Note di Matematica Vol.VI, 273-284(1986)

SHIFTED MOMENTS OF GAUSSIAN MEASURES IN HILBERT SPACES [1)]

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If H is a Hilbert space, then a measure μ is completely described by the function

$$y \rightarrow \int_H \|x+y\|^p d\mu(x) , \qquad y\in H,$$

provided that $p\neq 2,4,6,\ldots$. *We prove that for Gaussian measures on a Hilbert space* H *this is valid for all* $p>0$ *with* $p\neq 2$.

1. NOTATION AND DEFINITIONS

Let $[E,\|.\|]$ be a Banach space. A number $p>0$ is said to be E-regular or regular whenever the following is valid:
If μ and ν are two Radon measures on E with

$$\int_E \|x\|^p d\mu(x) < \infty \text{ as well as } \int_E \|x\|^p d\nu(x) < \infty ,$$

then these two measures coincide iff

$$(+) \qquad \int_E \|x+y\|^p d\mu(x) = \int_E \|x+y\|^p d\nu(x) \qquad \text{for all } y\in E.$$

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1) Part of this paper has been written during the author's stay in Lecce. He is grateful for the hospitality during his visit at the University of Lecce.

A general characterization of E-regular numbers is not known. Only is some concrete spaces as l_q^n, $C(K)$, L_q or $C_o(\Omega)$, Ω locally compact, the set of regular numbers is completely described (cf. [3],[4] and [6]). The first result in this direction goes back to W.Rudin ([10]) (a somewhat weaker result is due to A.I.Plotkin ([9])) who proved that $p>0$ is $\mathbb{C}$-regular, $\mathbb{C}$ complex plane, iff p is not an even integer. The same is valid in the case of arbitrary Hilbert spaces (cf. [11],[5],[7]).

A Radon probability measure μ on a Banach space E is said to be Gaussian whenever $a(\mu)$ is Gaussian for all $a \in E'$. Here E' denotes the topological dual space of E and $a(\mu)$ is the image measure of μ with respect to $a \in E'$.

The following properties of Gaussian measures are well-known:

(A) If μ is Gaussian, then for all $p>0$ we have

$$\int_E \|x\|^p d\mu(x) < \infty \qquad \text{(cf. [2])}.$$

(B) Given a Gaussian measure μ there exists a unique element $x_o \in E$ such that $\mu_o := \mu * \delta_{x_o}$ $(\mu_o(B) = \mu(B - x_o))$ is symmetric, i.e. $\mu_o(B) = \mu_o(-B)$ (cf. [1]).

(C) A symmetric Gaussian measure μ on E is completely described by its weak q-th moments

$$\int_E |\langle x, a\rangle|^q d\mu(x), \qquad a \in E',$$

for some fixed $q>0$. This is an easy consequence of

$$\{\int_E |\langle x,a\rangle|^q d\mu(x)\}^{1/q} = c_q \langle R_\mu a, a\rangle^{1/2}$$

where

$$R_\mu a := \int_E \langle x,a\rangle x \, d\mu(x)$$

denotes the covariance operator of μ (R_μ is an operator from E' into E) and $c_q > 0$ is a fixed constant only depending on q (cf. [12] and [8]).

Let us say that a real number $p>0$ is Gauss-regular whenever two Gaussian measures μ and ν on E coincide iff (+) holds for all $y \in E$. Of course, each regular number is Gauss-regular as well, but as we shall prove below in general the latter set is strictly larger

2. BASIC RESULTS

We shall prove now the announced result, first in the case of symmetric measures.

THEOREM 1. *Let* μ *and* ν *be two Gaussian symmetric measures on a Hilbert space* H. *If* $p>0$ *and* $p \neq 2$, *then*

$$\int_H \|x+y\|^p d\mu(x) = \int_H \|x+y\|^p d\nu(x), \qquad y \in H,$$

yields $\mu = \nu$.

Proof. Since the only non-H-regular numbers are the even integers (cf. [5] and [7]) we only have to prove the theorem in the case $p=2k$, $k=2,3,4,\dots$.

To do so, choose two symmetric Gaussian measures μ and ν, satisfying

$$\int_H \|x+y\|^{2k}\, d\mu(x) = \int_H \|x+y\|^{2k} d\nu(x)$$

for all $y\in H$. Fix $y\in H$ for a moment and define a function P on the real line by

$$P(\alpha) = \int_H \|x+\alpha y\|^{2k} d(\mu-\nu)(x), \qquad \alpha \in \mathbb{R}.$$

By assumption we have $P(\alpha) = 0$ for all $\alpha \in \mathbb{R}$.

On the other hand, since

$$P(\alpha) = \int_H (\|x\|^2 + 2\alpha\langle x,y\rangle + \alpha^2\|y\|)^k d(\mu-\nu)(x)$$

$$= \sum_{\substack{i+2j+\ell=k\\ i,j,\ell\geq 0}} c_{i,j,\ell}\,\alpha^{2j+2\ell}\|y\|^{2\ell} \int_H \langle x,y\rangle^{2j}\|x\|^{2i} d(\mu-\nu)(x)$$

where

$$c_{i,j,\ell} = \frac{2^{2j}k!}{i!(2j)!\ell!} \quad ,$$

P is a polynomial of degree $2k$. Observe that all terms with $\langle x,y\rangle^j$, j odd, vanish by the symmetry of μ and ν.

Thus, all coefficients of P are zero, especially, the coefficient of α^{2k-2}. It is easy to see that this coefficient equals

$$c_{0,1,k-2}\|y\|^{2k-4} \int_H \langle x,y\rangle^2\, d(\mu-\nu)(x)$$

$$+\ c_{1,0,k-1}\|y\|^{2k-2} \int_H \|x\|^2 d(\mu-\nu)(x).$$

Here we use the assumption $k \geq 2$.

Hence it follows

$$(++) \qquad \int_H [c_{0,1,k-2} \langle x,y\rangle^2 + c_{1,0,k-1} \|y\|^2 \|x\|^2] d(\mu-\nu)(x) = 0$$

for all $y \in H$.

Let us first assume $\dim H = \infty$. Then we find a sequence $(y_j) \subseteq H$ tending weakly to zero yet $\|y_j\|=1$. Putting the y_j's into (++) from Lebesgue's theorem we derive

$$\int_H \|x\|^2 d(\mu-\nu)(x) = 0$$

and, consequently, by (++) we have

$$\int_H \langle x,y\rangle^2 d\mu(x) = \int_H \langle x,y\rangle^2 d\nu(x)$$

for all $y \in H$. Property (C) ends the proof in this case, i.e. it follows $\mu = \nu$ as asserted. To complete the proof assume $\dim H < \infty$.

Of course, it suffices to investigate the case $H = \mathbb{R}^n$ endowed with the Euclidean norm. Let σ_{n-1} be the normalized Haar measure on the sphere S^{n-1} of $\mathbb{R}^n$ and integrate (++) over $y \in S^{n-1}$ with respect to σ_{n-1}. Then this yields

$$\left[\frac{c_{0,1,k-2}}{n} + c_{1,0,k-1}\right] \int_H \|x\|^2 d(\mu-\nu)(x) = 0,$$

and we can proceed as in the proof for infinite dimensional Hilbert spaces. This ends the proof of Theorem 1.

Our next aim is to prove Th.1 in the case of arbitrary Gaussian measures.

THEOREM 2. *Two Gaussian measures* μ *and* ν *on a Hilbert space* H *coincide iff*

$$\int_H \|x+y\|^p d\mu(x) = \int_H \|x+y\|^p d\nu(x), \qquad y\in H,$$

for some fixed $p>0$ *with* $p\neq 2$.

Proof. Let μ and ν be as in the statement of the theorem. Then there are elements $x_o, y_o \in H$ for which $\mu_o := \mu * \delta_{x_o}$ as well as $\nu_o := \nu * \delta_{y_o}$ are Gaussian symmetric.

By assumption we have

$$\int_H \|x+y-x_o\|^p d\mu_o(x) = \int_H \|x+y-y_o\|^p d\nu_o(x)$$

for all $y\in H$. Setting $z_o := x_o - y_o$ this implies

$$\int_H \|x+y\|^p d\mu_o(x) = \int_H \|x+y+z_o\|^p d\nu_o(x), \qquad y\in H.$$

Using the symmetry of μ_o and ν_o, respectively, it follows

$$\int_H \|x+y+2z_o\|^p d\nu_o(x) = \int_H \|x+y+z_o\|^p d\mu_o(x) = \int_H \|x-y-z_o\|^p d\mu_o(x)$$

$$= \int_H \|x-y\|^p d\nu_o(x) = \int_H \|x+y\|^p d\nu_o(x),$$

which clearly yields

$$\int_H \|x\|^p d\nu_o(x) = \int_H \|x+2nz_o\|^p d\nu_o(x)$$

for all natural numbers n. Dividing both sides of the last equation by n^p and taking the limit $n \to \infty$ we obtain $z_o=0$, i.e. $x_o=y_o$, from which we easily derive

$$\int_H \|x+y\|^p d\mu_o(x) = \int_H \|x+y\|^p d\nu_o(x)$$

for all $y \in H$. Hence, we are in the situation of Th.1 and get $\mu_o = \nu_o$, i.e. $\mu = \nu$ ending the proof.

REMARKS. (1) The preceding theorems are no longer valid for p=2 and $\dim(H) \geq 2$. Observe that two symmetric measures on a Hilbert space satisfy (+) with p=2 iff their second absolute moments coincide. But it is easy to construct two different Gaussian measures possessing the same second moments provided that the dimension of the underlying space is larger than 2.

(2) A careful inspection of the proof of Th.2 shows that we did not use the fact that the measures are defined on a Hilbert space. In other words, if E is an arbitrary Banach space, then a number $p > 0$ is Gauss-regular w.r.t. E iff (+) implies $\mu=\nu$ for all Gaussian *symmetric* measures μ and ν on E. Consequently, in order to decide whether a real number is Gaussian-regular one only has to investigate symmetric measures.

(3) Very little seems to be known about Gauss-regular numbers in arbitrary Banach spaces. C.Borell ([1]) proved the following general fact: if two Gaussian measures μ and ν on a

Banach space satify

$$\int_E \|x+y\|^{2k} d\mu(x) = \int_E \|x+y\|^{2k} d\nu(x)$$

for all $y \in E$ and *all* natural numbers k, then $\mu = \nu$.

(4) Th.2 suggests that the number 2 is probably the only non-Gauss-regular number in arbitrary Banach spaces. We shall disprove this now.

PROPOSITION 3. Let q be an even integ[illegible] Then there are different Gaussian measures μ and ν on R^2 with

$$\int_{R^2} \|x+y\|_q^q d\mu(x) = \int_{R^2} \|x+y\|_q^q d\nu(x)$$

for all $y \in R^2$. Here as usual $\|x\|_q := (|\xi_1|^q + |\xi_2|^q)^{1/q}$, $x=(\xi_1,\xi_2) \in R^2$. In other words, q is not a Gauss-regular number w.r.t. $[R^2, \|.\|_q]$.

Proof. Setting $e_1:=(1,0)$ and $e_2:=(0,1)$ we have

$$\|x+y\|_q^q = |\langle x+y,e_1\rangle|^q + |\langle x+y,e_2\rangle|^q,$$

and, if q=2k for some natural number k, then this coincides with

$$\sum_{j=0}^{2k} \binom{2k}{j} [\langle x,e_1\rangle^j \langle y,e_1\rangle^{2k-j} + \langle x,e_2\rangle^j \langle y,e_2\rangle^{2k-j}].$$

Integrating this expression with respect to a symmetric measure all terms with j odd vanish. Consequently, two symmetric measures μ and ν on R^2 satisfy

$$\int_{R^2} \|x+y\|_q^q d(\mu-\nu)(x) = 0$$

iff

$$\sum_{j=0}^{k} \binom{2k}{2j} \langle y,e_i\rangle^{2k-2j} \int_{R^2} \langle x,e_i\rangle^{2j} d(\mu-\nu)(x) = 0, \quad i=1,2,$$

for all $y \in R^2$.

Let μ and ν be Gaussian. Then for some constants $c_j > 0$

$$\int_{R^2} \langle x,e_i\rangle^{2j} d(\mu-\nu)(x) = c_j \{ (\int_{R^2} \langle x,e_i\rangle^2 d\mu(x))^j - (\int_{R^2} \langle x,e_i\rangle^2 d\nu(x))^j \}$$

where $i=1,2$ and $j=1,2,\ldots,k$ (use property (C) stated above). Hence, if two Gaussian symmetric measures μ and ν satisfy

$$\int_{R^2} \langle x,e_i\rangle^2 d\mu(x) = \int_{R^2} \langle x,e_i\rangle^2 d\nu(x), \qquad i=1,2,$$

then necessarily

$$\int_{R^2} \|x+y\|_q^q d(\mu-\nu)(x) = 0 .$$

To end the proof take two different Gaussian symmetric measures on R^2 possessing the same diagonal elements in their covariance matrices. These measures have the desired properties.

The following general problem remains open:

PROBLEM. Describe the Gauss-regular numbers of a given Banach space. Which are those numbers in L_q, $1 \le q \le \infty$, $q \neq 2$?

REMARK. There exist Banach spaces E where each positive real number is E-regular (C(K) with K without isolated points and $C_0(\Omega)$, Ω locally compact but not compact, are such examples). Of course, in this case all positive real numbers are Gauss-regular as well.

Added in proof

Recently M. Lewandowski (Technical University of Wroclaw, Poland) solved the problem of our paper in the case $q>2$. He proved that q is the only non-Gauss-regular number in L_q, $q\geq 2$. Recall that the non-regular numbers are q,2q,3q,... (cf. [6]). The characterization of Gauss-regular numbers in L_q, $1 \leq q<2$, remains open.

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