Note di Matematica Vol. VI, 205-236(1986)

ON FINITE GEOMETRIES OF TYPE C_3 WITH THICK LINES

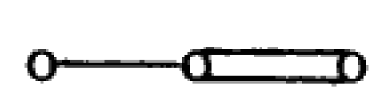
Antonio PASINI

Summary. *Several facts suggest the conjecture that every finite geometry of type* C_3 *with thick lines is either a building or flat (see* [4], [5], [8] *and* [9]). *That conjecture plays a central role in the problem of classifying finite thick geometries of type* C_n *and* F_4 *(see* [4], [7] *and* [8]). *In this paper we find very simple conditions that hold in a finite geometry of type* C_3 *with thick lines if and only if the geometry is either a building or flat. Then the previous conjecture can be restated so: prove that the conditions discussed in this paper are theorems in the case of finite geometries with thick lines.*

1. DEFINITIONS AND STATEMENTS OF THE THEOREMS.

In this paper Γ will always be a connected geometry of type C_3. We observe that Γ is residually connected, because it is connected and it has rank 3. Then it is strongly connected, by [3].

We mark types as below



0 1 2

The elements of type 0 are said to be *points*, those of type 1 *lines* and those of type 2 *planes*. If a point b is incident with a line x we say that b lies on x and that x passes through b. Similarly for incident point-plane and line-plane pairs. For every i=0,1,2 the symbol S_i denotes the set of elements of Γ of type i and, for every element x of Γ, we denote the i-shadow of x by the symbol $\sigma_i(x)$.

We use the symbol $*$ to denote the incidence relation, as in [15]. A line in *thin* if it has exactly two points. Otherwise it is *thick*. The geometry Γ is *flat* if we have $\sigma_0(u) = S_0$ for every plane u of Γ (equivalently, $\sigma_2(b) = S_2$ for every point b of Γ). If a,b are distinct points, we say that they are *colinear* if there is a line incident with both them. If a and b are colinear, we write $a \perp b$. Given a point x, we define $x^{\perp} = \{y \mid y \perp x \text{ or } y=x\}$ and given a set of points X we define $X^{\perp} = \bigcap_{x \in X} x^{\perp}$. The *radical* of Γ is the set $S_0^{\perp}$.

If the geometry Γ is flat, then we have $S_0^{\perp} = S_0$. I do not know whether the converse is true or not.At any rate, let us say that Γ is *almost flat* if $S_0^{\perp} = S_0$.

Given two distinct planes u and v, we say that they are *cocolinear* if there is a line incident with both them. If two planes u and v are cocolinear, then we write $u \top v$.

We have proved in [9] (Proposition 3) that the following property holds in Γ if and if Γ is either a building or flat:

(BF) (BUILDING-FLAT PROPERTY). *Given a point* b *and a plane* u *if there are two distinct planes both incident with* b *and cocolinear with* u, *then* b *lies on* u .

We have considered also the following two properties in [9]:

(PS) *Given a line* x *and a plane* u *, if* $\sigma_0(x) \cap \sigma_0(u)$ *contains at least two points, then we have* $\sigma_0(x) \subseteq \sigma_0(u)$ (that is, planes are 'subspaces').

(PS*)(dual of PS). *Given a line* x *and a point* b*, if* $\sigma_2(x) \cap \sigma_2(b)$ *contains at least two planes, then we have* $\sigma_2(x) \subseteq \sigma_2(b)$.

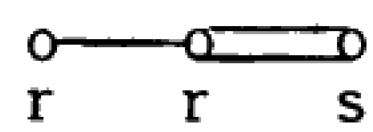
The property PS holds in buildings and flat geometries of type C_3. Then it is a consequence of BF. Nevertheless, it holds in every geometry of type C_3 in which all lines are thin. But there are a lot of such geometries that are neither buildings nor flat (see [11]; see also [9]). Then the property PS is weaker than BF. The property PS* is a specialization of BF. We get it from BF if, in the statement of BF, we assume that the two planes,incident with b and cocolinear with u, and the plane u itself pass through the same line. We observe that PS* is actually weaker than BF. Indeed it holds in Γ if every line of Γ lies on exactly two planes. But there are a lot of such geometries that are neither buildings nor flat (see [12]).

Nevertheless, the properties PS and PS* together are equivalent to BF. Indeed we have proved in [9] (Theorem 2) that Γ is either a building or flat if and only if both the properties PS and PS* hold in it. In this paper we shall prove the following theorem.

THEOREM 1. *Let* Γ *be finite and let us assume that all lines of* Γ *are thick and that the property* PS *holds in* Γ *. Then* Γ *is either a building or flat.*

We shall prove this theorem in Section 3. As to PS*, things look a little more difficult. We have

PROPOSITION 1. *Let* Γ *be finite and let it admit parameters*



where $r \leq s$.

The geometry Γ *is either a building or flat if the property* PS* *holds in it.*

We shall prove this proposition in Section 4. We shall consider also another property, suggested by the axion system given in [2] for the system of points and lines of a polar space.

(BS) (BUEKENHOUT-SHULT PROPERTY). *Given a point* b *and a line* x *, if* $\sigma_0(x) \cap b^{\perp}$ *contains at least two points, then* $\sigma_0(x) \subseteq b^{\perp}$.

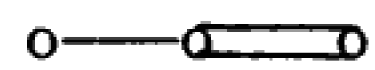
We observe that we always have $\sigma_0(x) \cap b^{\perp} \neq \emptyset$ (see Lemma 1 of Section 2 of this paper). The property BS holds in buildings and flat geometries of type C_3. Then the property BF implies BS. But the converse is false. Indeed the property BS holds in all finite geometries with thin lines (see [11]). But there are a lot of such geometries that are neither buildings nor flat. Then BF is not a consequence of BS. At any rate, we have the following results:

PROPOSITION 2. *Let* Γ *be finite and let us assume that all lines of* Γ *are thick and that the property* BS *holds in* Γ . *Then* Γ *is either a building or almost flat.*

PROPOSITION 3. *Let* Γ *be finite and let it admit parameters. The geometry* Γ *is either a building or flat if both the properties* PS* *and* BS *hold in it.*

Proposition 2 will be proved in Section 5 and Proposition 3 will be proved in Section 4.

Given two colinear points b and c, let n(b,c) be the number of lines incident with both b and c. We observe that, if Γ is finite and admits parameters



then $n(b,c) \leq 1+rs$ and we have $n(b,c) = 1+rs$ iff $\sigma_2(b) = \sigma_2(c)$ (see [13]). Then Γ is flat iff $n(b,c) = 1+rs$ for every colinear pair (b,c) and it is a building iff $n(b,c)=1$ for every colinear pair (b,c) (see Proposition 9 of [15]). Then, if Γ is a building or flat, the number n(b,c) does not depend on the choice of the colinear pair (b,c). This is still true if all lines of Γ are thin, even if Γ does not admit parameters or if it is not a building nor flat (see [11]). But it need not be true if Γ has some thin and some thick lines (see Section 6 of [11]).

Given a point b, we say that b is *homogeneous* if the number n(b,c) does not depend on the choice of the point c colinear with b. We have observed above that, if Γ is finite, admits parameters and is either a building or flat, then all points of Γ are homogeneous.

We have the following result:

THEOREM 2. *Let Γ be finite and let us assume that all lines of Γ are thick. The geometry Γ is either a building or flat if there is some homogeneous point in Γ.*

This theorem will be proved in Section 3.

As a conclusion, we have found two properties, besides BF, which, in the case of finite geometries with thick lines, are equivalent to the property of being either a building or flat: the property PS and the property of admitting some homogeneous points.

2. LEMMAS.

LEMMA 1. *Given a plane* u *and a point* b *, there is a plane through* b *cocolinear with* u.
(see Lemma 1 of [7]).

LEMMA 2. *Given any two distinct planes, there is at most one line incident with both them.*

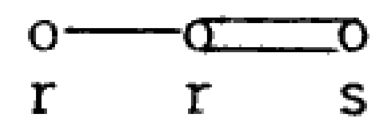
The proof is easy. It is left to the reader.

Let δ be the mapping from the set of point-plane pairs (b,u) to the set of integers {0,1} that takes the value 1 precisely when $b * u$. Given a point-plane pair (b,u), let $\alpha(b,u)$ be the number of planes v through b, cocolinear with u and such that the line incident with u and v (uniquely determined by Lemma 2) does not pass through b. The number $\alpha(b,u)-\delta(b,u)+1$ does not depend on the choice of the pair (b,u) (see [9], Theorem 1). We warn that, if $\alpha(b,u)$ is infinite, then we have $\alpha(b,u)+1 = \alpha(b,u) = \alpha(b,u)-1$.

Let us write α instead of $\alpha(a,u)-\delta(a,u)+1$. We say that α is the *Ott-Liebler number* of Γ, as in [9].

We have $\alpha = 0$ if and only if Γ is a building (see [9], Proposition 1).

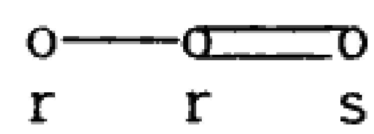
Moreover, if Γ is finite and admits parameters



then we have $\alpha = r^2 s$ iff Γ is flat (see [4] and [5]; see also [9], Corollary 2).

The following lemma is implicit in Proposition 3.9 of [4]:

LEMMA 3. *Let Γ be finite and let it admit parameters*



Let d *be the greatest common divisor of* r *and* s. *The number* rd *divides* α, *the number* $1+\alpha$ *divides both* $(1+r^2 s)(1+r+r^2)$ *and* $(1+s)(1+rs)(1+r^2 s)$ *and there are exactly* $(1+r+r^2)(1+r^2 s)/(1+\alpha)$ *points and* $(1+s)(1+rs)(1+r^2 s)/(1+\alpha)$ *planes in* Γ.

Let m be the multiplicity of the reflection representation of the Hecke algebra of Γ. By Proposition 3.9 of [4] we have

$$m = \frac{(1+r+r^2)(1+rs)(r^2 s-\alpha)}{r(r+s)(1+\alpha)} .$$

Then rd divides α.

The number of planes of Γ is $(1+s)(1+rs)(1+r^2 s)/(1+\alpha)$, by Proposition 3.9 of [4] (see also [9], Section 4). Then $1+\alpha$ divides $(1+s)(1+rs)(1+r^2 s)$ and there are exactly $(1+r+r^2)(1+r^2 s)/(1+\alpha)$ points in Γ. Then $1+\alpha$ divides also $(1+r+r^2)(1+r^2 s)$.

Q.E.D.

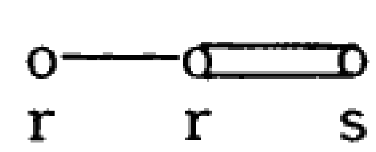
Let us state the following conventions. We have denoted the number of lines incident with two colinear points b and c by the symbol $n(b,c)$. We can define $n(b,c)$ also if $b = c$ or $b \neq c$ and $b \not* c$. If $b=c$ then $n(b,c)$ is the number of lines incidend with b. If $b \neq c$ and $b \not* c$, then $n(b,c)=0$.

Given a point b and a line x, let $n_2(b,x)$ be the number of planes incident with both b and x and let $n_0(b,x)$ be the number of points belonging to $\sigma_0(x) \cap b^{\perp}$. Given a point b and a plane u, let $L(b,u)$ be the set of lines x incident with u and such that there is a plane v incident with both b and x. Let $n_1(b,u)$ be the number of lines in $L(b,u)$. Finally, we denote by $n_0(b,u)$ the number of points in $\sigma_0(u) \cap b^{\perp}$.

Let δ' be the mapping from the set of point-line pairs (b,x) to the set of integers $\{0,1\}$ that takes the value 1 precisely when $b * x$.

The following Lemma could be stated in general. But we prefer to limit ourselves to the case of finite geometries admitting parameters.

LEMMA 4. *Let Γ be finite and let it admit parameters*



Let (b,x) *be a point-line pair. We have*

$$\sum_{c \in \sigma_0(x)} n(b,c) = \alpha + 1 + r \cdot n_2(b,x) + r(rs+1) \cdot \delta'(b,x) .$$

We must distinguish three cases.

Case 1. We have $b * x$. Let u be a plane through x. Let us compute the number N of quadruples (c,y,z,v) where c is a point on x different from b, y is a line through b and c different from x, v

is a plane through y and z is a line through c incident with both u and v. We can compute N in two ways. We can take any point c on x different from b, then any line y through c and b different from x. The line y does not lie on u. Then there is exactly one incident line-plane pair (z,v) in the residue Γ_c of c such that $y*v$ and $z*u$. So we get

$$N = \sum_{\substack{c*x \\ c\neq b}} (n(b,c)-1)$$

But we can also take any plane v cocolinear with u and such that the line z incident with both u and v does not pass through b (the line z is uniquely determined by Lemma 2). In Γ_u there is exactly one point c incident with both z and x. In Γ_v there is exactly one line y incident with both b and c. We have $y\neq z$ because z does not pass through b. Then $x\neq y$ by Lemma 2. So we get $N=\alpha$. Then

$$\alpha = \sum_{\substack{c*x \\ c\neq b}} (n(b,c)-1)$$

Then we have

$$\sum_{c*x} n(b,c) = \alpha+r+(rs+1)(r+1) = \alpha+1+r(s+1)+r(rs+1) =$$

$$= \alpha+1+r\cdot n_2(b,x)+ \delta'(b,x)\cdot r(rs+1).$$

Case 2. We have $b\notin\sigma_0(x)$ but there is a plane u incident with b and x. Let us compute the number N of all quadruples (c,y,z,v) where c is a point on x, y is a line through b and c not incident with u, (z,v) is an incident line-plane pair in Γ_c and $y*v$ and $z*u$. We can compute N in two ways. We can take any point c on x then any line y through b and c not incident with u and we find exactly one incident line plane pair (z,z) in Γ_c such that $y*v$ and $z*w$. So we have

$$N = \sum_{c*x} (n(b,c)-1).$$

But we can also take any plane v cocolinear with u, incident with b and such that the line z incident with both u and v (uniquely determined by Lemma 2) does not pass through b. If $z \neq x$, then there is just one point c incident with both x and z in Γ_u and just one line y incident with both b and c in Γ_v and y does not lie on u. If z=x, then we can take any point c on x and the line y through b and c in Γ_v. The line y does not lie on u. Otherwise we have x=y by Lemma 2 and $b*x$, whereas $b \notin \sigma_0(x)$. Then we get $N = (\alpha - (n_2(b,x)-1)) + (n_2(b,x)-1)(r+1)$. Then we have

$$\sum_{c*x} (n(b,c)-1 = \alpha + 1 + r \cdot n_2(b,x) - (r+1).$$

Then

$$\sum_{c*x} n(b,c) = \alpha + 1 + r \cdot n_2(b,x) =$$

$$= \alpha + 1 + r \cdot n_2(b,x) + r(rs+1) \cdot \delta'(b,x).$$

Case 3. There is no plane incident with both x and b. Let u be a plane incident with x. Let us compute the number N of quadruples (c,y,z,v) where c is a point in $\sigma_0(x) \cap b^{\perp}$,y is a line through b and c, (z,v) is an incident line-plane pair in Γ_c and $y*v$ and $z*u$. We can compute N in two ways. We can take any point c in $\sigma_0(x)\cap b^{\perp}$, any line y through b and c and in Γ_c there is exastly one incident line-plane pair (z,v) such that $y*z$ and $z*u$. So we have

$$N = \sum_{\substack{c*x \\ c\perp b}} n(b,c) = \sum_{c*x} n(b,c).$$

But we can also take any plane v through b cocolinear with u. Let z be the line incident with u and v (the line z is uniquely determined by Lemma 2). We have $z\neq x$ because $x\notin L(b,u)$. Then there is just one point c incident with both x and z. We find c in Γ_u. The line y through b and c in Γ_v is uniquely determined. So we have $N = \alpha+1$. Then

$$\sum_{c*x} n(b,c) = \alpha+1 = \alpha+1 + r\cdot n_2(b,x) + r(rs+1)\cdot\delta'(b,x).$$

Q.E.D.

LEMMA 5. *Let* (a,u) *be an incident point-plane pair and let* b *be a point not incident with* u. *We have*

$$n(a,b) = \sum_{\substack{x*a \\ x*u}} n_2(b,x) \quad .$$

Let us compute the number N of triplets (y,v,x) where x is a line on u through a, v is a plane through x incident with b and y is a line on v through a and b. We can compute N in two ways. We can take any line y through a and b. Then we find just one incident line-plane pair (x,v) in Γ_a such that $y*z$ and $x*u$. So we have $N = n(a,b)$. But we can also take any line x through a on u and any plane v through x incident with b. The line y through a and b in Γ_v is uniquely determined. So we have

$$N = \sum_{\substack{x*a \\ x*u}} n_2(b,x) \quad .$$

Q.E.D.

LEMMA 6. *Let* b *and* c *be colinear points, let* x *be a line through them and let* u *be a plane through* x. *We have*

$$n(b,c)-1 = \sum_{\substack{y*c \\ y*u \\ y\neq x}} (n_2(b,y)-1) \quad .$$

Let us compute the number N of triplets (z,v,y) where z is a line through b and c different from x and (y,v) is an incident line-plane pair in Γ_c such that z∗v and y∗u. We can compute N in two ways. We can take any line y on u through c different from x and any plane v through y incident with b and different from u, if there is such a plane. In Γ_v we take the line z through b and c. We have z≠x, otherwise we get the contradiction y=x by Lemma 2.

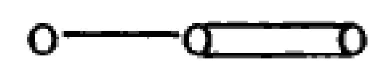
Then we have:

$$N = \sum_{\substack{y*c \\ y*u \\ y\neq x}} (n_2(b,y-1) \quad .$$

But we can also take any line z through b and c different from x. The line z does not lie on u. Then there is (just one) incident line-plane pair (y,v) in Γ_c such that z∗v and y∗u. Then we have N = n(b,c)-1.

Q.E.D.

COROLLARY 1. *Let* Γ *be finite and let it admit parameters*

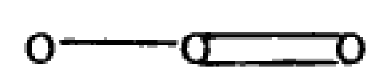


r r s

Then we have $\alpha \leq r^2 s$.

Trivial, by Lemma 4 and by the fact that $n(b,c) \leq rs+1$ for every colinear pair (b,c).

COROLLARY 2. *Let* Γ *be finite and let it admit parameters*



r r s

Let (b,u) *be a point-plane pair. Then we have*

$$\sum_{c \in \sigma_0(u)} n(b,c) = (r+1)(\alpha+1)+r(r+1)(s+1)\cdot\delta(b,u).$$

(Trivial, by Lemma 4 and 5).

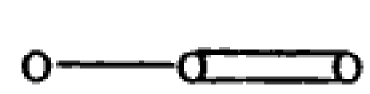
Given a point-line pair (b,x), let us set

$$n^*(b,x) = \mathrm{Max}(n(b,c) \mid c \in \sigma_0(x),\ c \neq b)$$

and

$$n_*(b,x) = \mathrm{Min}(n(b,c) \mid c \in \sigma_0(x) \cap b^\perp).$$

COROLLARY 3. *Let* Γ *be finite and let it admit parameters*



r r s

Let (b,x) *be a point-line pair. Then*

(i) *If* $b * x$, *we have* $n_*(b,x) \leq 1 + \alpha/x \leq n^*(b,x)$

(ii) *If* $b \notin \sigma_0(x)$ *but* $\sigma_2(x) \cap \sigma_2(b) \neq \emptyset$, *we have:*

$$n_*(b,x) \leq \frac{\alpha+1+r\cdot n_2(b,x)}{r+1} \leq n^*(b,x)$$

(iii) *If* $\sigma_2(b) \cap \sigma_2(x) = \emptyset$ *then we have* $n_*(b,x) \leq (\alpha+1)/n_0(b,x) \leq$
$\leq n^*(b,x)$.

This corollary easily follows from Lemma 4.

We observe that $n_0(b,x) \neq 0$ for every point-line pair (b,x). This fact follows from Lemma 4. But it can be easily got also by Lemma 1.

COROLLARY 4. *Let* Γ *be finite and let it admit parameters. Let* (b,u) *be a non incident point-plane pair. We have*

$$\alpha + 1 = \sum_{x \in \sigma_1(u)} n_2(b,x)$$

Indeed by Lemma 5 we have

$$\sum_{c*u} n(b,c) = \sum_{c*u} \sum_{\substack{x*c \\ x*u}} n_2(b,x) = \sum_{x*u} \sum_{c*x} n_2(b,x) = (r+1)\left(\sum_{x*u} n_2(b,x)\right)$$

where $r+1$ is the number of points on a line. By Corollary 2 we have $\sum_{c*u} n(b,c) = (r+1)(\alpha+1)$.

The conclusion follows.

Q.E.D.

LEMMA 7. *Let* (b,u) *be a non incident point-plane pair. We have*

$$\sigma_0(u) \cap b^{\perp} = \cup(\sigma_0) \mid x \in L(b,u)).$$

The proof is trivial. It is left to the reader.

LEMMA 8. *Let* Γ *be finite and let it admit parameters. Let* b *be a point and let* u,v *be planes not incident with* b . *We have* $\sigma_0(u) \cap b^{\perp} = \sigma_0(v) \cap b^{\perp}$ *if* $\sigma_0(u) \cap b^{\perp} \supseteq \sigma_0(v) \cap b^{\perp}$.

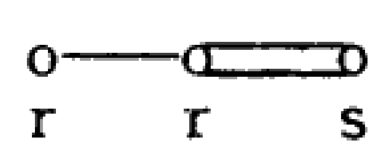
Indeed, if r+1 is the number of points on a line, we have

$$\sum_{a * u} n(a,b) = (\alpha+1)(r+1) = \sum_{c * v} n(c,b) \quad .$$

Let us set $U = (\sigma_0(u) - \sigma_0(v)) \cap b^{\perp}$ and $V = (\sigma_0(v) - \sigma_0(u)) \cap b^{\perp}$. By the previous equality we get $\sum_{a \in U} n(a,b) = \sum_{c \in V} n(c,b)$. Then $U = \emptyset$ if $V = \emptyset$.

Q.E.D.

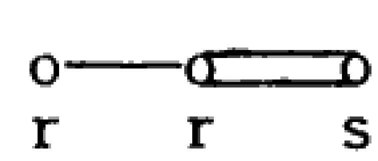
COROLLARY 5. *Let* Γ *be finite and let it admit parameters*



Let (b,u) *be a non incident point-plane pair. We have either* $n_0(b,u) = r+1$ *or* $n_0(b,u) \geq 2r+1$. *We have* $n_0(b,u) = r+1$ *precisely when* $n_1(b,u) = 1$.

The corollary trivially follows from lemmas 8 and 1.

LEMMA 9. *Let* Γ *be finite and let it admit parameters*

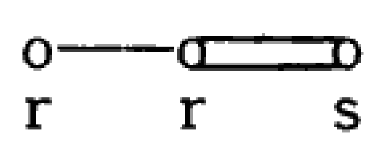


If $n_1(b,u) = 1$ *for some non incident point-plane pair, then* $\alpha < s$.

Indeed, if $n_1(b,u) = 1$ for a non incident point-plane pair (b,u), then L(b,u) contains just one line x and every plane through b cocolinear with u passes through x. Then $\alpha + 1 \leq s$.

Q.E.D.

COROLLARY 6. *Let* Γ *be finite and let it admit parameters*



where $s \leq r$.

The geometry Γ *is a building if there is a non incident point-plane pair* (b,u) *such that* $n_1(b,u) = 1$.

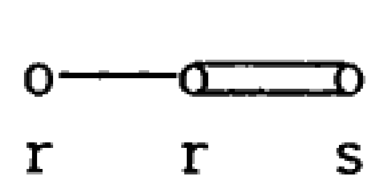
Indeed, if such a point-plane pair exists, we have $\alpha < s$ by Lemma 9. Then $\alpha = 0$ by Lemma 3, because r divides α and $r \geq s$. Then Γ is a building.

Q.E.D.

COROLLARY 7. *Let* Γ *be finite and let us assume that its lines are thick. The geometry* Γ *is a building if there is some point* b *such that* $\sigma_2(b) \neq S_2$ *and* $n_1(b,u)=1$ *for every plane* u *not incident with* b.

Let b be a point such that $n_1(b,u) = 1$ for every plane u not incident with b and $\sigma_2(b) \neq S_2$.

The geometry Γ admits parameters



because it has thick lines. We have $\alpha < s$ by Lemma 9. Then for every line x not incident with b, there is some point u through x that is not incident with b. Otherwise we have $\alpha \geq s$. Given a line x not incident with b, let u be a plane through x not incident with b. If $n_2(b,x) \neq 0$ then x is the unique in L(b,u), because $n_1(b,u) = 1$. Then every plane through b cocolinear with u passes through x. Then $n_2(b,x) = \alpha + 1$. Let us take a plane v through b and a point c on v different from b. Let y be the line on v through b and c. For every line x on v through c different from y, we have $n_2(b,x) = \alpha + 1$. Then we have

$$\sum_{d*v} n(d,c) = \sum_{\substack{x*c \\ x*u \\ x\neq y}} \sum_{\substack{d*x \\ d\neq c}} n(d,b) + \sum_{d*y} n(d,b) =$$

$$= r(r+1)(\alpha+1)-r\cdot n(c,b)+\alpha+1+r(s+1)+r(rs+1),$$

by Lemma 4. But we have also

$$\sum_{d*v} n(d,c) = (r+1)(\alpha+1) + r(r+1)(s+1)$$

by Corollary 2. Then $r(r+1)(\alpha+1)-r\cdot n(c,b)+\alpha+1+r(s+1)+r(rs+1) = (r+1)(\alpha+1) + r(r+1)(s+1)$. So we get $1+r\alpha = n(b,c)$. Then n(b,c) does not depend on the choice of c in $(b^{\perp})-\{b\}$. Then $n(b,c) = 1+\alpha/r$, by Corollary 3. Then $1+r\alpha = 1+\alpha/r$. Then either $r=1$ or $\alpha = 0$. But $r\neq 1$ because the lines of Γ are thick. Then $\alpha = 0$. The geometry Γ is a building.

Q.E.D.

Given a line-plane pair (x,u), let $n_0(x,u)$ be the number of points in $\sigma_0(x)\cap\sigma_0(u)$.

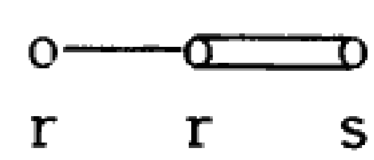
LEMMA 10. *Given a point-plane pair* (b,u) *, we have*

$$\sum_{c\in\sigma_0(u)} n(b,c) = \sum_{x\in\sigma_1(b)} n_0(x,u).$$

The proof is trivial. It is left to the reader.

3. PROOF OF THEOREMS 1 AND 2.

In this section Γ is finite and its lines are thick. Then it admits parameters



(where $r > 1$)

It is convenient to prove Theorem 2 for the first.

3.1. PROOF OF THEOREM 2. Let b be a homogeneous point of Γ.

If $\sigma_2(b) = S_2$ then there are exactly $(rs+1)(s+1)$ planes in Γ . Then $\sigma_2(c) = S_2$ for every point c and Γ is flat.

Let us assume that Γ is not flat. Then there is some plane u not incident with b.

Let us set $n = n(b,c)$, where c is any point of Γ. By Corollary 2 we have

(1) $$n_0(b,u) = (r+1)(\alpha+1)/n.$$

By (i) of Corollary 3 we have

(2) $$n = 1 + \alpha/r.$$

Then $n_0(b,u) = (r+1)(\alpha+1)/(1+\alpha/r)$ by (1) and (2). So it is easily seen that $n_0(b,u) < 1+r+r^2$. Then $L(b,u) \neq \sigma_1(u)$ and $\sigma_0(u) \not\subseteq b^\perp$. Let x be a line in $\sigma_1(u)-L(b,u)$. By (iii) of Corollary 3 we have

(3) $$n = (\alpha+1)/n_0(b,x).$$

Moreover, we have $L(b,u) \neq \emptyset$ by Lemma 1. Let y be a line in $L(b,u)$. By (ii) of Corollary 3 we have

$$(4) \qquad n = (\alpha+1+r\cdot n_2(b,y))/(r+1).$$

By (2) and (4) we easily get that $\alpha \equiv 0 \pmod{r^2}$. By (2) and (3) we easily get that $n_0(b,x) \equiv 1 \pmod{r}$ because $\alpha \equiv 0 \pmod{r^2}$. Then either $n_0(b,x) = 1$ or $n_0(b,x) = r+1$, because $n_0(b,x) \leq 1+r$. If $n_0(b,x) = r+1$, then $n_0(b,u) = 1+r+r^2$ because $n_0(b,x) = (\alpha+1)/n$ does not depend on the choice of x in $\sigma_1(u)-L(b,u)$. But we have already proved that $n_0(b,u) < 1+r+r^2$. Then $n_0(b,x) = 1$. Then $1+\alpha/r = 1+\alpha$ by (2) and (3). Then $\alpha = 0$ because $r>1$. Then Γ is building.

Q.E.D.

3.2. PROOF OF THEOREM 1.

Let the property PS hold in Γ.

Let x,y be distinct lines such that $\sigma_0(x) \neq \sigma_0(y)$ and let us assume that there are distinct points b and c incident with both x and y.

All planes incident with both b and c have the same 0-shadow. Indeed let u,v be planes through x and y respectively. We have $\sigma_0(y) \subseteq \sigma_0(u)$ and $\sigma_0(x) \subseteq \sigma_0(v)$ by PS. Then $\sigma_0(u) \supseteq \sigma_0(v)$ by PS, because $\sigma_0(u)$ contains the 0-shadow of the line y of v and a point d on v not incident with y. We find d in $\sigma_0(x)-\sigma_0(y)$ because $\sigma_0(x) \neq \sigma_0(y)$ and $\sigma_0(x) \subseteq \sigma_0(v)$. Similarly, $\sigma_0(v) \supseteq \sigma_0(u)$. Then $\sigma_0(u) = \sigma_0(v)$. Now let w be any other plane incident with both b and c. Let z be the line on w through b and c. We have either $\sigma_0(z) \neq \sigma_0(x)$ or $\sigma_0(z) \neq \sigma_0(y)$ because $\sigma_0(x) \neq \sigma_0(y)$. Then we have either $\sigma_0(w) = \sigma_0(u)$ or $\sigma_0(w) = \sigma_0(v)$, by the previous argument. Then $\sigma_0(w) = \sigma_0(u) = \sigma_0(v)$ in both cases, because we have already proved that $\sigma_0(u) = \sigma_0(v)$. Then all planes incident with both b and c have the same 0-shadow.

Moreover, the points b and c have the same 2-shadow. Indeed, let w be a plane through c. If $x*w$ or $y*w$, then we have $b*w$. Let us assume that $w \notin \sigma_2(x) \cap \sigma_2(y)$. In Γ_c there are incident line-plane pairs (v,z) and (v',z') such that $x*v$, $z*w$, $y*v'$ and $z'*w$. We have $\sigma_0(v) \supseteq \sigma_0(z')$ because $\sigma_0(v) = \sigma_0(v')$. if $z \neq z'$, then $\sigma_0(v)$ contains the 0-shadows of two distinct lines of w. Then $\sigma_0(v) = \sigma_0(w)$ by PS. Then $b*w$ because $b*v$. Let us assume that $z = z'$. We have assumed that all lines of Γ are thick. Then we find a line z" on v through c distinct from both x and z. Let u be any plane through z" different from v. We have $b*u$ by the previous argument (we observe that $v \neq v'$ because $x \neq y$ and $\sigma_0(x) \cap \sigma_0(y$ contains both b and c). Let x' be the line on u through b and c and let $\bar{u}$ be the plane through x' cocolinear with w in Γ_c. If $\sigma_0(x') \neq \sigma_0(x)$, then we can substitute v' with $\bar{u}$ and y with x' in the previous argument and we get $b*w$. If $\sigma_0(x') = \sigma_0(x)$, then $\sigma_0(x') \neq \sigma_0(y)$. Then we can substitute v with $\bar{u}$ and x with x' in the previous argument and we get $b*w$.

So we have proved that $\sigma_2(c) \subseteq \sigma_2(b)$. Similarly, $\sigma_2(b) \subseteq \sigma_2(c)$. Then $\sigma_2(b) = \sigma_2(c)$.

Let us assume that Γ is not flat. Then there is some plane w not incident with b. Let z be a line in L(b,w) and let u be a plane in $\sigma_2(z)\cap\sigma_2(b)$. We have c*u because $\sigma_2(b)=\sigma_2(c)$. Let z' be the line on u through b and c. We have $z'\neq z$ because $b\notin\sigma_0(w)$. Let d be the intersection point of z and z' in Γ_u. We have $w\in\sigma_2(d)-\sigma_2(b)$. Then all lines through b and d have the same 0-shadow. Then all lines through b and d are incident with c, because z' is a line through b and d and it passes through c. But every plane through b passes through d. Indeed it passes through c, then it has the same 0-shadow as u, then it passes through d because u passes through d. Then there are exactly rs+1 lines through b and d and they are incident with the same points because $b\notin\sigma_0(w)$ and d*w. Then each of these lines passes through c, because z' passes through c. Then there are rs+1 pairwise distinct lines through b and c with the same 0-shadow. But we have assumed that there are lines x,y through b and c such that $\sigma_0(x)\neq\sigma_0(y)$. Then there are at least rs+2 lines through b and c. They are too many. We have a contradiction.

Therefore, given two colinear points b and c, all lines through both b and c have the same 0-shadow.

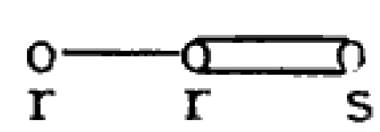
Now let (b,x) be an incident point-line pair. For every point c on x different from b and for every line y through b and c, we have $\sigma_0(x)=\sigma_0(y)$. Then, for every choice of the points c,d on x different from b, we have $\sigma_1(c)\cap\sigma_1(b)=\sigma_1(d)\cap\sigma_1(b)$. That is, the set of lines $\sigma_1(b)\cap\sigma_1(c)$ does not depend on the choice of c in $\sigma_0(x)-\{b\}$. Thus n(b,c) does not depend on the choice of c in $\sigma_0(x)-\{b\}$. So we have $n(b,c)=1+\alpha/r$, by Corollary 3. Then n(b,c) does not depend even on the choice of x. Thus b is homogeneous. Therefore Γ is a building by Theorem 2 (indeed we have assumed that it is not flat). So, Γ is a building if it is not flat.

Q.E.D.

REMARK. It is possible to give also a proof of Theorem 1 that does not depend on Theorem 2. It is similar to that given here, but a little different in the final part. In particular, the relation $\alpha \equiv 0$ (mod. r^2), that is the crucial point in the proof of Theorem 2, can be established, under the assumption that PS holds, by an argument more geometric than that used in the proof of Theorem 2.

4. ON THE PROPERTY PS*

In this section Γ is finite, admits parameters



and the property PS* holds in it.

If $r=1$, then, by the classification of finite geometries of type C_3 with thin lines given in [11], we easily get that Γ is either a building or flat.

Let us assume that $r \neq 1$.

LEMMA 11. *Given a point-line pair* (b,x), *we have either* $n_2(b,x)=0$, *or* $n_2(b,x) = 1$ *or* $n_2(b,x) = s+1$.

(Trivial. This lemma is nothing but a restatement of PS*).

LEMMA 12. *The least common multiple of* r *and* s *divides* α.

The statement is trivial if $\alpha = 0$. Let us assume that $\alpha \neq 0$. Let (b,u) be an incident point-plane pair. By the property PS* we have $\sigma_2(x) \subseteq \sigma_2(b)$ for every line x on u not incident with b and such that $\sigma_2(x) \cap \sigma_2(b)$ contains a plane different from u. Then s divides α, by Lemma 2. Then the least common multiple of r and s divides α, by Lemma 3.

Q.E.D.

LEMMA 13. *Given two colinear points* b *and* c *, we have* $n(b,c) \equiv 1$ (mod. s).

(Trivial, by Lemmas 6 and 11).

Lemma 13 allows us to establish the following convention. Given two colinear points b and c, we set $k_{bc} = (n(b,c)-1)/s$.

LEMMA 14. *Let* (b,x) *be a point-line pair. Then:*

(i) *If* $b * x$ *we have*

$$\alpha = \sum_{\substack{c*x \\ c \neq b}} k_{bc} s$$

(ii) *If* $b \notin \sigma_0(x)$ *but* $\sigma_2(x) \subseteq \sigma_2(b)$, *then we have:*

$$\alpha + rs = \sum_{c*x} k_{bc} s$$

(iii) *If* $b \notin \sigma_0(x)$ *and* $\sigma_2(x) \not\subseteq \sigma_2(b)$ *but* $\sigma_2(x) \cap \sigma_2(b) \neq \emptyset$, *then*

$$\alpha = \sum_{c*x} k_{bc} s$$

(iv) *If* $\sigma_2(b) \cap \sigma_2(x) = \emptyset$, *then we have*

$$\alpha+1 - n_0(b,x) = \sum_{\substack{c*x \\ c\perp b}} k_{bc}s$$

(Trivial, by Lemma 4 and 11).

LEMMA 15. *Let* (b,u) *be a point-plane pair. Then:*

(i) *If* $b*u$ *then we have*

$$\alpha(r+1) = \sum_{\substack{c*u \\ c\neq b}} k_{bc}s$$

(ii) *If* $b\notin\sigma_0(u)$ *then we have*

$$(\alpha+1)(r+1)-n_0(b,u) = \sum_{\substack{c*u \\ c\perp b}} k_{bc}s$$

and

$$\alpha+1 = n_1(b,\mu)$$

(Trivial, by Corollary 2 and Lemma 11).

LEMMA 16. *Let* (b,u) *be a non incident point-plane pair. Let* c *be a point in* $\sigma_0(u)\cap b^{\perp}$. *Then* $1+k_{bc}s$ *is equal to the number of lines in* $\sigma_1(c)\cap L(b,u)$.

(Trivial, by Lemmas 5 and 11).

LEMMA 17. *Let* (b,u) *be an incident point-plane pair and let* c *be a point on* u *different from* b . *Then* $1+k_{bc}$ *is equal to the number of lines* x *through* c *on* u *such that* $\sigma_2(x)\subseteq\sigma_2(b)$.

(Trivial, by Lemmas 6 and 11).

COROLLARY 8. *Let* (b,u) *be a point-plane pair. Then:*

(i) *If* $b*u$ *we have*

$$\sum_{\substack{c*u \\ c\neq b}} (k_{bc}s)^2 = \alpha^2+rs\alpha .$$

(ii) *If* $b\notin\sigma_0(u)$ *we have*

$$\sum_{\substack{c*u \\ c\perp b}} ((k_{bc}s)^2 + k_{bc}s) = \alpha^2 + \alpha .$$

Let us assume that $b*u$. For every point c on u different from b, let X_c be the set of lines x on u, incident with c but not with b and such that $\sigma_2(x) \subseteq \sigma_2(b)$ and let us set

$$X = \bigcup_{\substack{c*u \\ c\neq b}} X_c .$$

The set X_c contains k_{bc} lines, by lemma 17. The set X contains α/s lines, by Lemma 11. Then we have

$$\sum_{\substack{c*u \\ c\neq b}} k_{bc}^2 s = \sum_{\substack{c*u \\ c\neq b}} \sum_{x\in X_c} k_{bc}s = \sum_{x\in X} \sum_{c*x} k_{bc}s =$$

$= (\alpha+rs)\alpha/s$. Then (i) follows.

Let us assume that $b\notin\sigma_0(u)$. By Lemma 16, by the second relation of (ii) of Lemma 15 and by (iii) of Lemma 14 we have:

$$\sum_{\substack{c*u \\ c\perp b}} (k_{bc}s+1)k_{bc}s = \sum_{\substack{c*u \\ c\perp b}} \sum_{\substack{x*c \\ x\in L(b,u)}} k_{bc}s = \sum_{x\in L(b,u)} \sum_{c*x} k_{bc}s =$$

$= (\alpha+1)\alpha$.

And (ii) follows.

Q.E.D.

COROLLARY 9. *We have* $\alpha \leq r^2 - r$ *if* Γ *is not flat.*

If Γ is not flat, there is a non incident point-plane pair (b,u). Given a line x in $L(b,u)$ (we recall that $L(b,u) \neq \emptyset$ by Lemma 1) and given a point c on x, we have:

$$\sum_{\substack{d * x \\ d \neq c}} (k_{db}s)^2 \geq (\alpha - k_{bc}s)^2/r \quad \text{by (iii) of Lemma 14.}$$

Given a line y on u not in $L(b,u)$ and given a point c on y, we have

$$\sum_{\substack{d * x \\ d \perp b \\ d \neq c}} (k_{db}s)^2 \geq (\alpha - r - k_{bc}s)^2/r$$

by (iv) of Lemma 14, because $n_0(b,y) \leq r+1$.

Then, given a point c on u, by (ii) of Corollary 8 we get:

$$(k_{bc}s+1)(\alpha - k_{bc}s)^2/r + (r-k_{bc}s)(\alpha - r - k_{bc}s)^2/r + (k_{bc}s+1)(\alpha - k_{bc}s) +$$

$$+ (r-k_{bc}s)(\alpha - r - k_{bc}s) + (k_{bc}s)^2 + k_{bc}s \leq \alpha^2 + \alpha .$$

By this inequality we get, by easy computations, that

$$(\alpha - k_{bc}s)^2 \leq r^2(\alpha - k_{bc}s).$$

If $\alpha \leq k_{bc}s$, then $\alpha \leq r$ by Lemma 16. We are done.

Let us assume that $\alpha > k_{bc}s$. Then we have $\alpha \leq r^2 + k_{bc}s$. But $k_{bc}s \leq r$ by Lemma 16. Then either $\alpha = r^2+r$, or $\alpha = r^2$ or $\alpha = r^2 - hr$ for some positive integer h, because r divides α by Lemma 3.

Let us assume that $\alpha = r^2+r$. By Lemma 3 we know that, if d is the greatest common divisor of r and s, then rd divides α So d divides r+1 and hence d=1. Thus the least common multiple of r and s is rs. Then s divides r+1 by Lemma 12. But we can have $\alpha = r^2+r$ only if $k_{bc}s = r$ for every choice of c on u. So s divides also r. Then s=1 and Γ is a building by [12] and [1]. Then $\alpha = 0$. We have a contradiction, therefore $\alpha \leq r^2$.

Let us assume that $\alpha = r^2$. By Lemma 3 we have that $1+r^2$ divides $(1+r^2s)(1+r+r^2)$. Then $1+r^2$ divides s-1. Then either s=1 or $1+r^2 \leq$ s-1. But if s = 1 then Γ is a building and $\alpha = 0$. And we have a contradiction. On the other hand, $r^2 \geq s$ by a well known restriction on parameters of generalized quadrangles, because $r \neq 1$ (see [10]). Then $1+r^2 > s-1$. We have a contradiction anyway. Then $\alpha \neq r^2$. Then $\alpha \leq r^2-r$.

Q.E.D.

COROLLARY 10. *Let* Γ *be not a building. Then we have* $n_0(b,x) > 1$ *for every point-line pair* (b,x).

Let (b,x) be a point-line pair. If $\sigma_2(b) \cap \sigma_2(x) \neq \emptyset$, then $n_0(b,x) = r+1$ and there is nothing to prove. Let us assume that $\sigma_2(b) \cap \sigma_2(x) = \emptyset$. By (iv) of Lemma 14 and by Lemma 16 we have $n_0(b,x) \geq (\alpha+1)/(r+1)$. If Γ is not a building, then $r \leq \alpha$ because r divides α by Lemma 3.

Let us assume that $\alpha = r$. Then 1+r divides $(1+r^2s)(1+r+r^2)$ by Lemma 3. So 1+r divides 1+s and hence $r \leq s$. But s divides r by Lemma 12 because $\alpha = r$, so that r=s. Thus Γ is either a building or flat, by [5]. So, $\alpha \neq r$. Then $\alpha > r$ and $n_0(b,x) \geq (\alpha+1)/(r+1) > 1$.

Q.E.D.

Now we give the proofs of Propositions 1 and 3.

PROOF OF PROPOSITION 1. Let Γ be not flat. Then there is a non incident point-plane pair (b,u). If $k_{bc} = 0$ for every point c in $\sigma_0(u)\cap b^\perp$, then $\alpha = 0$ by (iii) of Lemma 14, because $L(b,u)\neq\emptyset$ by Lemma 1. Then Γ is a building. Let us assume that Γ is not a building. Then $k_{bc}s\neq 0$ for some point c in $\sigma_0(u)\cap b^\perp$. By Lemma 16 we have $k_{bc}s\leq r$. Then $s\leq r$. But we have $s\neq r$ by [15] because we have assumed that Γ is neither a building nor flat. Then $s<r$.

Q.E.D.

PROOF OF PROPOSITION 3. Let the property BS hold in Γ. Let Γ be not a building. Then it is almost flat by Corollary 10 and Property BS.

Let us assume that Γ is not flat. Then there is a non incident point-plane pair (b,u). Let c be a point on u. By (iii) and (iv) of Lemma 14 and by the fact that Γ is almost flat we get that

$$\sum_{d*x} k_{bd}s = \alpha$$

for every line x on u through c. By (ii) of Lemma 15 we have

$$\sum_{d*u} k_{bd}s = \alpha(r+1) - r^2$$

because Γ is almost flat. But we have also

$$\sum_{d*u} k_{bd}s = k_{bc}s + \sum_{\substack{x*c\\ x*u}} \sum_{\substack{d*x\\ d\neq c}} k_{bd}s = k_{bc}s+(r+1)(\alpha-k_{bc}s).$$

Then $(r+1)\alpha-rk_{bc}s = (r+1)\alpha -r^2$. Then $k_{bc}s= r$. Then $\alpha = r^2+r$ by (iii) or (iv) of Lemma 14. But we have $\alpha \leq r^2-r$ by Corollary 9.

We have the contradiction.

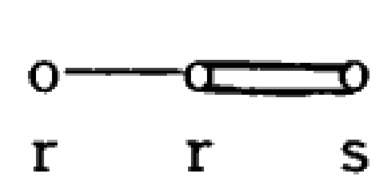
Q.E.D.

REMARK. There are not many chances to improve the inequality of Corollary 9 working only with relations (i) - (iv) of Lemma 14 and (i) and (ii) of Corollary 8. For instance, we can try to maximize the sum of (ii) of Corollary 8, rather than minimize it as we have done in the proof of Corollary 9. We can substitute in that sum k_{bc}s with r (by Lemma 16). We get the inequality $(r^2+r)^2 + (r^2+r) \geq \alpha^2+\alpha$, that again leads to the inequality $\alpha \leq r^2+r$ that can be improved up to $\alpha \leq r^2-r$. We can try to use tricks more refined than the mere substitution of k_{bc}s with r, of course. But we do not make any real progress. Things are even worse with (i) of Corollary 8.

5.ON THE PROPERTY BS.

In this section Γ is finite, the property BS holds in it and all lines of Γ are thick.

The geometry Γ admits parameters



because it has thick lines.

LEMMA 18. *Let* (b,u) *be a point-plane pair such that* $n_1(b,u) = 1$. *Then we have* $n_1(b,v) = 1$ *for every plane* v *not incident with* b.

Let us assume that $n_1(b,v) \neq 1$ for some plane v not incident with b, by contradiction. Then we have $n_0(b,v) = 1+r+r^2$ by Lemma 7, Corollary 5 and by the property BS.

The geometry Γ is strongly connected. Then we find a sequence $(v_0,v_1,\dots,v_n)$ of planes and a sequence $(x_1,x_2,\dots,x_n)$ of lines such that $v = v_0*x_1*x_1*,\dots*x_n*v_n = u$. Let v_{i-1} be the last plane in the sequence $(v_0,v_1,\dots v_n)$ that does not pass through b and such that $n_1(b,v_{i-1})\neq 1$. If v_i does not pass through b, then we have $\sigma_0(x_i) = \sigma_0(v_i)\cap b^\perp$ by Lemma 7 because $n_1(b,v_i)=1$. Then we have $\sigma_0(v_i)\cap b^\perp \subseteq \sigma_0(v_{i-1})\cap b^\perp$. Then $\sigma_0(v_i)\cap b^\perp = \sigma_0(v_{i-1})\cap b^\perp$ by Lemma 8. We have the contradiction $\sigma_0(v_{i-1})\cap b^\perp = \sigma_0(x_i)$. Then v_i passes through b. Let v_j be the first plane in the sequence $(v_{i+1},v_{i+2},\dots v_n)$ that does not pass through b. Then all planes $v_{i+1},v_{i+2},\dots v_{j-1}$ pass through b and we have $n_1(b,v_j)=1$ by the previous choice of v_{i-1}. We have to distinguish two cases.

Case 1. We have $j=i+1$. Let c be the intersection point in Γ_{v_i} of x_i and x_{i+1}. In Γ_c we find two lines y_1,y_2 and a plane w such that y_1 lies on both v_{i-1} and w, y_2 lies on both v_{i+1} and w, and $y_2\neq x_{i+1}$.

Let us assume that $b*w$. We have $\sigma_0(x_{i+1}) \subseteq b^\perp$ because $b*v_i$. Then $\sigma_0(x_{i+1}) = \sigma_0(v_{i+1})\cap b^\perp$ because $n_1(b,v_{i+1}) = 1$. But we have also $\sigma_0(y_2) \subseteq b^\perp$ because $b*w$. Then we have the contradiction $y_2=x_{i+1}$ because $n_0(b,v_{i+1}) = r+1$. Then $b\notin\sigma_0(w)$. Now we have a contradiction again, as in the case when v_i did not pass through b, discussed above. Indeed, if $n_0(b,w) = 1$, then w plays the role of v_i in that argument. If $n_0(b,w) = 1+r+r^2$, then w plays the role of v_{i-1} and v_{i+1} palys the role of v_i in that argument. The contradiction is got anyway. Then we are led to the second case.

Case 2. We have $j>i+1$. Let c be the intersection point of x_j and x_{j-1} in $\Gamma_{v_{j-1}}$.

In Γ_c there are lines y_1, y_2 and a plane w such that y_1 lies on both w and v_{j-2}, y_2 lies on both w and v_j and $y_2 \neq x_j$. We have $\sigma_0(x_j) \subseteq b^{\perp}$ because $x_j * v_{j-1}$ and $b * v_{j-1}$. Then $\sigma_0(x_j) = \sigma_0(v_j) \cap b^{\perp}$ because $n_0(b,v_j)=r+1$. Then $b \notin \sigma_0(w)$. Otherwise we have $\sigma_0(y_2) \subseteq b^{\perp}$ and we get the contradiction $y_2 = x_j$. We have $n_1(b,w)=1$. Otherwise we get a contradiction as in the case when v_i did not pass through b, putting w into the role of v_{i-1} and v_j into the role of v_i in that argument.

Then we can substitute x_{j-1} with y_1, v_{j-1} with w and w_j with y_2 in the sequence of planes and lines

$$(v_0, x_1, \ldots v_{i-1}, x_i, v_i, \ldots v_{j-2}, x_{j-1}, v_{j-1}, x_j, v_j, \ldots x_n, v_n).$$

Now w plays the role that was played by v_j. Iterating this argument we finally get the situation of Case 1. And we have the contradiction.

Q.E.D.

The proof of Proposition 2 is quite trivial now.

PROOF OF PROPOSITION 2. If there is some point-plane pair (b,u) such that $n_1(b,u) = 1$, then $n_1(b,v) = 1$ for every plane v not incident with b, by Lemma 18. The geometry Γ is a building by Corollary 7.

Otherwise we have $n_0(b,u)=1+r+r^2$ for every point-plane pair (b,u). Then Γ is almost flat.

Q.E.D.

BIBLIOGRAPHY

[1] A.BROUWER & A.COHEN: Some remarks on Tits' geometries, *Ind. Math.* 45 (1983), 393-402.

[2] F.BUEKENHOUT & E.SHULT: On the foundations of polar geometry, *Geom. Ded.* 3(1974), 155-170.

[3] F.BUEKENHOUT & W.SCHWARZ: A simplified version of strong connectivity in geometries, *J.Comb.Th.A*, 37, (1984), 73-75.

[4] R.LIEBLER: A representation theoretic approach to finite geometries of spherical type, to appear.

[5] U.OTT: On finite geometries of type B_3, *J.Comb.Th.A*, 39(1985), 209-221.

[6] U.OTT: *Some remarks on representation theory in finite geometry, in "Geometry and groups"*, L.N. 893, Springer (1981).

[7] A.PASINI: On Tits geometries of type C_n, to appear in *Europ.J. Comb.*

[8] A.PASINI: Some results on Tits geometries of type F_4, *Note Mat.* V (1985), 51-81.

[9] A.PASINI: On geometries of type C_3 that are either buildings or flat, to appear in *Bull. Soc. Math. Belgique.*

[10] S.PAYNE & J.THAS: *Finite generalized quadrangles*, Pitman, Boston 1984.

[11] S.REES: Finite C_3 geometries in which all lines are thin, *Math.Z.*, 189 (1985) 263-271.

[12] S.REES: A classification of a class of C_3 geometries, to appear in *J.Comb.Th.*

[13] S.REES: *On diagram geometry*, Ph.D.Thesis, Oxford, May 1984.

[14] J.TITS: *Buildings of spherical type and finite BN-pairs*, L.N.386, Springer, 1984.

[15] J.TITS: *A local approach to buildings, in "The geometric vein"*, Springer 1981, 519-547.

Antonio PASINI
Dipartimento di Matematica, Univesità di Siena
Via del Capitano 15,
53100 SIENA (Italy)