

(5.6) Remarks.

(a) In [33;Thm.1] (see also [34; Thm.2]) D.N.Zarnadze had proved that a Fréchet space satisfies (5.5)(d) if and only if it is the reduced projective limit of a projective sequence of Banach spaces satisfying (5.4) (d). Moreover, in [34;Remark 2] he had asked whether (5.5)(d) and (5.4)(a) are. equivalent. S.F.Bellenot and E.Dubinsky [4] call a strict projective limit of a projective sequence of Banach spaces a "quojection".

(b) Let E be a Fréchet space satisfying one of the equivalent conditions in (5.5). Then the following statements hold (cf.[35]).

(α) E is quasinormable and hence distinguished;

(β) E is a Montel space if and only if E is finite dimensional or topologically isomorphic to ω (See D.N.Zarnadze [33;p.825].)

(γ) E is reflexive if and only if E is the projective limit of a projective sequence of reflexive Banach spaces. Consequently, if E is reflexive, then E is totally reflexive (i.e. all quotients of E are reflexive). (See A.Grothendieck [14;Prop.10].)

We would like to take the opportunity and mention that the formulation of Proposition 10 in A.Grothendieck [14;p.100] is not correct:

Let $E_1 := \varphi\omega\ (:= \bigoplus_{n\in\mathbb{N}} \omega)$ and $E_n := \varphi$ $(n > 1)$. If $K \subset \mathbb{N}$ is finite, then $\prod_{n\in K} E_n$ is either topologically isomorphic to φ, $\varphi\omega$, or $\varphi\omega \times \varphi$. If follows from a result of V.Eberhardt (Beispiele topologischer Vektorräume mit der Komplementarraumeigenschaft, Arch.Math. 26(1975),627-636, 1.2 Satz) that φ, $\varphi\omega$ and $\varphi\omega\times\varphi$ are all totally reflexive. On the other hand, it is well known that $\prod_{n\in\mathbb{N}} E_n = \varphi\omega \times \omega\varphi$ has a non-reflexive quotient (G.Köthe [20;p.120]). A.Grothendieck's result is clearly true for every sequence $(E_n)_{n\in\mathbb{N}}$ of Fréchet spaces, and his proof was meant for that case only, since he used the theorem of Banach-Dieudonné in his proof.

(c) The twisted Fréchet spaces constructed by V.B.Moscatelli in [23] all satisfy (5.5)(a).

(d) Let $F=\text{ind}_{n\to} F_n$ be a strict LB-space. Then there is $(B_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}(\mathcal{U}_o(F_n) \cap \mathcal{B}(F_n))$ such that $B_{n+1} \cap F_n = B_n = \Gamma B_n$ for all $n\in\mathbb{N}$. The sets $U_n := \frac{1}{n}B_n^\circ$ $(n\in\mathbb{N})$ form a basis of $\mathcal{U}_o(F'_b)$. Moreover, $B := \bigcup_{n\in\mathbb{N}} B_n$ belongs to $\mathcal{U}_o(F)$. Let $V = \overline{\Gamma V} \in \mathcal{U}_o(F)$ be such that $\frac{1}{2}B \subset V \subset B$. Then $U_n \subset \frac{1}{n}(V \cap F_n)^\circ = \frac{1}{n}\overline{(V^\circ + F_n^\circ)}^{F'_s} =$

$= \frac{1}{n}(V^\circ + F_n^\circ) \subset 2U_n$ $(n\in\mathbb{N})$.

This proves that the Fréchet space F'_b satisfies condition (5.5)(b).

(e) Let E be a Fréchet space satisfying one of the equivalent conditions in (5.5). If E admits a continuous norm, then E is a Banach space as follows immediately from (5.5)(b).

For Fréchet spaces satisfying one of the equivalent conditions in (5.5) we can prove the following structure theorem.

(5.7) PROPOSITION. *Let E be a Fréchet space and let* $((E_n)_{n\in\mathbb{N}}, (P_{nm})_{m\geq n})$ *be a projective sequence of Banach spaces such that* $P_{nm}: E_m \to E_n$ *is surjective for all* $m > n$ *and such that E is equal to the projective limit* $\text{proj}_{\leftarrow n} E_n$.

Moreover, let F be a Banach space with the λ *extension property for some* $\lambda \geq 1$ *(i.e. for all Banach spaces* $(X, \|.\|)$ *and closed linear subspaces* $(Y, \|.\|\,|\,Y)$ *each linear continuous map* $T: Y \to F$ *has a continuous linear extension* $\check{T}: X \to F$ *such that* $\|\check{T}\| \leq \lambda\|T\|$ (see H.E.Lacey [22;p.86])).

Then the canonical map $\Psi : \operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F) \to \mathscr{L}_b(E,F)$ *defined in* (5.2) *is a topological isomorphism and* $\mathscr{L}_b(E,F)$ *is a strict LB-space.*

In particular, for every Fréchet space E *satisfying one of the conditions of (5.5) and every Banach space* F *with the* λ *estension property, the space* $\mathscr{L}_b(E,F)$ *is a bornological DF-space*

Proof. Clearly, for every $n\in\mathbb{N}$, the canonical projection $P_n : E \to E_n$ is surjective, and the projective sequence $((E_n)_{n\in\mathbb{N}},(P_{nm})_{m>n})$ satisfies (5.4)(c). Therefore, by (5.2), (b) and (d), the map Ψ is a continuous linear bijection and the inductive limit $\operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F)$ is a strict LB-space. Thus we have only to show that Ψ is open.

Choose $(B_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}(\mathscr{U}_o(E_n) \cap \mathscr{B}(E_n))$ such that $P_{n\,n+1}(B_{n+1}) = B_n = \Gamma B_n$ for all $n\in\mathbb{N}$, and let A denote the closed unit ball in F. Let $\mathscr{W} \in \mathscr{U}_o(\operatorname{ind}_{n\to} \mathscr{L}_b(E_n,F))$ be given. Then there is a decreasing sequence $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \rho_n(\mathscr{W}(B_n,A)\circ P_n) \subset \Psi(\mathscr{W})$ where $\mathscr{W}(B_n,A) := \{T\in \mathscr{L}(E_n,F) : T(B_n) \subset A\}$ $(n\in\mathbb{N})$. $B := E \cap (\prod_{n\in\mathbb{N}} \frac{1}{\rho_n}B_n)$ is a bounded set in E and $\mathscr{V} := \{T\in\mathscr{L}(E,F) : T(B) \subset \frac{1}{2\lambda}A\}$ belongs to $\mathscr{U}_o(\mathscr{L}_b(E,F))$. Thus it suffices to show that $\mathscr{V} \subset \Psi(\mathscr{W})$.

Let $T\in \mathscr{V}$. Then there are $k\in\mathbb{N}$ and $S\in \mathscr{L}_b(E_k,F)$ such that $T = S\circ P_k$. We first show that $S(\bigcap_{n\le k} P_{nk}^{-1}(\frac{1}{\rho_n}B_n)) \subset \frac{1}{2\lambda}A$.

Let $x_k \in \bigcap_{n\le k} P_{nk}^{-1}(\frac{1}{\rho_n}B_n)$. Then for every $n\le k$, the element

$x_n:=P_{nk}(x_k)$ belongs to $\frac{1}{\rho_n}B_n$. Moreover, because of $P_{l,l+1}(B_{l+1})=B_l$ $(l\in\mathbb{N})$, we inductively find a sequence $(x_l)_{l>k}\in\prod_{l>k}\frac{1}{\rho_k}B_l$ such that $B_{l,l+1}(x_{l+1})=x_l$ $(l\geq k)$. Because of $\rho_k\geq\rho_l$ $(l\geq k)$ we have that $x_l\in\frac{1}{\rho_l}B_l$ $(l>k)$, whence $x:=(x_l)_{l\in\mathbb{N}}\in$ $\in E\cap\prod_{l\in\mathbb{N}}\frac{1}{\rho_l}B_l$.
Because of $T\in\mathscr{V}$ we get that $S(x_k)=T(x)\in\frac{1}{2\lambda}A$.

Let us now consider the space $\prod_{n<k}E_n$ which is a Banach space with respect to the Minkows-ki functional of the set $\prod_{n\leq k}\frac{1}{\rho_n}B_n$.
$J: E_k\to\prod_{n<k}E_n$, $x\to(P_{nk}(x))_{n<k}$, is linear and injective and $J^{-1}(\prod_{n\leq k}\frac{1}{\rho_n}B_n)=\bigcap_{n\leq k}P_{nk}^{-1}(\frac{1}{\rho_n}B_n)$. Since F has the X-extension property, there exists a linear map $\check{S}:\prod_{n<k}E_n\to F$ such that $\check{S}\circ J=S$ such that $\check{S}(\prod_{n\leq k}\frac{1}{\rho_n}B_n)\subset A$.
For every $n\leq k$ let $J_n: E_n\to\prod_{l<k}E_l$ denote the natural inclusion. Then $T_n:=\check{S}\circ J_n$ belongs to $\rho_n\mathscr{W}(B_n,A)$ $(n\leq k)$ and $T=\sum_{n\leq k}T_n\circ P_n$ since $T((x_n)_{n\in\mathbb{N}})=S(x_k)=\check{S}((x_n)_{n\leq k})=\check{S}(\sum_{n<k}J_n(x_n))$
$=\sum_{n\leq k}(\check{S}\circ J_n\circ P_n)((x_l)_{l\in\mathbb{N}})=\sum_{n\leq k}(T_n\circ P_n)((x_l)_{l\in\mathbb{N}})$.
Thus $T\in\sum_{n\leq k}(\rho_n\mathscr{W}(B_n,A)\circ P_n)\subset\Psi(\mathscr{W})$.

1 would like to thank K.Floret for eliminating a superfluous sling from a former version of the above proof.

In contrast to (5.7) we. have the following proposition (recall that the Banach spaces $l^\infty(I)$ have the 1 extension property, see H.E.Lacey [22;p.89]).

(5.8) PROPOSITION. *Let E be a metrizable locally convex space containing a decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex zero-neighbourhoods such that*

(a) *the sets* $\frac{1}{n}U_n$ $(n\in\mathbb{N})$ *form a basis of* $\mathscr{U}_o(E)$;

(b) B & (E) $\exists U\in\mathscr{U}_o(E)$ $\forall n\in\mathbb{N}$ “ n $\not\subset B+U$.

Then the space $\mathscr{L}_b(E,l^\infty)$ *is not quasibarrelled.*

Consequently, if in addition E *is the projective limit* $\text{proj}_{\leftarrow n} E_n$ *of a projective sequence* $((E_n)_{n\in\mathbb{N}}, (P_{nm})_{m\geq n})$ *of normed spaces, then the canonical map* $\Psi : \lim_{\rightarrow} J_{mn}(\mathscr{L}_b(E_n,l^\infty)) \to \mathscr{L}_b(E,l^\infty)$ (*see* (5.2)) *is not open.*

Proof. Let A denote the closed unit ball in l^∞, and for every $n\in\mathbb{N}$ put $\mathscr{W}_n := \{T\in\mathscr{L}(E,l^\infty) : T(U_n) \subset A\}$. Then the set $\mathscr{W} := \overline{\bigcup_{n\in\mathbb{N}} \mathscr{W}_n}$ (where the closure is taken in $\mathscr{L}_b(E,l^\infty)$) is clearly a bornivorous barrel in $\mathscr{L}_b(E,l^\infty)$.

Assume that $\mathscr{W}$ is a zero-neighbourhood in $\mathscr{L}_b(E,l^\infty)$. Then there is $B = \Gamma B \in \mathscr{B}(E)$ such that $\mathscr{V} := \{T\in\mathscr{L}(E,l^\infty) : T(B) \subset A\}$ is contained in $\frac{1}{2}\mathscr{W}$. According to hypothesis (b) there exist $U = \Gamma U \in \mathscr{U}_o(E)$, a sequence $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} U_n$ and a sequence $(f_n)_{n\in\mathbb{N}}$ in $(B + \frac{1}{2}U)^\circ$ such that $f_n(x_n) > 1$ for every $n\in\mathbb{N}$.

$T : E \to l^\infty$, $x \mapsto (f_n(x))_{n\in\mathbb{N}}$, is linear and continuous because the sequence $(f_n)_{n\in\mathbb{N}}$ is equicontinuous. Moreover, $T(B) \subset A$ whence $T \in \mathscr{V} \subset \frac{1}{2}\mathscr{W}$.

The set $D := \{x_n : n\in\mathbb{N}\}$ is bounded in E, since $x_n \in U_n \subset U_k$ for

all $n \geq k$. Thus $\mathscr{U} := \{S \in \mathscr{L}(E, l^\infty) : S(D) \subset \frac{1}{2} A\}$ belongs to $\mathscr{U}_o(\mathscr{L}_b, l^\infty))$ and $T \in \frac{1}{2} \bigcup_{n \in \mathbb{N}} \mathscr{W}_n + \mathscr{U}$.
Consequently there is $n \in \mathbb{N}$ and $R \in \frac{1}{2} \mathscr{W}_n$ such that $T - R \in \mathscr{U}$ hence $T(x_n) \in R(x_n) + \frac{1}{2} A \subset A$, which is clearly a contradiction to $f_n(x_n) > 1$.

Next we will construct (a class of) Fréchet spaces E satisfying the hypotheses of (5.8). The construction of these Fréchet' spaces is rather dual to the construction of the LB-spaces of (4.7).

(5.9) Example.

Let $(X_n, r_n)_{n \in \mathbb{N}}$ and $(Y_n, s_n)_{n \in \mathbb{N}}$ be two sequences of Banach spaces such that for every $n \in \mathbb{N}$

(a) X_n is a linear subspace of Y_n and $r_n \geq s_n | X_n$;

(b) the set $\{x \in X_n : r_n(x) \leq 1\}$ is closed in (Y_n, s_n).

Moreover, just as in (4.7), let $(Z, \|.\|)$ be a normal Banach sequence space, i.e. $\mathbb{K}^{(\mathbb{N})} \subset Z \subset \mathbb{K}^{\mathbb{N}}$ algebraically, the inclusion $(Z, \|.\|) \to \mathbb{K}^{\mathbb{N}}$ is continuous, and whenever $(a_k)_{k \in \mathbb{N}} \in Z$, $(b_k)_{k \in \mathbb{N}} \in \mathbb{K}^n$ are such is continuous, and whenever $(a_k)_{k \in \mathbb{N}} \in Z$, $(b_k)_{k \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ are such that $|b_k| \leq |a_k|$ $(k \in \mathbb{N})$, then $(b_k)_{k \in \mathbb{N}} \in Z$ and $\|(b_k)_k\| \leq \|(a_k)_k\|$. For every $k \in \mathbb{N}$ there is $\rho_k > 0$ such that $\|(\delta_{kl} a)_{l \in \mathbb{N}}\| = \rho_k |a|$ for all $a \in K$ (cf. (4.7). Moreover, let condition (c) in (4.7) be satisfied.

Now, for all $n \in \mathbb{N}$, the linear space

$$E_n := \{(y_k)_{k \in \mathbb{N}} \in \prod_{k < n} X_k \times \prod_{k \geq n} Y_k : ((r_k(y_k))_{k < n}, (s_k(y_k))_{k \geq n}) \in Z\}$$

is a Banach space with respect to the norm

$$(y_k)_{k\in\mathbb{N}} \mapsto \|((r_k(y_k))_{k<n},(s_k(y_k))_{k\geq n})\|,$$

as has been proved in (4.7). Let

$$B_n := \{(y_k)_{k\in\mathbb{N}} \in E_n : \|((r_k(y_k))_{k<n},(s_k(y_k))_{k\geq n})\| \leq 1\}$$

denote the closed unit ball in E_n.

Then $E_{n+1} \subset E_n \subset \prod_{k\in\mathbb{N}} Y_k$, $B_{k+1} \subset B_k$, and the inclusions $E_{n+1} \hookrightarrow E_n \hookrightarrow \prod_{k\in\mathbb{N}}(Y_k,s_k)$ are continuous ($n\in\mathbb{N}$).

Let $E:=\bigcap_{n\in\mathbb{N}} E_n$ be equipped with the initial topology with respect to the inclusion maps $E \hookrightarrow E_n$ ($n\in\mathbb{N}$). Then E is topologically isomorphic to the projective limit $\text{proj}_{\leftarrow n} E_n$, hence a Fréchet space.

Let $U_n := B_n \cap E$ ($n\in\mathbb{N}$). Then the sets $\frac{1}{n}U_n$ ($n\in\mathbb{N}$) form a basis of $\mathcal{U}_0(E)$.

Let us suppose from now on that in addition the following condition is satisfied

(d) r_k generates a strictly stronger topology on X_k than $s_k|X_k$ for every $k\in\mathbb{N}$.

We will show that the sequence $(U_n)_{n\in\mathbb{N}}$ satisfies condition (b) in (5.8). For this purpose it obviously suffices to prove

$$U_n \not\subset m\, U_{n+1} + \frac{1}{2}\, U_1 \text{ for all } n,m\in\mathbb{N}.$$

Let $m,n\in\mathbb{N}$ be given, and let $A(r):= \{x\in X_n : r_n(x) \leq 1\}$, $A(s):= \{y\in Y_n : s_n(y) \leq 1\}$.

Assume that $A(s) \cap X_n \subset mA(r) + \frac{1}{2}A(s)$. Then for every $k\in\mathbb{N}$

we have that $A(s) \cap X_n \subset \sum_{0 \le j \le k} m \frac{1}{2^j} A(r) + \frac{1}{2^{k+1}} A(s)$, hence $A(s) \cap X_n \subset \bigcap_{k \in \mathbb{N}} (2mA(r) + \frac{1}{2^{k+1}} A(s)) = 2m \overline{A(r)}^{\,n} = 2m\, A(r)$, which is a contradiction to hypothesis (d). Consequently, there exists $x \in A(s) \cap X_n$ such that $x \notin m\, A(r) + \frac{1}{2} A(s)$. The element $(\delta_{kn} \frac{1}{\rho_n} x)_{k \in \mathbb{N}}$ belongs to E and to B_n, since $\|(\delta_{kn} \frac{1}{\rho_n} s_n(x))_{k \in \mathbb{N}}\| = s_n(x) \le 1$. Assume that $(\delta_{kn} \frac{1}{\rho_n} x)_{k \in \mathbb{N}} \in m\, U_{n+1} + \frac{1}{2} U_1$. Then - by the monotony of $\|.\|$ - there are $y \in X_n$, $s \in Y_n$ such that $\frac{1}{\rho_n} x = y + z$ and such that $(\rho_n r_n(y) =) \|(\delta_{kn} r_n(y))_{k \in N}\| \le m$ and $(\rho_n s_n(z) =) \|(\delta_{kn} s_n(z))_{k \in N}\| \le \frac{1}{2}$. Thus $Y. \in \frac{m}{\rho_n} A(r)$ and $z \in \frac{1}{2\rho_n} A(s)$ whence $x = \rho_n (y+z) \in m\, A(r) + \frac{1}{2} A(s)$ which is a contradiction to the choice of x.

Thus we have proved that $U_n \not\subset m\, U_{n+1} + \frac{1}{2} U_1$. Now, by (5.8), the space $\mathscr{L}_b(E, l^\infty)$ is not quasibarrelled.

Remarks.

(a) We just managed to realize the hypotheses of (5.8). In fact, instead of (5.8)(b), the Fréchet spaces E constructed in (5.9) satisfy the (formally) stronger condition

$$\exists\, U \in \mathscr{U}_0(E) \quad \forall\, B \in \mathscr{B}(E) \quad \forall\, n \in \mathbb{N} \quad U_n \not\subset U + B\,.$$

Corollary (5.15) will show that quasinormable Fréchet spaces E never satisfy the hypotheses of (5.8).

(b) For a nondistinguished Fréchet space E the space $\mathscr{L}_b(E, l^\infty)$ is clearly never quasibarrelled (as it contains E_b' as a complemented subspace). Theref ore the examples which we obtained with the help of (5.8) and (5.9) are only interesting if we make sure that we can get distinguished Fréchet spaces E with $\mathscr{L}_b(E,l^\infty)$ non-quasibarrelled in that way. - Taking (X_n,r_n), (Y_n,s_n) $(n\in\mathbb{N})$, and $(Z,\|.\|)$ reflexive, we obtain ref-lexive Banach spaces E_n $(n\in\mathbb{N})$ (we do not prove this statement, cf. also V.B.Moscatelli [23;Thm.1 and its proof]) and hence a reflexive Fréchet space E, which is in particular distinguished.

(c) For a quasibarrelled space E the following statements are equivalent:

(α) There exists a decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex zero-neighbourhoods in E such that (5.8),(a) and (b), are satisfied.

(β) There exists an increasing sequence $(B_n)_{n\in\mathbb{N}}$ of absolutely convex $\sigma(E',E)$-closed, $\beta(E',E)$-bounded subsets in $F:=E_b'$ such that (4.5).(a) and (b), are satisfied.

Proof. (α) ⟹ (β). Let $(U_n)_{n\in\mathbb{N}}$ according to (α) be given, and define $B_n:=U_n^\circ$ $(n\in\mathbb{N})$. Then the sequence $(B_n)_{n\in\mathbb{N}}$ satisfies (4.5)(a). Moreover, let $V=V^{\circ\circ}\in\mathscr{U}_o(E_b')$ be given. Then there is $U=\overline{\Gamma U}\in\mathscr{U}_o(E)$ such that $U_n\not\subset V^\circ+2U$ $(n\in\mathbb{N})$, whence $B_n=U_n^\circ\not\supset(V^\circ+U)^\circ$ and in particular $B_n\not\supset(V^\circ\cup U)^\circ=V\cap U^\circ$ $(n\in\mathbb{N})$.

(β) ⟹ (α). Let $(B_n)_{n\in\mathbb{N}}$ according to (β) be given, and define $U_n:=B_n^\circ$ $(n\in\mathbb{N})$. Then the sequence $(U_n)_{n\in\mathbb{N}}$ satisfies (5.8)(a). Moreover, let $B\in\mathscr{B}(E)$ be given. Then there is $A\in\mathscr{R}(E_b')$ such that $\frac{1}{2}B^\circ\cap A\not\subset B_n$ $(n\in\mathbb{N})$. We may assume that $A=A^{\circ\circ}$ (using again the quasibarrelledness of E). Then $U_n=B_n^\circ\not\subset(\frac{1}{2}B^\circ\cap A)^\circ$

($n\in\mathbb{N}$). Since $B + \frac{1}{2}A^\circ \subset (\frac{1}{2}B^\circ \cap A)^\circ$ we obtain that $U_n \not\subset B + \frac{1}{2}A^\circ$ ($n\in\mathbb{N}$).

Quite analogously one proves the following statement

(d) For a quasibarrelled space F the following are equivalent

(α) There exists an increasing sequence $(B_n)_{n\in\mathbb{N}}$ of absolutely convex closed bounded sets in F such that (4.5),(a) and (b), are satisfied.

(8) There exists a decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex zero-neighbourhoods in $E:=F'_b$ such that (5.8),(a) and (b) are satisfied.

(e) Let E be a nondistinguished metrizable locally convex space. Then E satisf ies the hypotheses of (5.8). (Confer however part (b) of this remark).

Proof. We have thut $\beta(E',E'')$ is strictly stronger than $\beta(E',E)$ and that E'_b is a DF-space. Consequently, by G.Köthe [20;p.398,(7)], there is $U \in \mathscr{U}_0(E)$, $U=U^{\circ\circ}$, such that $\beta(E',E'')|U^\circ \supsetneq \beta(E',E)|U^\circ$.

Let $(A_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in E'_b such that every A_n is absolutely convex and $\sigma(E',E)$-compact. There exists a sequence $(\rho_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $B^\circ \cap U^\circ \not\subset \sum_{n\in\mathbb{N}} \rho_n A_n$ for all $B \in \mathscr{B}(E)$. The sets $B_n := \sum_{m<n} \rho_m A_m$ ($n\in\mathbb{N}$) form an increasing sequence of $\sigma(E',E)$-closed, $\beta(E',E)$-bounded, absolutely convex sets in E'_b which obviously satisfy (4.5)(a) as well as a condition which implies (4.5)(b) (Cf. Remark (1) after (4.7)). Now we obtain the assertion with the help of (c).

Another method of combining the examples of section four with those of section five is based on the following simple fact.

(5.10) LEMMA. *Let* X,Y *be Hausdorff quasibarrelled spaces. Then the map*

$$\Theta : \mathscr{L}_b(X,Y'_b) \to \mathscr{L}_b(Y,X'_b), T \mapsto T^t|Y,$$

(where we consider Y as a subspace of Y'' in the usual way) *is a topolcgical isomorphism.*

Proof. If $T \in \mathscr{L}(X,Y'_b)$ then $T^t : (Y'_b)_{b'} \to X'_b$ is continuous, thus $T^t|Y$ belongs to $\mathscr{L}(Y,X'_b)$, and Θ is well defined.

Since the map $\mathscr{L}_b(Y,X'_b) \to \mathscr{L}_b(X,Y'_b)$, $S \mapsto S^t|X$ is the inverse map to Θ, it suffices - because of symmetry - to show that Θ is continous. But the continuity follows at once from

$$\underset{A\in\mathscr{B}(X)}{\forall}\ \underset{B\in\mathscr{B}(Y)}{\forall}\ \underset{T\in\mathscr{L}(X,Y'_b)}{\forall} (T(A) \subset B^\circ \Rightarrow T^t(B) \subset A^\circ).$$

As an application of (5.10) we would like to mention the following two examples.

(5.11) Examples.

(a) Let F be a reflexive DF-space such that $\mathscr{L}_b(l^1,F)$ is not quasibarrelled (cf. (4.7),(4.8)). Then $E:=F'_b$ is a reflexive Fréchet space such that $\mathscr{L}_b(E,l^\infty)$ is not quasibarrelled.

(b) For every $n\in\mathbb{N}$ let

$$a^{(n)}_{i,k} := \begin{cases} k & \text{whenever } (i,k) \in \mathbb{N}\times\mathbb{N},\ i \le n \\ 1 & \phantom{\text{whenever }} (i,k) \in \mathbb{N}\times\mathbb{N},\ i > n, \end{cases}$$

and let $1 \leq p \leq q < \infty$. For every $n \in \mathbb{N}$ let

$$E_n := \{(x_{i,k})_{(i,k)\in\mathbb{N}\times\mathbb{N}} \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}} : (a_{i,k}^{(n)} x_{i,k})_{(i,k)\in\mathbb{N}\times\mathbb{N}} \in l^p(\mathbb{N}\times\mathbb{N})\}$$

be provided with the norm

$$(x_{i,k})_{(i,k)\in\mathbb{N}\times\mathbb{N}} \mapsto \|(a_{i,k}^{(n)} x_{i,k})_{(i,k)\in\mathbb{N}\times\mathbb{N}}\|_p ,$$

where $\|\cdot\|_p$ denotes the usual norm on $l^p(\mathbb{N}\times\mathbb{N})$.
Then E_n is a Banach space, $E_{n+1} \subset E_n \subset \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ and the inclusions $E_{n+1} \hookrightarrow E_n$ are continuous ($n\in\mathbb{N}$). The projective limit $E := \operatorname{proj}_{\leftarrow n} E_n$ is an echelon space of order p.
We will show that $\mathscr{L}_b(E,l^q)$ is not quasibarrelled.

In fact, if p=1, then E is not distinguished (see G.Köthe [20;p.435]), whence E_b' and hence $\mathscr{L}_b(E,l^q)$ are not quasibarrelled. On the other hand, let $p\in(1,\infty)$. Then E is reflexive and its strong dual E_b' is equal to the LB-space F constructed in (4.8) with respect to $p' := \frac{p}{p-1}$ (see K.D.Bierstedt, R.G.Meise, W.H.Summers [5;2.8 Cor.]). In (4.8) we had proved that $\mathscr{L}_b(l^r,F)$ is not quasibarrelled whenever $1 \leq r \leq p'$. Because of $1 < p \leq q < \infty$ we have that $q' := \frac{q}{q-1}$ belongs to the interval $(1,p']$. By Lemma (5.10) the space $\mathscr{L}_b(E,l^q)$ is topologically isomorphic to $\mathscr{L}_b(l^{q'},F)$. Thus $\mathscr{L}_b(E,l^q)$ is not quasibarrelled.
Finally, if $q=\infty$, then by (5.10) the space $\mathscr{L}_b(E,l^\infty)$ is topologically isomorphic to $\mathscr{L}_b(l^1,F)$ and hence not quasibarrelled.

In order to prove a statement similar to that in (4.10) we need the following simple fact.

(5.12) *Remark.* Let F be a locally convex space and let I be a set. Let $l_I^\infty(F) := \{(x_\iota)_{\iota\in I} \in F^I : \{x_\iota : \iota \in I\} \in \mathscr{B}(F)\}$ be provided with the (locally convex) topology of uniform convergence on I. Then the sets $U^I \cap l_I^\infty(F)$ $(U \in \mathscr{U}_o(F))$ form a basis of $\mathscr{U}_o(l_I^\infty(F))$, and the sets B^I $(B \in \mathscr{B}(F))$ form a fundamental system of bounded sets in $l_I^\infty(F)$.

Furthermore let - as in (4.10) - $\mathbb{E}^1 := \mathbb{K}^{(I)}$ be provided with the norm $(a_\iota)_{\iota\in I} \mapsto \sum_{\iota\in I} |a_\iota|$. Then the map

$$\Theta : \mathscr{L}_b(\mathbb{E}^1, F) \to l_I^\infty(F),\ T \mapsto (T((\delta_{\iota\kappa})_{\kappa\in I}))_{\iota\in I},$$

is a topological isomorphism.

In fact, Θ is clearly linear and injective. Moreover, if $x = (x_\iota)_{\iota\in I} \in l_I^\infty(F)$, then $T: \mathbb{E}^1 \to F$, $(a_\iota)_{\iota\in I} \mapsto \sum_{\iota\in I} a_\iota x_\iota$ belongs to $\mathscr{L}(\mathbb{E}^1, F)$ and $\Theta(T) = x$.

Let A denote the closed unit ball in $\mathbb{E}^1$. Then $A = \Gamma\{(\delta_{\iota\kappa})_{\kappa\in I} : \iota \in I\}$ and consequently we have the equivalence

$$T(A) \subset U \Longleftrightarrow \Theta(T) \in U^I \cap l_I^\infty(F) \quad (U = \Gamma U \in \mathscr{U}_o(F),\ T \in \mathscr{L}(\mathbb{E}^1, F)).$$

This proves that Θ is a topological isomorphism.

Now, Proposition (4.10) immediately implies the following statement.

(5.13) PROPOSITION. *Let F be a DF-space and let I be a set. Then the space $l_I^\infty(F)$ is also a DF-space.*

(*But - according to (4.7), (4.8) - there exist reflexive LB-spaces F such that $l^\infty(F)$ $(:= l_{\mathbb{N}}^\infty(F))$ is not quasibarrelled though it is a DF-space by (5.13).*) - *We would like to mention* *that according to a result of A.Marquina and J.M.Sanz Serna* (Barrelledness Conditions on $c_o(E)$, Arch.Mat. 31(1978),589-596) *the space $c_o(F)$ is quasibarrelled whenever F is a quasibarrelled* DF-space. *This shows that the functors* $F \rightsquigarrow l^\infty(F)$ and $F \rightsquigarrow c_o(F) \subset l^\infty(F))$ *behave quite differently.*

On the other hand, Proposition (5.13) is in accordance with the following result of J.Schmets.
If F is a DF-space, then also $c_o(F)$ is a DF-space.
(More generally, J.Schmets proves: Let K be a compact Hausdorff space and F a DF-space, then $\mathscr{C}(K;F)$ is a DF-space.)

Next we will prove the following analogue to (4.10).

(5.14) PROPOSITION. *Let E be a metrizable locally convex space and let I be a set. Then $\mathscr{L}_b(E,l_I^\infty)$ is a DF-space.*
(Here we use the notation $l_I^\infty := l_I^\infty(\mathbb{K})$ is the sense of (5.12).)

Proof. Because of (5.13) it suffices to show that the map

$$\Theta : \mathscr{L}_b(E,l_I^\infty) \to l_I^\infty(E_b'),\ T \mapsto (P_\iota \circ T)_{\iota\in I},$$

where $P_\iota : l_I^\infty \to \mathbb{K}$ denotes the projection onto the ι'th coordinate ($\iota\in I$), is a topological isomorphism.
In fact, this latter statement is true for spaces E which are just quasibarrelled instead of being metrizable.

For every $T\in \mathscr{L}(E,l_I^\infty)$ the family $(P_\iota \circ T)_{\iota\in I}$ is bounded in E_b'. Consequently, Θ is well defined. Moreover, Θ is clearly linear and injective. Given $(f_\iota)_{\iota\in I} \in l_I^\infty(E_b')$, the map $T:E \to l_I^\infty$, $T(x):=(f_\iota(x))_{\iota\in I}$ ($x\in E$), is well defined (since $(f_\iota)_{\iota\in I}$ is a fortiori bounded in E_s') and linear. By the quasibarrelledness of E, the family $(f_\iota)_{\iota\in I}$ is equicontinuous, which implies the continuity of T. Since $\Theta(T)=(f_\iota)_{\iota\in I}$, we have proved the surjectivity of Θ. Finally, for every $B \in I(E)$ we have that

$$\Theta(\{T\in\mathscr{L}_b(E,l_I^\infty) : \sup_{x\in B}\|T(x)\|_\infty \leq 1\}) = (B^\circ)^I \cap l_I^\infty(E_b').$$

Consequently Θ is a homeomorphism.

From (5.14) and (5.3) we immediately obtain the following

(5.15) ***Corollary.*** Let E be a quasinormable Fréchet space and I a set. Then $\mathscr{L}_b(E,l_I^\infty)$ is a retractive LB-space, and for every projective seouence $((E_n)_{n\in\mathbb{N}}, (P_{nm})_{m>n})$ of Banach spaces such that E is the reduced projective limit $\operatorname{proj}_{\leftarrow n} E_n$, the canonical map

$$\Psi : \operatorname{ind}_{n\to} \mathscr{L}_b(E_n,l_I^\infty) \to \mathscr{L}_b(E,l_I^\infty)$$

is a topological isomorphism.

(5.16) ***Remarks.***

(a) Let E be a Hausdorff locally convex space and $G \subset E$ a dense linear subspace such that on $E'=G'$ the topologies $\beta(E',E)$ and $\beta(E',G)$ coincide. Then for every Hausdorff locally convex space F the linear injection

$$\mathscr{L}_b(E,F) \to \mathscr{L}_b(G,F),\ T \mapsto T|G,$$

is a topological isomorphism onto its range, since the hypothesis about the strong topologies is equivalent to the following condition

$$\forall B\in\mathscr{B}(E)\quad \exists A\in\mathscr{B}(G)\quad B \subset \bar{A}^E$$

Consequently, if F is complete, then $\mathscr{L}_b(E,F)$ and $\mathscr{L}_b(G,F)$ are topologically isomorphic.

Now, let E be a metrizable locally convex space, and let G be a dense linear subspace such that G is distinguished. Then $\beta(E',E'')'$ and $\beta(E',G)$ are LB-space topologies and $\beta(E',E'') \supset \beta(E',E) \supset \beta(E',G)$ whence $\beta(E',E'') = \beta(E',E) = \beta(E',G)$ (and also E is distinguished).

Combining these statements with (5.15) and the fact that a quasinormable metrizable locally convex space is in particular distinguished (see A.Grothendieck [14;p.108,Prop.14]), we obtain that for every quasinormable metrizable locally convex space E and for every set I the space $\mathscr{L}_b(E,l_I^\infty)$ is a bornological DF-space.

(b) It should be mentioned that (5.15) can be derived directly from (4.11). In fact, if E is a quasinormable Fréchet space, then E_b' has a representation as a retractive LB-space $\operatorname{ind}_{n\to} F_n$, whence $\mathscr{L}_b(l^1(I),E_b')$ is a retractive LB-space by (4.12), and by (5. 10), the spaces $\mathscr{L}_b(l^1(I),E_b')$ and $\mathscr{L}_b(E,l_I^\infty)$ are topologically isomorphic.

(We apologize for the disharmony of the notations $l^1(I)$ and l_I^∞.)

(c) The hypotheses of (5.7) and (5.15) are incomparable. In fact, not every quasinormable Fréchet space E satisfies the conditions of (5.5) (cf. (5.6)(b)(β)). On the other hand, not every Banach space F with the λ extension property for some $\lambda \geq 1$ is of type l_I^∞. Indeed, the class of all Banach spaces with the 1 extension property consists precisely of all spaces $\mathscr{C}(K)$ where K is an extremally disconnected compact Hausdorff space (see M.M.Day [6;p.123, Thm.3]). There exists an extremally disconnected compact Hausdorff space K which is not homeomorphic to the Stone-Čech compactification of a discrete space (see Z.Semadeni, Banach spaces of continuous functions, Polish Scientific Publishers, Warsaw 1971,p.432 and p.284; I would like to thank E.Behrends for pointing out this reference to me). On account of the Theorem of Banach-Stone (M.M.Day [6;p.115]), $\mathscr{C}(K)$ is not of type l^{∞}_{I}.

(d) In connection with (4.12),(5.15), and (5.7) quite naturally the question arises what an analogue to (5.7) for strict LB-spaces would look like. Let us say for the moment that a Banach space has the λ lifting property for some $\lambda \geq 1$ if for every pair (Y,Z) of Banach spaces with $Y \subset Z$ and every $T \in \mathscr{L}(X,Z/Y)$ there exists $S \in \mathscr{L}(X,Z)$ such that $S \circ Q = T$ ($Q: Z \to Z/Y$ denoting the quotient map and such that $\|S\| \leq \lambda \|T\|$. Using similar methods as in the proof of (5.7) one can show:

If E is a Banach space with the λ lifting property for some $\lambda \geq 1$ and $F = \operatorname{ind}_{n\to} F_n$ is a strict LB-space, then the canonical map

$$\Phi : \operatorname{ind}_{n\to} \mathscr{L}_b(E,F_n) \to \mathscr{L}_b(E,F)$$

is a topological isomorphism and $\mathscr{L}_b(E,F)$ is a strict LB-space.

However, according to G.Köthe [19;p.188,(8)], every Banach space with the λ lifting property for some $\lambda \geq 1$ is topologically isomorphic to $l^1(I)$ for a suitable set I. Thus the above statement is just a special case of (4.11).

The following remark will show that Proposition (5.8) gives rise to a construction of nondistinguished Fréchet spaces.

(5.17) ***Remark.*** Let E be a locally convex space and let $\mathbb{P}$ denote the set of all continuous seminorms on E. According to A.Pietsch [24] we provide the space

$$l^1\{E\}:=\{(x_n)_{n\in\mathbb{N}}\in E^{\mathbb{N}}: \sum_{n\in\mathbb{N}} p(x_n)<\infty \text{ for every } p\in\mathbb{P}\}$$

with the locally convex topology generated by the seminorms

$$q_p : l^1\{E\}\to\mathbb{R},\ (x_n)_{n\in\mathbb{N}}\mapsto \sum_{n\in\mathbb{N}} p(x_n), \qquad (p\in\mathbb{P}).$$

If E is metrizable, then clearly also $l^1\{E\}$ is metrizable; if E is complete, then also $l^1\{E\}$ is complete (A.Pietsch [24 ;p. 28,1.4]) . Moreover, we have the following statement about the strong dual of $l^1\{E\}$.

PROPOSITION. *Let* E *be a quasibarrelled space such that* E ***has property*** (B) ***in the sense of*** *A.Pietsch* [24;p.30], (By loc.cit. every DF-space and every metrizable locally convex space have property (B).)

Then $(l^1\{E\})'_b$ ***is topologically isomorphic to*** $l^\infty(E'_b)$.

Proof. For every $n\in\mathbb{N}$ the map $J_n : E\to l^1\{E\}$, $x\mapsto(\delta_{mn}x)_{m\in\mathbb{N}}$, is linear and continuous. Moreover, for every $f\in(l^1\{E\})'$, the sequence $(f\circ J_n)_{n\in\mathbb{N}}\in(E'_b)^{\mathbb{N}}$ is equicontinuous and thus belongs to $l^\infty(E'_b)$. Consequently the map $\Phi:(l^1\{E\})'_b\to l^\infty(E'_b)$,

$f \mapsto (f \circ J_n)_{n\in\mathbb{N}}$, is well defined and linear. Φ is injective, as $E^{(\mathbb{N})}$ is dense in $l^1\{E\}$. Let $(f_n)_{n\in\mathbb{N}} \in l^\infty(E'_b)$ be given. As E is quasibarrelled, the sequence $(f_n)_{n\in\mathbb{N}}$ is equicontinuous. Therefore the map $f : l^1\{E\} \to \mathbb{K}$, $(x_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} f_n(x_n)$, is well defined, linear, and continuous. Moreover, $\Phi(f) = (f_n)_{n\in\mathbb{N}}$. This proves that Φ is surjective.

Let $B = \Gamma B \in \mathscr{B}(E)$ be given, and let p_B denote its Minkowski functional. Then $\mathbb{B} := \{(x_n)_{n\in\mathbb{N}} \in l^1\{E\} \cap [B]^{\mathbb{N}} : \sum_{n\in\mathbb{N}} p_B(x_n) \leq 1\}$ is obviously bounded in $l^1(E)$ and $\Phi(\mathbb{B}^\circ) \subset (B^\circ)^{\mathbb{N}}$. This proves the continuity of Φ.

We finally prove that Φ is open. Let $\mathbb{B} \subset l^1\{E\}$ be bounded. Since E has property (B) of Pietsch, there is $B = \Gamma B \in \mathscr{B}(E)$ such that

$$\mathbb{B} \subset \{(x_n)_{n\in\mathbb{N}} \in l^1\{E\} \cap [B]^{\mathbb{N}} : \sum_{n\in\mathbb{N}} p_B(x_n) \leq 1\}.$$

We show that $\Phi(\mathbb{B}^\circ) \supset (B^\circ)^{\mathbb{N}} \cap l^\infty(E'_b)$ (which will finish the proof). Let $(f_n)_{n\in\mathbb{N}} \in (B^\circ)^{\mathbb{N}} \cap l^\infty(E'_b)$ be given. Then $f : l^1\{E\} \to \mathbb{K}$, $f((x_n)_{n\in\mathbb{N}}) := \sum_{n\in\mathbb{N}} f_n(x_n)$, belongs to $(l^1\{E\})'$ and to $\mathbb{B}^\circ$. In fact, if $(x_n)_{n\in\mathbb{N}} \in \mathbb{B}$, then $|f((x_n)_{n\in\mathbb{N}})| \leq \sum_{n\in\mathbb{N}} |f_n(x_n)| \leq$ $\leq \sum_{n\in\mathbb{N}} p_B(x_n) \leq 1$, whence $f \in \mathbb{B}^\circ$. Clearly, $\Phi(f) = (f_n)_{n\in\mathbb{N}}$.

Now let E be a Fréchet space satisfying the hypotheses of Proposition (5.8). Then $\mathscr{L}_b(E, l^\infty)$ is a DF-space by (5.14) but not quasibarrelled by (5.8). As it was shown in the proof of (5.14), the spaces $\mathscr{L}_b(E, l^\infty)$ and $l^\infty(E'_b)$ are topologically isomorphic. By the above proposition, the space $l^{\infty}(E_b')$ is topologically isomorphic to the strong dual of the Fréchet space $l^1\{E\}$. Consequently, the Fréchet space $l^1\{E\}$ is not distinguished. ▪ As was shown in (5.9), the space E may be chosen to be reflexive and separable. Thus, for a separable and reflexive Fréchet space E the space $l^1\{E\}$ may be nondistinguished.

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Fachbereich IV - Mathematik
der Universität Trier
Postfact 3825
D-5500 TRIER
FEDERAL REPUBLIC OF GERMANY