DIRICHLET PROBLEM FOR THE LAPLACE OPERATOR IN A RECTANGLE AND IN A STRIP

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Summary. Bounds for the solution of the Dirichlet problem for the Laplace operator in a rectangle and in a strip are given by means of the solution of a "symmetrized" problem.

INTRODUCTION. Many authors have considered the effect of Schwartz symmetrization on elliptic (see, e.g., [10], [9], [11]) and parabolic (see [3], [2]) problems.

The principal aim of these papers is to obtain some optimal bounds for the solutions of these problems.

A typical case in this setting is the following (see [9]).

Consider the problem

\[
\begin{align*}
- \Delta u &= f \quad \text{in } G \\
  u &= 0 \quad \text{on } \partial G
\end{align*}
\]

and look for the

\[
\sup_{p,q \text{ suitable}} \frac{\|u\|_p}{\|f\|_q}
\]

where the supremum is taken when f ranges through the functions with a fixed rearrangement and G ranges through the domains of \(\mathbb{R}^n\) with fixed measure.

This least upper bound is attained for the solution of the problem

\[
\begin{align*}
- \Delta w &= f^# \quad \text{in } G^# \\
  w &= 0 \quad \text{on } \partial G^#
\end{align*}
\]
where $f^#$ is the spherically symmetric rearrangement of $f$ in the sense of Hardy-Littlewood and $G^#$ is the ball centered at the origin with same measure of $G$.

In fact, this is a useful point of view because now we deal with majorization formulas for symmetric and then simpler problems. Moreover it is well-known that symmetrization results are of particularly relevant interest in many fields of Mathematical Physics (see, e.g., the classical book of Polya-Szego [12]).

In this paper, following the previous point of view, we study the effect of a Steiner symmetrization on the following problems. Consider the problem

\[
\begin{aligned}
\begin{cases}
-\Delta u(x_1,x_2) = f(x_1,x_2) & \text{in } G \\
u(x_1,x_2) = 0 & \text{on } \partial G,
\end{cases}
\end{aligned}
\]

(0.1)

where $G = (-a,a) \times (0,b)$ with $a, b>0$ and $\Delta$ is the Laplacian operator. Our aim is to give some bounds for $u(x_1,x_2)$ by using the solution of a "symmetrized" problem of the type

\[
\begin{aligned}
\begin{cases}
-\Delta U(x_1,x_2) = f^#(x_1,x_2) & \text{in } G \\
U(x_1,x_2) = 0 & \text{on } \partial G,
\end{cases}
\end{aligned}
\]

(0.1)'

where, for any fixed $x_2 \in [0,b]$, $f^#(\cdot,x_2)$ is the symmetrically decreasing rearrangement of $f(\cdot,x_2)$ as defined by Hardy and Littlewood.\(^{(1)}\)

\(^{(1)}\) For $x_2 \in [0,b]$, $f^#(x_1,x_2) = f^*(C_1|x_1|,x_2)$ where $f^*(t,x_2)$ ($t\geq 0$) denotes the decreasing rearrangement of $f(\cdot,x_2)$ in $[0,\infty)$, $C_1=2$. For more details, see [3] and [9].
We suppose \( f \) and \( f^\# \) sufficiently smooth so that there is existence and uniqueness for the solutions of problems (0.1), (0.1)# in \( C^2(G) \cap C^0(\bar{G}) \).

In section 1 we will give, for any \( x_2 \in [0, b] \), a bound for the \( L^1 \)-norm of \( u(\cdot, x_2) \) in terms of the \( L^1 \)-norm of \( U(\cdot, x_2) \) and then we will obtain also a bound of the \( L^1 \)-norm of \( u \) in terms of the \( L^1 \)-norm of \( U \) in all \( G \).

Consider then the problem

\[
\begin{align*}
- \Delta u(x_1, x_2) &= f(x_1, x_2) \quad \text{in } G \\
u(x_1, x_2) &= 0 \quad \text{on } \partial G,
\end{align*}
\]

(0.2)

where \( G = (-\infty, +\infty) \times (0, b) \) with \( b > 0 \), and the "symmetrized" problem

\[
\begin{align*}
- \Delta U(x_1, x_2) &= f^\#(x_1, x_2) \quad \text{in } G \\
U(x_1, x_2) &= 0 \quad \text{on } \partial G,
\end{align*}
\]

(0.2)\#

with \( f \) and then \( f^\# \) belonging to \( L^2(G) \).

It is well known that there exist unique solutions to (0.2) and (0.2)# in \( W^{1, 2}_0(G) \cap W^{2, 2}_{\text{loc}}(G) \); see [5].

In section 2 we will obtain for a strip a result which is similar to that obtained for a rectangle; moreover we will give, for any \( x_2 \in [0, b] \), a bound for \( \sup_{-\infty < x_1 < +\infty} |u(x_1, x_2)| \) in terms of \( \sup_{-\infty < x_1 < +\infty} U(x_1, x_2) \) and also a bound for \( \sup_{\bar{G}} |u(x_1, x_2)| \) in terms of \( \sup_{\bar{G}} U(x_1, x_2) \).

To obtain these results we will use a technique developed by
C. Bandle in her treatment of parabolic operators (see [2]). In the same framework, we can quote the work of C. Borell, see [4], where a symmetrization like that of Steiner is used.

SECTION 1.

Consider the boundary problem

$$\begin{cases}
-\Delta u(x_1,x_2) = f(x_1,x_2) & \text{in } G \\
u(x_1,x_2) = 0 & \text{on } \partial G
\end{cases}$$

(1.1)

where $G = (-a,a) \times (0,b)$, $a,b>0$.

We suppose $f$ smooth enough to guarantee existence and uniqueness of the solution of (1.1) in $C^2(G) \cap C^0(\overline{G})$ (see [6], th. 4.3, pag. 55 and [7], th. 3.1, pag. 328).

Moreover we suppose $f$ non-negative and so, by the maximum principle, $u$ is positive in $G$. The assumption $f(x_1,x_2) \geq 0$ is not restrictive for our aims, since, for an arbitrary $f$, the modulus of the solution relative to $f$ is less than, or equal to, the solution relative to $|f|$.

Put

$$H(s,x_2) = \int_{-s}^{s} u(x_1,x_2)dx_1$$

(1.2)

with $s \in [0,a]$ and $x_2 \in [0,b]$.

**Lemma 1.1.** The following differential inequality holds:

$$\frac{\partial^2 H(s,x_2)}{\partial s^2} + \frac{\partial^2 H(s,x_2)}{\partial x_2^2} + \int_{0}^{2s} f^*(s',x_2)ds' \geq 0,$$

(1.3)
Dirichlet problem for the Laplace ...

in \((0,a) \times (0,b)\).

Moreover we have:

\[
\begin{cases}
H(0,x_2) = 0, \\
H(s,0) = H(s,b) = 0, \\
\frac{\partial H}{\partial s}(a,x_2) = 0,
\end{cases}
\]

with \(s \in [0,a]\), \(x_2 \in [0,b]\).

**Proof.** Fix \(x_2 \in (0,b)\); we obtain from (1.1) by integration

\[(1.5) \quad -\int_{-s}^{s} \frac{\partial^2 u}{\partial x_1^2} dx_1 - \int_{-s}^{s} \frac{\partial^2 u}{\partial x_2^2} dx_1 = \int_{-s}^{s} f dx_1,
\]

for every \(s \in (0,a)\). We obtain easily:

\[
\int_{-s}^{s} \frac{\partial^2 u}{\partial x_1^2} dx_1 = -\frac{\partial^2 H}{\partial s^2}(s,x_2)
\]

and then, observing that from Hardy's inequality \((2)\)

\[
\int_{-s}^{s} f dx_1 \leq \int_{0}^{2s} f^*(s',x_2) ds',
\]

from (1.5) we obtain:

\[
\frac{\partial^2 H}{\partial s^2}(s,x_2) + \frac{\partial^2 H}{\partial x_2^2}(s,x_2) + \int_{0}^{2s} f^*(s',x_2) ds' \geq 0,
\]

in \((0,a) \times (0,b)\), that is (1.3).

\(^{(2)}\) The Hardy's inequality is (see [9])

\[
\int_{-r}^{r} fg dx \leq \int_{0}^{2r} f^* g^* ds.
\]
The equalities (1.4) are easily obtained.

Consider now the "symmetrized" problem

\[
\begin{align*}
\begin{cases}
-\Delta U(x_1, x_2) &= f^\#(x_1, x_2) \quad \text{in } G \\
U(x_1, x_2) &= 0 \quad \text{on } \partial G.
\end{cases}
\end{align*}
\]

(1.1)\

We suppose \( f^\# \) sufficiently smooth so that problem (1.1) has a unique solution \( U \in C^2(G) \cap C^0(\bar{G}) \). For example, if \( f > 0 \) in \( G \), \( f = 0 \) on \( \partial G \) and \( f \in C^0_{\text{loc}}(G) \), then also \( f^\# \in C^0_{\text{loc}}(G) \) (see [2], proposition 1.2) and problems (1.1) and (1.1) have unique solutions \( u, U \) respectively in \( C^2(G) \cap C^0(\bar{G}) \).

The function \( f^\# \) is non-negative in \( G \), and so \( U \) is positive in \( G \).

Put

\[
(1.6) \quad \tilde{H}(s, x_2) = \int_{-s}^{s} U(x_1, x_2) \, dx_1
\]

with \( s \in [0, a] \), and \( x_2 \in [0, b] \).

**Lemma 1.2.** The following equality holds:

\[
(1.7) \quad \frac{\partial^2 \tilde{H}}{\partial s^2}(s, x_2) + \frac{\partial^2 \tilde{H}}{\partial x_2^2}(s, x_2) + \int_{0}^{2s} f^\#(s', x_2) \, ds' = 0,
\]

in \( (0, a) \times (0, b) \).

Moreover we have:
\[
\begin{align*}
\begin{cases}
\tilde{H}(0,x_2) = 0 , \\
\tilde{H}(s,0) = \tilde{H}(s,b) = 0 , \\
\frac{\partial \tilde{H}}{\partial s}(a,x_2) = 0 ,
\end{cases}
\end{align*}
\]

(1.8)

with \( s \in [0,a] \) and \( x_2 \in [0,b] \).

**Proof.** The proof of (1.7) is similar to that of (1.3). lemma 1.1, if we note that

\[
\int_{-s}^{s} f^\#(x_1,x_2) dx_1 = \int_{0}^{2s} f^*(s',x_2) ds'.
\]

The equalities (1.8) are obvious.

Put now

\[
d(s,x_2) = H(s,x_2) - \tilde{H}(s,x_2) ,
\]

(1.9)

with \( s \in [0,a] \) and \( x_2 \in [0,b] \).

From lemmas 1.1 and 1.2 we have

\[
\frac{\partial^2 d}{\partial s^2} + \frac{\partial^2 d}{\partial s^2} = \Delta d > 0
\]

in \((0,a) \times (0,b)\) with

\[
\begin{align*}
\begin{cases}
d(0,x_2) = 0 \\
d(s,0) = d(s,b) = 0 , \\
\frac{\partial d}{\partial s}(a,x_2) = 0 ,
\end{cases}
\end{align*}
\]

(1.11)

for \( s \in [0,a] \) and \( x_2 \in [0,b] \).

Now we prove the following
LEMMA 1.3. If \( d(s,x_2) \neq 0 \), then \( d(s,x_2) < 0 \) in \((0,a) \times (0,b)\).

Proof. We have \( d(s,x_2) \neq 0 \) and \( \Delta d \geq 0 \) in \((0,a) \times (0,b)\), and so by the maximum principle, the maximum of \( d(s,x_2) \) must be attained on the sides of the rectangle \((0,a) \times (0,b)\) of \( \mathbb{R}^2 \).

But the maximum cannot be attained on \( \{s=a, 0 < x_2 < b\} \); in fact \( \frac{\partial d}{\partial s}(a,x_2) = 0 \) for \( x_2 \in (0,b) \), while if \((a,x_2)\) is a maximum point, we would have \( \frac{\partial d}{\partial s}(a,x_2) > 0 \) (see th.7 of [8], pag. 65). Then it follows from the first two relations of (1.11) that \( \max d(s,x_2) = 0 \).

So we proved that \( d(s,x_2) < 0 \) in \((0,a) \times (0,b)\).

We can now prove:

THEOREM 1.1. Let \( u \) be the solution of problem (1.1) and \( U \) the solution of problem (1.1)#. Then, for every \( x_2 \in [0,b] \),

\[
(1.12) \quad \|u(\cdot,x_2)\|_{L^1((-a,a) \times \{x_2\})} \leq \|U(\cdot,x_2)\|_{L^1((-a,a) \times \{x_2\})},
\]

and then

\[
(1.13) \quad \|u\|_{L^1(G)} \leq \|U\|_{L^1(G)}.
\]

Proof. From lemma 1.3 and equalities (1.11) we have

\[
(1.14) \quad \frac{\partial d}{\partial s}(s,x_2) \leq 0,
\]

in \([0,a] \times [0,b]\).

By (1.9) and (1.14) we have

\[
(1.15) \quad H(s,x_2) \leq \tilde{H}(s,x_2),
\]

that is
(1.16) \[ \int_{-S}^{S} u(x_1, x_2) dx_1 \leq \int_{-S}^{S} U(x_1, x_2) dx_1. \]

Then, for every \( x_2 \in [0, b] \):
\[ \int_{-h}^{a} u(x_1, x_2) \ dx_1 \leq \int_{-h}^{a} U(x_1, x_2) \ dx_1, \]

that is (1.12).

SECTION 2.

Consider the boundary problem:

\[
\begin{cases}
-\Delta u(x_1, x_2) = f(x_1, x_2) & \text{in } G \\
u(x_1, x_2) = 0 & \text{on } \partial G,
\end{cases}
\]

with \( G = (-\infty, +\infty) \times (0, b), b > 0 \), and the "symmetrized" one:

\[
\begin{cases}
-\Delta U(x_1, x_2) = f^\#(x_1, x_2) & \text{in } G \\
U(x_1, x_2) = 0 & \text{on } \partial G.
\end{cases}
\]

We suppose \( f \in L^2(G) \) and consequently \( f^\# \in L^2(G) \). Then, by theorem 5.4 of [5], page 632, problems (2.1) and (2.1)\# have unique weak solutions \( u, U \) respectively, belonging to \( W^{1, 2}_0(G) \cap W_{loc}^{2, 2}(G) \). Moreover, we take \( f, f^\# \) in \( C^{0, \lambda}(\bar{G}) \), with \( 0 < \lambda \leq 1 \), so that \( u, U \in C^2(G) \cap C^0(\bar{G}) \) and finally we suppose \( f \geq 0 \) and so, by the maximum principle, we have \( u \geq 0 \) in \( G \).

Also we obtain \( U \geq 0 \) in \( G \), since \( f^\# \geq 0 \) in \( G \).

As in section 1, we put
(2.2) \[ H(s, x_2) = \int_{-s}^{s} u(x_1, x_2) dx_1, \]

(2.3) \[ \tilde{H}(s, x_2) = \int_{-s}^{s} U(x_1, x_2) dx_1, \]

(2.4) \[ d(s, x_2) = H(s, x_2) - \tilde{H}(s, x_2) \]

with \( s \geq 0 \) and \( x_2 \in [0, b]. \)

We obtain, as in lemmas 1.1 and 1.2,

(2.5) \[ \Delta d \geq 0, \text{ in } (0, +\infty) \times (0, b), \]

and

(2.6) \[
\begin{cases}
  d(0, x_2) = 0, & x_2 \in [0, b] \\
  d(s, 0) = d(s, b) = 0, & s \geq 0.
\end{cases}
\]

We will obtain the following

**Lemma 2.1.** \( d(s, x_2) \leq 0 \) in \([0, +\infty) \times [0, b].\)

In order to prove lemma 2.1 we will use the following

**Theorem (Phragmèn-Lindelöf)** - Let \((r, \theta)\) be polar coordinates such that the polar semiaxis is coincident with the positive \( x_1 \) axis and let \( V = \{(r, \theta), r > 0, -\frac{\pi}{2a} < \theta < \frac{\pi}{2a}\} \), be the open sector of the angle \( \frac{\pi}{4} \).

Let \( \nu \) be a function in \( C^2(V) \cap C^0(V) \) such that...
Dirichlet problem for the Laplace ...

\[ \Delta \nu \geq 0 \quad \text{in} \ V, \]

and assume \( \nu \leq M \) on the boundary \( \theta = \pm \frac{\pi}{2\alpha} \) and

\[
(2.7) \quad \min \lim_{R \to +\infty} (\max_{r=R} \nu(r, \theta)) \leq 0.
\]

Then \( \nu \leq M \) on \( V \).

**Remark 2.1.** Let \( V' \) be an open subset of \( V \), and \( \nu \) a function in \( C^2(V') \cap C^0(\bar{V}') \) such that \( \Delta \nu \geq 0 \) in \( V' \), \( \nu \leq M \) on \( \partial V' \) and suppose that (2.7) holds relatively to \( V' \). Then it is easily seen that \( \nu \leq M \) in \( V' \).

**Proof of Lemma 2.1.** Since (2.6) hold, it is sufficient to prove that \( d(s, x_2) \leq 0 \) in \( (0, +\infty) \times (0, b) \).

Let \( \alpha = 1 \), \( V' = (0, +\infty) \times (0, b) \), \( \nu = d \).

By (2.5), (2.6), the Phragmèn-Lindelöf theorem and remark 2.1, it is sufficient to prove that (2.7) holds in \( (0, +\infty) \times (0, b) \).

We will prove that:

\[
(2.8) \quad \lim_{R \to +\infty} \left( \frac{\max_{(s, x_2) \in \Gamma_R} d(s, x_2)}{R} \right) = 0,
\]

where \( \Gamma_R \) is the arc of circle centered in \( (0, 0) \) and with radius \( R \), contained in \( (0, +\infty) \times (0, b) \).

In fact, for any \( (s, x_2) \in \Gamma_R \),

\( ^3 \) For the proof of Phragmèn-Lindelöf theorem and of remark 2.1, see [8], pp. 93-96.
(2.9) \[ |d(s, x_2)| \leq \int_{-s}^{s} u(x_1, x_2) dx_1 + \int_{-s}^{s} U(x_1, x_2) dx_1. \]

Then, by the Schwartz-Hölder inequality,

(2.10) \[ \int_{-s}^{s} u(x_1, x_2) dx_1 \leq (2s)^{\frac{3}{2}} \left( \int_{-s}^{s} u(x_1, x_2)^2 dx_1 \right)^{\frac{1}{2}} \leq \]

\[ \leq (2s)^{\frac{3}{2}} \left( \int_{-\infty}^{+\infty} u(x_1, x_2)^2 dx_1 \right)^{\frac{1}{2}}. \]

Moreover, by the embedding of \( W^{1,2}(G) \) in \( L^2(-\infty, +\infty \times \{ x_2 \}) \) (see [1], lemma 5.19), a positive constant \( C \) exists, which is independent of \( u \) and \( x_2 \) and such that

(2.11) \[ \left( \int_{-\infty}^{+\infty} u(x_1, x_2)^2 dx_1 \right)^{\frac{1}{2}} \leq C \| u \|_{W^{1,2}(G)}. \]

From (2.10) and (2.11), we obtain

(2.12) \[ \int_{-s}^{s} u(x_1, x_2) dx_1 \leq K s^{\frac{1}{2}}, \]

where \( K = 2^{\frac{3}{2}} C \| u \|_{W^{1,2}(G)}. \)

In the same way we obtain for \( U \):

(2.13) \[ \int_{-s}^{s} U(x_1, x_2) dx_1 \leq K' s^{\frac{1}{2}}, \]

where \( K' = 2^{\frac{3}{2}} C \| U \|_{W^{1,2}(G)}. \)

From (2.9), (2.12), and (2.13), it follows
Dirichlet problem for the Laplace ...

\[ |d(s,x_2)| \leq C' s^{\frac{1}{2}}, \]

where \( C' = K + K' \) is independent of \( s \) and \( x_2 \).

Then we easily obtain

\[ \max_{R} d(s,x_2) \leq C' R^{\frac{1}{2}} \]

and so (2.8) follows.

From lemma 2.1 and in the same way as in the proof of theorem 1.1, we obtain

**THEOREM 2.1.** Let \( u \) be the solution of problem (2.1) and \( U \) the solution of problem (2.1)#. Then, \( \forall x_2 \in [0,b] \),

\[ \|u(\cdot,x_2)\|_{L^1((-\infty, +\infty) \times \{x_2\})} \leq \|U(\cdot,x_2)\|_{L^1((-\infty, +\infty) \times \{x_2\})}, \]

and so also

\[ \|u\|_{L^1(G)} \leq \|U\|_{L^1(G)}. \]

Moreover the following theorem holds:

**THEOREM 2.2.** Let \( u,U \) the solutions of problems (2.1) and (2.1)# respectively. Then, \( \forall x_2 \in [0,b] \),

\[ \sup_{-\infty < x_1 < +\infty} u(x_1,x_2) \leq \sup_{-\infty < x_1 < +\infty} U(x_1,x_2), \]

and so also

\[ \sup_{G} u(x_1,x_2) \leq \sup_{G} U(x_1,x_2). \]
Proof. Fix \( x_2 \in (0, b) \); from lemma 2.1 and the first of (2.6) we obtain

\[
\frac{\partial d}{\partial s}(0, x_2) \leq 0.
\]

(2.18)

Since \( \frac{\partial H}{\partial s}(0, x_2) = 2u(0, x_2) \) and \( \frac{\partial \tilde{H}}{\partial s}(0, x_2) = 2U(0, x_2) \), we have

\[
\frac{\partial d}{\partial s}(0, x_2) = 2(u(0, x_2) - U(0, x_2))
\]

and then, from (2.18).

(2.19)

\[ u(0, x_2) \leq U(0, x_2) \]

Now, let \((k, x_2)\) be a point of \( G \), with \( k \in \mathbb{R} \), at height \( x_2 \). We write \( f|_k(x_1, x_2) \) for the translation of \( f(x_1, x_2) \) of the quantity \( |k| \), in the direction of the positive \( x_1 \)-axis if \( k < 0 \), and in the direction of the negative \( x_1 \)-axis if \( k \geq 0 \). If we consider \( f|_k \) instead of \( f \), then also \( u \) will be translated. We write \( u|_k \) for this translation.

Now, \( Vx_2 \in (0, b) \), \( f|_k(x_1, x_2) = (f|_k)(x_1, x_2) \).

Then, by (2.19),

\[ u(k, x_2) = u|_k(0, x_2) \leq U(0, x_2) \leq \sup_{-\infty < x_1 < +\infty} U(x_1, x_2) \]

and then (2.16) follows.
REFERENCES


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