Note di Matematica Vol. VI, 155-203(1986)

# SOME DENSE BARRELLED SUBSPACES OF BARRELLED SPACES WITH DECOMPOSITION PROPERTIES

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1. INTRODUCTION. There are many results on the barrelledness of subspaces of a barrelled topological vector space X. For example, by M.Valdivia [14], Theorem 3. and independently by S.A.Saxon and M.Levin [11], every subspace of countable codimension in X is barrelled.

Recalling that locally convex Baire spaces are always barrelled, it is an interesting fact that every infinite dimensional Banach space contains a dense barrelled subspace which is not Baire, by S.A.Saxon [10]. From the point of view of constructing dense barrelled subspaces' L of a barrelled space X the smaller subspaces L are the more interesting ones by the simple fact that if L is dense and barrelled so are all subspaces M between L and X. For barrelled sequence spaces X some known constructions of dense barrelled subspaces use "thinness conditions" on the spacing of the non-zero terms $x_n$ in the sequences $(x_n)_{n\in\mathbb{N}}\in X$. Section 3 of this article contains a unification and generalization of these constructions. which we describe now in the special case of sequence spaces.

Let X be a sequence space over the field $\mathbb{K}$ of real or complex numbers. For every $x=(x_n)_{n\in\mathbb{N}}\in X$ and $k\in\mathbb{N}:=\{1,2,3,...\}$ let $g_k(x)$ denote the number of indices $n\in\{1,2,...,k\}$ such that $x_n\neq 0$. Clearly the "thinness condition"

$$(\Theta) \qquad \lim_{k\to\infty} \frac{g_k(x)}{k} = 0$$

on the sequence of non-zero components of x defines a linear subspace L of X. If X is the Banach space $l^1$, then L is a proper dense barrelled subspace of X by G.Köthe. [7], [8], §27.1, where this fact is deduced essentially from the following simple lemma.

1.1. LEMMA. ***A subspace Y of a (locally convex Hausdorff) barrelled space X is dense and barrelled if and only if in the topological dual X' of X every $\sigma(X',Y)$-bounded sequence (or, equivalently, set) is $\sigma(X',X)$-bounded.***

Lateron, thinness condition (0) and variants thereof were used for similar purposes by J.H.Webb [16] in his Lemma D.(1) on perfect sequence spaces and by M.Valdivia [13], Theorem 1 and by M.Valdivia and P.P.Carreras [15], Example 1. Webb's Lemma D.(1) reads as follows.

LEMMA D.(1). ***If X is a perfect sequence space which is barrelled under the Mackey topology $\tau(X,X^\times)$, where $X^\times$ denotes the $\alpha$-dual of X in the sense of G. Köthe [7], [8], §30, then the subspace L of X defined by (0) is barrelled under the topology $\tau(L,X^\times)$.***

We shall deduce this result from our Corollary 3.4, which says (roughly speaking): Suppose that X is a projection invariant barrelled sequance space such that $x = \sum_{n=1}^{\infty} x_n e_n$ for all $x=(x_n)_{n\in\mathbb{N}}\in X$, where $e_n = (\delta_{nk})_{k\in\mathbb{N}}$. Then the corresponding subspace L and even a certain smaller dense subspace M of L are barrelled. This result is a special case of our general main Theorem 3.1 in which we assume a kind of decomposition, called pseudodecomposition which is more general than a Schauder decomposition.

The notion of pseudodecomposition is introduced and discussed in Section 2. A ***pseudodecomposition of*** an l.c. (locally convex Hausdorff topological vector) space X is a pair $((X_n)_{n\in\mathbb{N}},D)$ where the $X_n$ $(n\in\mathbb{N})$ are (not necessarily closed) subspaces of X and $D \subset \prod_{n\in\mathbb{N}} X_n$ is a subspace such that $\sum_{n=1}^{\infty} x_n$ converges in X for all $(x_n)_{n\in\mathbb{N}}\in D$ and such that every $x\in X$ has an expansion $x = \sum_{n=1}^{\infty} x_n$ for some $(x_n)_{n\in\mathbb{N}}\in D$. (This expansion need not be unique.) Obviously every Schauder decomposition of X gives rise to an associated pseudodecomposition. The pseudodecomposition $((X_n)_{n\in\mathbb{N}},D)$ is called ***projection invariant*** if $(\varepsilon_n x_n)_{n\in\mathbb{N}} \in D$ for all $(x_n)_{n\in\mathbb{N}} \in D$ and $(\varepsilon_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$. Projection invariance is characterized in Lemma 2.4 by a summability property, and its relation with the equicontinuity of an associated set of projectors is described in Lemma 2.5 for Schauder decompositions $(X_n)_{n\in\mathbb{N}}$.

Large parts of Section 2 are devoted to demonstrating in a series of examples the very broad generality of the concept of pseudodecomposition. The Main Example 2.8 proves that there even exist Banach sequence spaces X that have a projection invariant pseudodecomposition $((\mathbb{K}e_n)_{n\in\mathbb{N}},D)$ with $\bigoplus_{n\in\mathbb{N}} \mathbb{K}e_n \subset D$ and with unique expansions but such that $(\mathbb{K}e_n)_{n\in\mathbb{N}}$ is not a decomposition of X in the usual sense.

The ***main result*** of Section 3 is Theorem 3.1 stating that if X is a barrelled (l.c.) space with a projection invariant pseudodecomposition $((X_n)_{n\in\mathbb{N}},D)$, then also the dense subspace L of X defined by thinness condition (8) and even a certain smaller dense subspace $M \subset L$ are barrelled. The scope of Theorem 3.1 and Corollaries 3.3.3.4 is analyzed in Sections 4 and 5 by means of some classes of examples on classica1 Banach or Fréchet spaces. In particular, we shall show:

(a) The hypothesis of projection invariance cannot be dropped in Theorem 3.1. and the hypothesis on the equicontinuity of the set of projectors $q_J$ ($J \subset \mathbb{N}$ finite) cannot be omitted in Corollary 3.3.

(b) There exist special Banach spaces with Schauder basis for which L and even a smaller subspace are barrelled although projection invariance does not hold.

(c) There are other thinness conditions more restrictive than (8) which yield dense barrelled subspaces for some special Banach or Fréchet spaces with Schauder decomposition but not for others.

(d) Thinness condition $(\theta)$ is in some sense best possible (cf. Theorem 5.6).

## 2. PROJECTION INVARIANT PSEUDODECOMPOSITIONS.

We define pseudodecompositions of an l.c. (locally convex Hausdorff topological vector) space $X$ by weakening the familiar defining conditions on Schauder decompositions.

2.1. DEFINITION. A *pseudodecomposition* of an l.c. space $X$ is a pair $((X_n)_{n\in\mathbb{N}},D)$ where the $X_n$ are (not necessarily closed) subspaces of $X$ and $D$ is a subspace of

$$(2.1)\qquad E := \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \sum_{n=1}^{\infty} x_n \text{ converges}\}$$

such that, for all $x\in X$ there is a sequence $(x_n)_{n\in\mathbb{N}} \in D$ with $\sum_{n=1}^{\infty} x_n = x$. Here, $(x_n)_{n\in\mathbb{N}}$ will be called an *expansion* of $x$ (in $D$). The set of all expansions of $x$ in $D$ is denoted by $Dx$.

Our concept of pseudodecomposition goes over into that of 1. Singer [12], p.538, Definition 15.21, by considering only Banach spaces for $X$, closed subspaces for $X_n$, and $D=E$. We will discuss this rather general concept in some detail. For all $m\in\mathbb{N}$ let $pr_m : \prod_{n=1}^{\infty} X_n \to X_m$, $pr_m((x_n)_{n\in\mathbb{N}}) := x_m$, denote the m-th canonical projection. A given pseudodecomposition $((X_n)_{n\in\mathbb{N}},D)$ may always be "reduced" to the pseudodecomposition $((pr_n(D))_{n\in\mathbb{N}},D)$. Definition 2.1 does not require that $\bigoplus_{n\in\mathbb{N}} pr_n(D)$ lies in $D$. - Consider now the canonical linear surjection $f: D \to X$, $f((x_n)_{n\in\mathbb{N}}) := \sum_{n=1}^{\infty} x_n$. If $\bigoplus_{n\in\mathbb{N}} pr_n(D)$ lies in $D$, then its image under $f$ is dense in $X$. The set $DD$ of all expansinns of $0$ in $D$ is the kernel of $f$. Every element $x\in X$ has a unique expansion $(x_n)_{n\in\mathbb{N}}$ in D if and only if $D0=(0)$. If $((X_n)_{n\in\mathbb{N}},D)$ is a pseudodecomposition with unique expansions, one has the associated projectors

$$(2.2)\qquad q_n : X \to X,\ q_n(x) := x_n \quad (n \in \mathbb{N}),\ \text{and}$$

$$q_J := \sum_{n\in J} q_n \quad \text{for finite } J \subset \mathbb{N}\ .$$

$D = E$ and uniqueness of the expansions together mean that $(X_n)_{n\in\mathbb{N}}$ is a *decomposition of* X in the sense of J. Marti [9], Chapter VII; and if, in addition, the projectors $q_n$ are continuous one obtains exactly the *Schauder Secompositions* $(X_n)_{n\in\mathbb{N}}$ of X (with $X_n = \{0\}$ admitted). The (Schauder) decompositions of X can in this way be considered as special pseudodecnmpositions. Applying the abnve to X endowed with the weak topology $\sigma(X,X')$ one nbtains weak pseudodecompositions and weak (Schauder) decompositions. Patently, every pseudodecomposition is a weak pseudodecomposition, and it is easy to see that every Schauder decompnsition is a weak Schauder decomposition. Conversely, for a barrelled space X, every weak Schauder decomposition is a Schauder decompositinn (cf. J. Marti [9], p.128, Theorem 5).

If $((X_n)_{n\in\mathbb{N}},D)$ is a pseudodecomposition of X with $\bigoplus_{n\in\mathbb{N}} X_n \subset D$, with unique expansions and cnntinuous projectors $q_n$ (cf. (2.2)), then it is a Schauder decomposition. (To prove $D = E$, let $(y_n)_{n\in\mathbb{N}}\in E$. Put $x := \sum_{n=1}^{\infty} y_n$ and consider its expansion $(x_n)_{n\in\mathbb{N}} \in Dx$. Then, for all $n,r \in \mathbb{N}$ and $r \geq n$, the sequence $(y_1,\ldots,y_r,0,0,\ldots)$ is in D, and the continuity of $q_n$ implies $y_n=q_n(\sum_{i=1}^{r} y_i)=q_n(\sum_{i=1}^{\infty} y_i)=$

$= q_n(x) = x_n$, hence $(y_n)_{n\in\mathbb{N}} \in D$.)

2.2. EXAMPLE. *There exists a Banach sequence space $X$ with pseudodecomposition $((\mathbb{K}e_n)_{n\in\mathbb{N}}, D)$ with $e_n := (\delta_{nm})_{m\in\mathbb{N}}$, $\bigoplus_{n=1}^{\infty} \mathbb{K}e_n \subset D$ and with unique expansions, but such that $(\mathbb{K}e_n)_{n\in\mathbb{N}}$ is not a decomposition of $X$, i.e. such that $D \neq E$.*

(a) Let $Z$ be an l.c. space with a topological basis $(b_n)_{n\in\mathbb{N}}$. That is, for every $z \in Z$ there exists exactly one sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ such that $z = \sum_{n=1}^{\infty} \lambda_n b_n$. (If, in addition, $z \mapsto \lambda_n$ is continuous for all $n \in \mathbb{N}$, then $(b_n)_{n\in\mathbb{N}}$ is called a Schauder basis.) Let $N \subset Z$ be a closed subspace such that $N \neq \{0\}$ and $N \cap \operatorname{span}\{b_n : n\in\mathbb{N}\} = \{0\}$. Let $Y$ be a complementary subspace of $N$ in $Z$ containing $\operatorname{span}\{b_n : n\in\mathbb{N}\}$. Thus $Y$ is a proper dense subspace of $Z$. Transferring the Hausdorff quotient topology from $Z/N$ to $Y$ by means of the canonical isomorphism $Y \cong Z/N$ one obtains an l.c. space $X$ with the same underlying vector space as $Y$. The topology of $X$ is strictly coarser than that of $Y$, since for $a \in N$, $a \neq 0$ and $a = \sum_{n=1}^{\infty} \lambda_n b_n$ in $Z$, the series $\sum_{n=1}^{\infty} \lambda_n b_n$ is not convergent in $Y$, but it is convergent in $X$ with sum $0$. Consider now the decomposition $(\mathbb{K}b_n)_{n\in\mathbb{N}}$ of $Y$ as a pseudodecomposition $((\mathbb{K}b_n)_{n\in\mathbb{N}}, D)$ of $Y$, where

$$D := \{(x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} \mathbb{K}b_n : \sum_{n=1}^{\infty} x_n \text{ converges in } Y\} .$$

This is at the same time a pseudodecomposition of $X$ with unique expansions in $D$. (Uniqueness holds since for $(y_n)_{n\in\mathbb{N}} \in D$ the sum $\sum_{n=1}^{\infty} y_n$ in $X$ agrees with the sum in $Y$, and $(\mathbb{K}b_n)_{n\in\mathbb{N}}$ is a decomposition of Y.) However. it is not a decomposition of X: The space E from (2.1.) for this pseudodecomposition is strictly larger than D since E contains the subspace

$$F := \{ (z_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} \mathbb{K} b_n : \sum_{n=1}^{\infty} z_n \text{ converges in } Z \}.$$

If we use fnr Z a barrelled space then X is also barrelled; and if we choose Z as a Banach sequence space with Schauder basis $(b_n)_{n\in\mathbb{N}} = (e_n)_{n\in\mathbb{N}}$, then X and $(\mathbb{K} e_n)_{n\in\mathbb{N}}, D)$ have the required properties. If $Z = l^2$, then a Hilbert space structure is induced on X by that of $l^2/N$.

Returning to the case of an l.c. space Z, we note that the projectors $q_n$ (see (2.2)) for our pseudodecomposition of X are not all continuous by the remark before 2.2. (This also becomes obvious if one takes $a \in N$, $a \neq 0$, $a = \sum_{n=1}^{\infty} a_n$ in Z with $a_n \in \mathbb{K} b_n$ $(n \in \mathbb{N})$, chooses $m \in \mathbb{N}$ such that $a_m \neq 0$ and notes that $\sum_{n=1}^{\infty} a_n = 0$ in X. but $q_m(\sum_{n=1}^{r} a_n) = a_m$ for all $r \geq m$.)

(b) We now compare the spaces E and F of the above construction more closely. Clearly we have the interna1 algebraic direct **sums** $E = D \oplus E_0$ and $F = D \oplus G$ where

$$E_0 := \{(x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} \mathbb{K} b_n : \sum_{n=1}^{\infty} x_n = 0 \text{ in } X\},$$

$$G := \{(x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} \mathbb{K} b_n : \sum_{n=1}^{\infty} x_n \text{ converges in } Z, \text{ with sum in } N\},$$

and clearly $G \subset E_0$. Therefore F = E holds if nnd only if $G = E_0$. We assume now that $(b_n)_{n\in\mathbb{N}}$ is a Schauder basis of Z, and we claim:

If dim N is finite, then $F=E$, i.e., $G=E_0$; and if dim N is infinite and $Z=l^p$ with $1 \leq p < \infty$, one may have $F \neq E$.

1. Let dim N be finite and $(x_n)_{n\in\mathbb{N}} \in E_0$. To prove $(x_n)_{n\in\mathbb{N}} \in G$ we may assume that $(x_n)_{n\in\mathbb{N}} \neq 0$. Let C be the compact unit sphere for some norm on N, and let $\mathcal{U}_0(Z)$ denote the filterof all neighbourhoods of 0 in Z. For every $U\in\mathcal{U}_0(Z)$ we choose a natural number $r_U$ such that, for all $r \geq r_U$, there exist some elements $z_{U,r} \in C$ and $\lambda_{U,r} \in \mathbb{K}$ such that

$$(2.3) \qquad \sum_{n=1}^{r} x_n - \lambda_{U,r} z_{U,r} \in U.$$

The set $T := \{(U,r) \in \mathcal{U}_0(Z) \times \mathbb{N} : r \geq r_U\}$ is directed by $(U,r) \leq (U',r')$ if and only if $U \supset U'$ and $r \leq r'$. The net $(z_{U,r})_{(U,r)\in T}$ in C has a subnet $(z_{U_s,r_s})_{s\in S}$ converging to some point $c\in C$. Applying now to (2.3) the continuous projectors $q_n$ for the Schauder decomposition $(\mathbb{K}b_n)_{n\in\mathbb{N}}$ of Z, one sees that the subnet $(\lambda_{U_s,r_s})_{s\in S}$ of $(\lambda_{U,r})_{(U,r)\in T}$ converges to some $\lambda\in\mathbb{K}$ and that $x_n = \lambda q_n(c)$ for all $n\in\mathbb{N}$, i.e., $\sum_{n=1}^{\infty} x_n = \lambda c$ in Z. Hence $(x_n)_{n\in\mathbb{N}} \in G$, and this proves our first assertion $G=E_0$.

2. Let $Z=l^p$ with $1 \leq p < \infty$ and $b_n = e_n$ $(n\in\mathbb{N})$. We define a closed subspace N of Z such that $E \neq F$ as follows: Let $(I_k)_{k\in\mathbb{N}}$ be a partition of $2\mathbb{N}$ into infinite subsets Ik and let $d_k := \sum_{n\in I_k} 2^{-n} e_n$ $(k\in\mathbb{N})$. We define N to be the closure of span $\{a_n : n\in\mathbb{N}\}$ in Z. where $a_1 := e_1 - d_1$

and $a_n:=e_{2n-1}+d_{n-1}-d_n$ for $n\geq 2$. The space N consists of all elements of the form $\sum_{n=1}^{\infty}\lambda_n a_n$ with $(\lambda_n)_{n\in\mathbb{N}}\in l^p$, and $(\lambda_n)_{n\in\mathbb{N}}\to\sum_{n=1}^{\infty}\lambda_n a_n$ is a topologica1 isomorphism of $l^p$ onto N, whence $(a_n)_{n\in\mathbb{N}}$ is a Schauder basis of N. Obviously $N\cap\operatorname{span}\{e_n:n\in\mathbb{N}\}=\{0\}$ since the $I_k$ are infinite and disjoint. Choose a complementary subspace Y of N in $l^p$ which contains span $\{e_n: n\in\mathbb{N}\}$ and def ine X as in part (a). Then we see from $e_1+e_3+\ldots+e_{2n-1}=a_1+a_2+\ldots+a_n+d_n$ that $\sum_{k=1}^{\infty}e_{2k-1}$ converges in X with sum 0. Hence the sequence $(\frac{1}{2}(1-(-1)^n)e_n)_{n\in\mathbb{N}}$ belongs to $E_0$ but not to F. This proves nur secnnd assertion under (b).

We shall imprnve Example 2.2 in nur Main Example 2.8 by making the pseudndecnmpnsitinn in addition, prnjectinn invariant (cf. Definition 2.3). Mnst pseudodecompositions $((X_n)_{n\in\mathbb{N}},D)$ to be needed later will be invariant under the canonica1 prnjectnrs

(2.4)
$$p_J:\prod_{n\in\mathbb{N}}X_n\to\prod_{n\in\mathbb{N}}X_n\qquad(J\subset\mathbb{N}),$$
$$p_J((x_n)_{n\in\mathbb{N}}):=(y_n)_{n\in\mathbb{N}},\ \text{where } y_n=x_n \text{ fnr } n\in J \text{ and } y_n=0 \text{ fnr } n\notin J.$$

2.3. DEFINITION. A pseudndecnmpnsitinn $((X_n)_{n\in\mathbb{N}},D)$ of an l.c. space X is called ***projection invariant*** if D is prnjectinn invariant, i.e., if $p_J(D)\subset D$ fnr all $J\subset\mathbb{N}$. $((X_n)_{n\in\mathbb{N}},D)$ is called *unconditional* if all series $\sum_{n=1}^{\infty} x_n$ with $(x_n)_{n\in\mathbb{N}}\in D$ are unconditionally convergent.

A projection invariant pseudodecomposition contains always the projection invariant subspace $\bigoplus_{n\in\mathbb{N}} pr_n(D)$ and one may reduce it by replacing the spaces $X_n$ by $pr_n(D)$.

If $((X_n)_{n\in\mathbb{N}}, D)$ is a projection invariant pseudodecomposition of X with unique expansions $x \leftrightarrow (x_n)_{n\in\mathbb{N}}$, there exist the projectors

(2.5) $q_J : x \to X,\ q_J(x) = \sum_{n=1}^{\infty} y_n$ with $y_n$ from (2.4), for all $J\subset\mathbb{N}$,

extending (2.2). If X is an l.c. space with a projection invariant pseudodecomposition $((X_n)_{n\in\mathbb{N}}, D)$, then for all $(x_n)_{n\in\mathbb{N}}\in D$, the series $\sum_{n=1}^{\infty} x_n$ is subseries convergent. By M.M. Day [1], p. 78, this is equivalent to saying that the sequence $(x_n)_{n\in\mathbb{N}}$ and all its subsequences are summable, or equivalently, that $\sum_{n=1}^{\infty} x_n$ and all its subseries are unconditionally convergent. By a form of the Dunford-Pettis Theorem (cf. H.Jarchow [5], p.308, Theorem 4), these statements about convergence and summability are equivalent to the corresponding ones with respect to the weak topology of X. Thus the nntinns of prnjectinn invariant pseudndecnmpnsitinn and of prnjectinn invariant weak pseudndecnmpnsitinn are equivalent. Alsn by the above, every prnjectinn invariant pseudndecompnsitinn is uncondi-tinnal.

The next lemma shows that under a mild cnmpleteness assumptinn on X, the prnjectinn invariance of a pseudndecnmpnsitinn $((X_n)_{n\in\mathbb{N}}, E)$ with E frnm (2.1) may be characterized by the summability of the sequences $(x_n)_{n\in\mathbb{N}}\in E$. We call a series $\sum_{n=1}^{\infty} y_n$ with $y_n\in X$ $(n\in\mathbb{N})$ a *Cauchy series* if its partial sums form a Cauchy sequence. A pseudodecomposition $((X_n)_{n\in\mathbb{N}},D)$ with $D=E$ is called *complete* if

(2.6) every Cauchy series $\sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ $(n\in\mathbb{N})$ converges

(and hence $(x_n)_{n\in\mathbb{N}}\in E$). This condition is trivially satisfied if X is sequentially complete. For Schauder decompositions it has been introduced by N.Kalton [6], p.35.

2.4.LEMMA. *A complete pseudodecomposition $((X_n)_{n\in\mathbb{N}},D)$ with $D=E$ (cf.(2.1)) of an l.c. space X is projection invariant if and only if all sequences $(x_n)_{n\in\mathbb{N}}\in E$ are summable, i.e., if and only if it is unconditional.*

Proaf. The condition is necessary as remarked after (2.5). Now let it be fulfilled. Let $(x_n)_{n\in\mathbb{N}}\in E$, $J\subset\mathbb{N}$ and $(y_n)_{n\in\mathbb{N}} :=$ $= p_J((x_n)_{n\in\mathbb{N}})$. For every neighbourhood U of 0 in X there is a finite set $I\subset\mathbb{N}$ such that for all finite sets $K\subset\mathbb{N}$ with $K\cap I=\emptyset$ one has $\sum_{n\in K} x_n \in U$. It follows that also $\sum_{n\in K} y_n\in U$ for these K. So the series $\sum_{n=1}^{\infty} y_n$ is Cauchy and by (2.6) converges, i.e., $(y_n)_{n\in\mathbb{N}} \in E$.

Consequently. if a topological basis $(b_n)_{n\in\mathbb{N}}$ of an l.c. space X yields a complete (pseudo-) decomposition $((\mathbb{K}b_n)_{n\in\mathbb{N}},E)$ of X, one can say that this (pseudo-) decomposition is projection invariant if and only if $(b_n)_{n\in\mathbb{N}}$ is an unconditional basis.

In the next lemma we give a relationship between projection invariance of a Schauder decomposition $(X_n)_{n\in\mathbb{N}}$ of an l.c. space X and equicontinuity of the set

(2.7) $$P := \{q_J : J \subset \mathbb{N} \text{ finite}\}$$

of continuous projectors $q_J$ from (2.2). We recall that by D.G.H. Garling [3], p.1001, Theorem 2. P is equicontinuous if and only if X has a base $\mathcal{B}$ of neighbourhoods of 0 such that $q(U) \subset U$ for all $q \in P$ and $U \in \mathcal{B}$.

**2.5. LEMMA.** *Let X be an l.c. space with Schauder decomposition $(X_n)_{n\in\mathbb{N}}$. If X is barrelled and $(X_n)_{n\in\mathbb{N}}$ is projection invariant, then the set P from (2.7) is equicontinuous, or, equivalently, $\{q_J : J \subset \mathbb{N}\}$ (cf. (2.5)) is equicontinuous. If P is equicontinuous and $(X_n)_{n\in\mathbb{N}}$ is complete (cf. (2.6)). then $(X_n)_{n\in\mathbb{N}}$ is projection invariant.*

Proof. To prove the first statement, suppose that P is not equicontinuous. Then. as X is barrelled, P is not pointwise bounded. So there exist some $x\in X$ with expansion $(x_n)_{n\in\mathbb{N}}$, and $u\in X'$ such that $\{u(q_J(x)) : J\subset\mathbb{N} \text{ finite}\}$ is unbounded. Since $\{u(q_J(x)) : J\subset\{1,2,\dots,m\}\}$ is bounded for each $m \in \mathbb{N}$, one obtains inductively a sequence $(J_k)_{k\in\mathbb{N}}$ of finite sets $J_k \subset \mathbb{N}$, $J_k \neq \emptyset$, such that $\max J_k < \min J_{k+1}$ and

$$|u(q_{J_k}(x))| = |\sum_{n\in J_k} u(x_n)| \geq k \qquad (k\in\mathbb{N}).$$

This means that the subseries of $\sum_{n=1}^{\infty} x_n$ corresponding to $J := \bigcup_{k=1}^{\infty} J_k$

is not a Cauchy series, whence $p_J((x_n)_{n\in\mathbb{N}})\notin E$, contrary to the assumption.

To prove the second statement, let $(x_n)_{n\in\mathbb{N}}\in E$ and $J\subset\mathbb{N}$. To show that $(y_n)_{n\in\mathbb{N}}:=p_J((x_n)_{n\in\mathbb{N}})\in E$, let $U$ be a neighbourhood of 0 in X. There exists a neighbourhood V of 0 in X such that $q_K(V)\subset U$ for all finite $K\subset\mathbb{N}$. Then there exists some $m\in\mathbb{N}$ such that $\sum_{n=r}^{s} x_n\in V$ for $m<r\leq s$. Applying the map $q_{\{r,...,s\}\cap J}$ one obtains $\sum_{n=r}^{s} y_n\in U$ for $m\leq r\leq s$. This means that $\sum_{n=1}^{\infty} y_n$ is a Cauchy series and hence converges since $(X_n)_{n\in\mathbb{N}}$ is complete.

A theorem of N.Kalton [6], p. 35 implies that a barrelled space with a complete Schauder decomposition $(X_n)_{n\in\mathbb{N}}$ is complete (resp. quasicomplete, resp. sequentially complete) if and only if the spaces $X_n$ $(n\in\mathbb{N})$ have these properties, respectively.

In Example 4.1 we show that the classical Banach space $bv_0$ has a Schauder basis $(b_n)_{n\in\mathbb{N}}$ such that the associated Schauder decomposition $(\mathbb{K}b_n)_{n\in\mathbb{N}}$ is not projection invariant.

2.6. EXAMPLE. In every separable metrizable and complete l.c. space $X\neq\{0\}$ one has the projection invariant pseudodecompositions $((X_n)_{n\in\mathbb{N}},D)$, where $(X_n)_{n\in\mathbb{N}}$ is any sequence of one-dimensional subspaces of X with $\bigcup_{n=1}^{\infty} X_n$ dense in X and

$$D:=\{(x_n)_{n\in\mathbb{N}}\in\prod_{n=1}^{\infty} X_n:(x_n)_{n\in\mathbb{N}} \text{ is subsequence summable}\}.$$

Certainly D is a projection invariant subspace of $\prod_{n=1}^{\infty} X_n$. Let d be a translation invariant metric for X. Then, for every $x\in X$, one finds inductively a strictly increasing sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ and $x_{n_j} \in X_{n_j}$ such that

$$d(x, \sum_{j=1}^{k} x_{n_j}) < 2^{-k} \qquad (k \in \mathbb{N}).$$

Then $d(x_{n_{k+1}}, 0) < 2^{1-k}$ and hence $(x_{n_j})_{j\in\mathbb{N}}$ is subsequence summable with sum x. Putting $x_n := 0$ for all $n \in \mathbb{N}\setminus\{n_j : j\in\mathbb{N}\}$ one has $(x_n)_{n\in\mathbb{N}} \in D$ and $x = \sum_{n=1}^{\infty} x_n$, as desired. - Plainly, the expansions in D are not unique.

Familiar examples of l.c. spaces with projection invariant Schauder decompositions are the products and the locally convex direct sums of l.c. spaces and the $l^p$-sums of normed spaces ($1 \leq p < \infty$). J.Marti [9], p.91, Corollary 3 gives an example of a projection invariant decomposition of the Banach space $l^\infty$ which is not a Schauder decomposition. A similar construction works for many separable metrizable spaces X:

2.7. EXAMPLE. Let X be a separable metrizable l.c. space which has a closed subspace Y with a projection invariant Schauder decomposition $(Y_n)_{n\in\mathbb{N}}$ such that $Y_n \neq \{0\}$ ($n \in \mathbb{N}$). Then $(Y_{2n})_{n\in\mathbb{N}}$ is a projection invariant Schauder decomposition of $Z := \overline{\text{span} \bigcup_{n\in\mathbb{N}} Y_{2n}}$. Since X is metrizable and separable and Z has infinite codimension, Z possesses a dense complementsry subspace $X_1 \neq X$. Letting $X_n := Y_{2n-2}$ for $n \geq 2$, it is easily seen that $(X_n)_{n\in\mathbb{N}}$ is a projection invariant decomposition of X, but not a Srhsuder decomposition.

The following Main Example 2.8 shows that the concept of a projection invariant pseudodecomposition $((X_n)_{n\in\mathbb{N}},D)$ is strictly more general than that of a projection invariant decomposition, even for Banach sequence spaces X and even under the additional restriction that the expansions in D are unique and the subspaces $X_n$ are one-dimensional.

**2.8. MAIN EXAMPLE.** *There exist l.c. sequence spaces and even Banach sequence spaces X which have a projection invariant pseudodecomposition $((\mathbb{K}e_n)_{n\in\mathbb{N}},D)$ (where $e_n=(\delta_{nk})_{k\in\mathbb{N}}$) with $\bigoplus_{n\in\mathbb{N}} \mathbb{K}e_n \subset D$ and with unique expansions but such that $(\mathbb{K}e_n)_{n\in\mathbb{N}}$ is not a decomposition of X.*

In order to exhibit such spaces we return to the construction in Example 2.2. We start with an l.c. sequence space Z which has $(e_n)_{n\in\mathbb{N}}$ as a topological basis such that $\varphi:=\text{span}\{e_n:n\in\mathbb{N}\}\neq Z$. As in Example 2.2. (a) we choose a closed subspace $N\neq\{0\}$ of Z with $N\cap\varphi=\{0\}$ and a complementary subspace Y of N in Z containing $\varphi$. Recall that, by carrying over the quotient topology from Z/N to Y by means of the canonical isomorphism $Z/N\simeq Y$, Y gives rise to an l.c. space X with the pseudodecomposition $((\mathbb{K}e_n)_{n\in\mathbb{N}},D)$ where

$$D:=\{(x_n)_{n\in\mathbb{N}}\in\prod_{n=1}^{\infty}X_n : \sum_{n=1}^{\infty}x_n \text{ converges in } Y\},$$

and that the expansions in D are unique. To make D projection invariant we choose Y invariant under the projectors $p_J:\mathbb{K}^{\mathbb{N}}\to\mathbb{K}^{\mathbb{N}}$ ($J\subset\mathbb{N}$; cf. (2.4)). Such Z,N,Y exist in abundance as subspaces of $l^p$

with $1 \leq p < \infty$ (and of all other Banach sequence spaces that have $(e_n)_{n\in\mathbb{N}}$ as a Schauder basis): Begin with any proper projection invariant subspace $Y$ of $l^p$ with $\varphi \subset Y$ (such as $Y:=\varphi$ itself) and choose any closed subspace $N \neq \{0\}$ of $l^p$ with $Y\cap N=\{0\}$. Then the subspace $Z:=Y+N$ of $l^p$, together with $N,Y$, is as desired. ($Z$ need not be projection invariant.) - The more difficult task is to make $X$ a Banach space. This is done by starting with a projection invariant Banach sequence space $Z$ with $(e_n)_{n\in\mathbb{N}}$ as Schauder basis and showing that $Z$ always has projection invariant hyperplanes $Y$ containing $\varphi$ ; $N$ may then be any complementary subspace of $Y$ in $Z$ (and $X$ and $(\mathbb{K}e_n)_{n\in\mathbb{N}}, D)$ are then obtained as above). The existence of such hyperplanes $Y$ is guaranteed by the following algebraic lemma that covers more general cases as well. (Alternatively, the last part of Lemma 2.10 could be invoked.)

**2.9.** LEMMA. *Let $(X_\iota)_{\iota\in I}$ be a family of vector spaces over a field* $\mathbb{K}$ *and let* $L$ *be a projection invariant subspace of* $\prod_{\iota\in I} X_\iota$ (cf. (2.4)). *Then every proper projection invariant subspace* $M \subset L$ *is the intersection of all projection invariant hyperplanes* $H \subset L$ *with* $M \subset H$. *In particular, if* $\bigoplus_{\iota\in I} X_\iota \subsetneq L$ *, there exists a projection invariant hyperplane of* $L$ *containing* $\bigoplus_{\iota\in I} X_\iota$.

Proof. For $A \subset I$, $x \in \prod_{\iota\in I} X_\iota$ and $T \subset \prod_{\iota\in I} X_\iota$ let $x_A:=p_A(x)$ and $T_A:=p_A(T)$, with $p_A$ analogous to (2.4). Let $M$ be as in Lemma 2.9 and $d\in L\setminus M$. The set of all projection invariant subspaces $U\subset L$ such that $M\subset U$ and $d\notin U$ contains a maximal element $H$ by Zorn's Lemma. If we show that $H$ is a hyperplane the proof will be finished. Suppose that $H$ is not a hyperplane. Choose a subspace $T \subset L$ with $H\subsetneq T$

and $d \notin T$. As $H$ is maximal, $T$ is not projection invariant. Choose $z \in T$ and $A \subset I$ with $z_A \notin T$. Let $A' := I \setminus A$. Then also $z_{A'} = z - z_A \notin T$. Plainly. $S := \mathrm{span}\{z_B : B \subset A\}$ is projection invariant. So also $H+S$ is projection invariant. As $z_A \in S \setminus H$ we have $H \neq H+S$. Because $H$ is maximal, we deduce that $d \in H+S$, whence $d_{A'} \in H_{A'} \subset H$. The same argument applied to $A'$ instead of $A$ gives $d_A \in H$. So $d = d_A + d_{A'} \in H$, a contradiction.

If $((X_n)_{n\in\mathbb{N}}, D)$ is a pseudodecomposition of an l.c. space $X$ and $C$ is a complementary subspace of $D_0$ (see 2.1) in $D$, then $((X_n)_{n\in\mathbb{N}}, C)$ is a pseudodecomposition of $X$ with unique representations. However if $((X_n)_{n\in\mathbb{N}}, D)$ is projection invariant, $((X_n)_{n\in\mathbb{N}}, C)$ need not be projection invariant. That $C$ may be chosen projection invariant under additional algebraic conditions is contained in the following lemma, to be applied with $\mathbb{N}, D, D_0, \bigoplus_{n\in\mathbb{N}} X_n$ instead of $I, L, N, M$.

2.10. LEMMA. *Let* $(X_\iota)_{\iota\in I}$ *be a family of* $K$*-vector spaces and* $L$ *a projection invariant subspace of* $\prod_{\iota\in I} X_\iota$. *A subspace* $N$ *of* $L$ *has a projection invariant complementary subspace* $S$ *in* $L$ *if one of the following conditions is satisfied.*

(a) $\dim N$ *is finite.*

(b) *The codimension* $m$ *of* $N$ *in* $L$ *is finite, and for all subsets* $J \subset I$ *of at most* $m$ *elements one has*

$$p_J\left(\prod_{\iota\in I} X_\iota\right) \cap N = \{0\}.$$

(c) *The set of all sets* $I_x := \{\iota\in I : x_\iota \neq 0\}$ *with* $x = (x_\iota)_{\iota\in I} \in N \setminus \{0\}$ *has the finite intersection property.*

*If* (a) *is satisfied and* $M \subset L$ *is a projection invariant subspace* *with* $M \cap N = \{0\}$, $S$ *may be chosen such that* $M \subset S$.

Proof. 1. Let (a) hold and let $M \subset L$ be a projection invariant subspace with $M \cap N = \{0\}$. We argue by inductinn on $n := \dim N$. The case n=0 is trivial. To pass frnm n-l to n, chnnse $a \in N, a \neq 0$. By Lemma 2.9 there is a prnjectinn invariant hyperplane H in L with $M \subset H$ and $a \notin H$. Hence we have the interna1 direct sum $N = (H \cap N) \oplus Ka$. By inductinn hypnthesis. $H \cap N$ has a prnjectinn invariant complementary subspace S in H with $M \subset S$. By adding $Ka$ to $H = S \oplus (H \cap N)$ nne nbtains $L = S \oplus N$.

2. Let (b) hnld. There are m linearly independent elements $d_1, \ldots d_m$ in L. Then there exists a subset $J \subset I$ with at most m elements such that $p_J(d_1), \ldots, p_J(d_m)$ are linearly independent. So there are $k_1, \ldots, k_m \in \{1, \ldots, m\}$ and $\iota_1, \ldots, \iota_m \in J$ such that $p_{\{\iota_1\}}(d_{k_1}), \ldots, p_{\{\iota_m\}}(d_{k_m})$ are linearly independent. These vectnrs span an m-dimensiona1 prnjectinn invariant subspace S of $p_J(\prod_{\iota \in I} X_\iota)$, and $S \cap N = \{0\}$ by assumptinn. Hence S is as desired.

3. If (c) is satisfied, there is an ultrafilter $\mathcal{G}$ on 1 such that $I_x \in \mathcal{G}$ fnr all $x \in N \setminus \{0\}$. The subspace $T := \{x \in L : p_A(x) = 0$ for some $A \in \mathcal{G}\}$ has intersectinn $\{0\}$ with N. Chnnse a cnmplementary subspace S of N in L with $T \subset S$. The prnnf will be finished if we show that S is prnjectinn invariant. Let $x \in S$ and $A \subset I$. If $A' := I \setminus A \in \mathcal{G}$, nne has $p_{A'}(p_A(x)) = 0$ and hence $p_A(x) \in T \subset S$. If $A' \notin \mathcal{G}$, then $A \in \mathcal{G}$, so, by the last argument, $p_{A'}(x) \in S$, and again nne has $p_A(x) = x - p_{A'}(x) \in S$.

The hypnthesis $p_J(\prod_{\iota \in I} X_\iota) \cap N = \{0\}$ is nnt superflunus in 2.10, (b) as is nbvinus frnm the example $I := \mathbb{N}$, $X_n := K$ $(n \in \mathbb{N})$, $L := K^{\mathbb{N}}$, and

N a hyperplane containing $\bigoplus_{n\in\mathbb{N}} K$. Case (b) is not interesting for the above appl ication to pseudodecompositions since from X $\simeq D/D_o =$ $= L/N$ and (b) one obtains dim X = m.

The first part of Lemma 2.9 and the part of Lemma 2.10 pertaining to condition (a) generalize to arbitrary vector spaces L (not necessarily embedded in a prnduct) fnr which there is given a Boolean algebra of prnjectnrs $u\in\mathrm{End}(L)$ with zero element $0\in\mathrm{End}(L)$, unit element $id_L$, and the usual prnperties $u\wedge v=u\circ v=v\circ u$, $u\vee v =$ $= u+v-u\circ v$ and cnmplement $u'=id_L-u$.

## 3. THINNBSS CONDITIONS AND DENSE BARRELLED SUBSPACES.

Let X be an l.c. space and $((X_n)_{n\in\mathbb{N}},D)$ a pseudndecnmpnsitinn of X. Similarly as in the Intrnductinn we define $g_k((x_n)_{n\in\mathbb{N}})$ fnr $(x_n)_{n\in\mathbb{N}}\in D$ and $k \in \mathbb{N}$ as the number of indices $n\in\{1,\dots,k\}$ such that $x_n \neq 0$. Then the thinness condition

$$(\Theta) \qquad \lim_{k\to\infty} \frac{1}{k}\, g_k((x_n)_{n\in\mathbb{N}}) = 0$$

determines a subspace $D(\Theta)$ of D whnse image under the canonica1 map $f : D \to X$, $(x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} x_n$ is the subspace

(3.1) $L := \{x\in X$: there exists $(x_n)_{n\in\mathbb{N}}\in D_x$ which satisfies $(\Theta)\}$

of X. Of course L need nnt be a prnper subspace, but in many cases nbvinusly it is. (If dim X $\geq$ 2 in Example 2.6, the subspace L is nnt prnper as can be easily checked. The same applies to the subspace M to be defined presently.) Besides L we cnnsider a smaller subspace $M := \operatorname{span} M_0$, where $M_0 \subset X$ is defined by a thinnes rond it ion more restrictive than ($\theta$), depending on an arbitrary preassigned sequence $(b_j)_{j\in\mathbb{N}}$ in IN as follows.

Let J be the set of all subsets $J \subset \mathbb{N}$ which are either finite or. if $J = \{n_j : j\in\mathbb{N}\}$ with $n_j < n_{j+1}$ $(j\in\mathbb{N})$, satisfy

(J.1) $n_{j+1}-n_j \leq n_{j+2}-n_{j+1}$ $(j\in\mathbb{N})$, and $\lim\limits_{j\to\infty} (n_{j+1}-n_j) = \infty$ ,

and

(5.2) the sequence $\left(\frac{n_{j+1}-n_j}{b_j}\right)_{j\in\mathbb{N}}$ is unbounded.

Cancelling the first condition under (J.l) leads to a larger class $\hat{J}$, which for increasing $(b_j)_{j\in\mathbb{N}}$ contains all subsets of sets $J \in \hat{J}$. - If $(b_j)_{j\in\mathbb{N}}$ is bounded. (J.l) implies (5.2).

Nnw we define

(3.2) $M_0 := \{x\in X$: there is $(x_n)_{n\in\mathbb{N}} \in D$ such that

$$\{n\in\mathbb{N} : x_n \neq 0\} \subset J \quad \text{for some } J \in J\}.$$

$M_0$ need not be a subspace of X as can be seen from the spaces $\mathbb{K}^{\mathbb{N}}$ or $l^p$ $(1\leq p<\infty)$ with their Schauder decompositions $(\mathbb{K}e_n)_{n\in\mathbb{N}}$. Because of (J.l) nne has $M_0 \subset L$, hence alsn

(3.3) $M := \operatorname{span} M_0$

is cnntained in L. This inclusion may be proper even if nne disregards

(5.2). (For $X:=l^1$ and Xn: Men the element $\sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^{-k} e_{2^k+j}$ 1 ies in L but not in M.)

If $((X_n)_{n\in\mathbb{N}}, D)$ is assumed to be projection invariant, then Mo and hence M and L are dense in' X since they contain the sums of the sequences $(x_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} pr_n(D)$.

Now we come to the main result of this sectinn.

3.1. THEOREM. *If X is a barrelled space with a projection invariant pseudodecomposition* $((X_n)_{n\in\mathbb{N}}, D)$, *then also the dense subspaces M and L from(3.3) and(3.1) are barrelled.*

**REMARK.** The fact that M and L are barrelled in their relative topologies implies that also $(M, \tau(M,X'))$ and $(L,\tau(L,X'))$ are barrelled, fnr a barrelled space carries the Mackey topology, and $\tau(M,M') = \tau(M,X')$ and $\tau(L,L') = \tau(L,X')$ hnld always true.

Because of $M \subset L$ and in view of Lemma 1.1, Thenrem 3.1 is an immediate cnnsequence of the follnwing proposition abnut not necessarily barrelled l.c. spaces X.

3.2. PROPOSITION. *If X is an l.c. space with a projection invariant pseudodecomposition* $((X_n)_{n\in\mathbb{N}}, D)$ *then every* $\sigma(X',M)$-*bounded subset of X' is* $\sigma(X',X)$-*bounded.*

(b) *More sharply, let X be any l.c. space, let* $(u_r)_{r\in\mathbb{N}}$ *be a sequence in X' and let* $x\in X$ *be such that* $(u_r(x))_{r\in\mathbb{N}}$ *is unbounded. suppose that there is a sequence* $(x_n)_{n\in\mathbb{N}}$ *in X such that* $x = \sum_{n=1}^{\infty} x_n$ *and that this series is subseries convergent. Then there exists a* $J \in \mathcal{J}$ (see (J.1) and (5.2)) *such that* $(u_r(\sum_{n\in J} x_n))_{r\in\mathbb{N}}$ *is unbounded.*

Proof. Clearly (a) follows from (b). If now under the hypotheses of (b) there is a finite set $J\subset\mathbb{N}$ such that the sequence $(u_r(\sum_{n\in J}x_n))_{r\in\mathbb{N}}$ is unbnunded, then J belnngs to J and is as desired. We assume nnw that

**(3.4)** $(u_r(\sum_{n\in I}x_n))_{r\in\mathbb{N}}$ is bnunded fnr every finite set $I\subset\mathbb{N}$.

The desired set J e J will be nbtained by a *sliding hump* argument in the fnrm $J = \{ n_i : i\in\mathbb{N} \}$ with $n_i < n_{i+1}$ $(i\in\mathbb{N})$. The rnnstructinn of the $n_i$ invnlves the inductive cnnstructinn of fnur sequences $(a_k)_{k\in\mathbb{N}}$, $(\rho_k)_{k\in\mathbb{N}}$, $(r_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$. It is cnnvenient to put

(3.5) $s_o:=0$ and $s_k:=\sum_{i=1}^{k}(\rho_i a_{i-1}+(m_i-1)a_i)$ fnr $k \in \mathbb{N}$.

Fnrming fnr every $k \in \mathbb{N}$ the finite arithmetic progression

(3.6) $$s_{k-1} + \rho_k a_{k-1} + \mu a_k \quad (\mu = 0,1,\ldots,m_k-1),$$

(whnse last term $s_k$ is strictly smaller than the first term $s_k + \rho_{k+1} a_k$ of the next progression) and putting these prngressinns nne after annther. the resulting sequence $(n_i)_{i\in\mathbb{N}}$ will be as desired.

We set nnw $c_o:=0$ and note that

(3.7) $$c_k:=\sup_{r\in\mathbb{N}}\left| u_r\left(\sum_{i=1}^{k}\sum_{\mu=0}^{m_i-1} x_{s_{i-1}+\rho_i a_{i-1}+\mu a_i}\right)\right| \in \mathrm{IR} \text{ fnr all } k \in \mathrm{IN}.$$

The sequences $(a_k)_{k\in\mathbb{N}}$, $(\rho_k)_{k\in\mathbb{N}}$, $(r_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ shall satisfy conditions $(\alpha_k)$-$(\varepsilon_k)$ fnr all $k \in \mathbb{N}$:

$(\alpha_k)$ $a_k = 2^{m_1+\ldots+m_{k-1}+1} a_{k-1}$, with $a_0 := 1$;

$(\beta_k)$ $\rho_k \in \{1,2,\ldots,a_k/a_{k-1}\}$;

$(\gamma_k)$ $\{u_r(\sum_{\mu=0}^{\infty} x_{s_{k-1}+\rho_k a_{k-1}+\mu a_k}) : r \in \mathbb{N}\}$ is unbounded;

$(\delta_k)$ $|u_{r_k}(\sum_{\mu=0}^{\infty} x_{s_{k-1}+\rho_k a_{k-1}+\mu a_k})| \geq k + c_{k-1}$;

$(\varepsilon_k)$ $|u_{r_k}(\sum_{n\in I} x_n)| \leq 1$ for all $I \subset \{n \in \mathbb{N} : n > s_k\}$.

The sequence $(a_k)_{k\in\mathbb{N}}$ is defined uniquely by $(\alpha_k)$. Since $(u_r(\sum_{n=1}^{\infty} x_n))_{r\in\mathbb{N}}$ is unbounded by assumption and since $\sum_{n=1}^{\infty} x_n$ is the sum of its subseries $\sum_{\mu=0}^{\infty} x_{\rho a_0+\mu a_1}$ $(\rho = 1,\ldots,a_1/a_0)$, there is some $\rho_1$ such that $(\beta_1)$ and $(\gamma_1)$ hold. Hence there is $r_1 \in \mathbb{N}$ such that $(\delta_1)$ holds. Since $\sum_{n=1}^{\infty} x_n$ is subseries convergent the subsequences of $(x_n)_{n\in\mathbb{N}}$ are summable (see the remarks after Definition 2.3). So there is $m_1 \in \mathbb{N}$, $m_1 \geq 2$ so large that $s_1$ becomes so large that $(\varepsilon_1)$ holds for all finite and hence, by continuity, for all $I \subset \{n \in \mathbb{N} : n > s_1\}$. Similarly, if now $k \in \mathbb{N}$ and if $\rho_i, r_i, m_i \in \mathbb{N}$ $(1 \leq i \leq k)$ are already determined in accordance with the above conditions we define $\rho_{k+1}, r_{k+1}, m_{k+1}$ as follows. Since $s_k = s_{k-1} + \rho_k a_{k-1} + (m_k - 1)a_k$, the series $\sum_{\mu=1}^{\infty} x_{s_k+\mu a_k}$ may be regarded as a section of the series in $(\gamma_k)$. Hence $(\gamma_k)$ and (3.4) imply that $\{u_r(\sum_{\mu=1}^{\infty} x_{s_k+\mu a_k}) : r \in \mathbb{N}\}$ is unbounded. Since $\sum_{\mu=1}^{\infty} x_{s_k+\mu a_k}$

is the sum of its subseries $\sum_{\mu=0}^{\infty} x_{s_k+\rho a_k+\mu a_{k+1}}$ $(\rho=1,\ldots,a_{k+1}/a_k)$ it fnllows that there is some $\rho_{k+1} \in \mathbb{N}$ such that $(\beta_{k+1})$ and $(\gamma_{k+1})$ hnld. Hence there is some $r_{k+1} \in \mathbb{N}$ such that $(6_{k+1})$ hnlds, and we can chnnse $m_{k+1} \in \mathbb{N}$, $m_{k+1} \geq 2$. so large that $(\varepsilon_{k+1})$ hnlds as well. This ends the cnnstructinn of the sequences

$$(a_k)_{k\in\mathbb{N}},\ (\rho_k)_{k\in\mathbb{N}},\ (r_k)_{k\in\mathbb{N}},\ (m_k)_{k\in\mathbb{N}}.$$

Nnw we build up the sequence $(n_i)_{i\in\mathbb{N}}$ as described after (3.5). To prove (J.1) note that the difference of the last term $s_k$ of the progression (3.6) and the first term in the next progression is $\rho_{k+1}\, a_k$ which lies between $a_k$ and $a_{k+1}$ by $(\beta_k)$. As $a_k \uparrow \infty$ by $(\alpha_k)$, (J.l) is nnw clear.

Putting nnw $l_k := m_1+\ldots+m_k$, the definition of the $n_i (i\in\mathbb{N})$ yields $n_{l_k+2} - n_{l_k+1} = a_{k+1} = 2b_{l_k+1}a_k$ whence (5.2) follows, as $a_k \to \infty (k\to\infty)$. Thus $J := \{n_i : i \in \mathbb{N}\} \in J$.

To finish the prnnf we will nnw deduce frnm $(\delta_k)$ and $(\varepsilon_k)$ that

$$(3.8) \qquad |u_{r_k}(\sum_{n\in J} x_n)| \geq k\text{-}l \qquad (k \in \mathbb{N}).$$

Fnr $k \in \mathbb{N}$ the set $K:=\{n \in J : n > s_{k-1}\}$ is cnntained in the set $\{s_{k-1}+\rho_k\, a_{k-1} + \mu a_k : \mu = 0,1,\ldots\}$ by cnnstructinn. and the difference 1 of these sets cnnsists of elements greater than $s_k$ so that $(\varepsilon_k)$ applies. whereas

$$|u_{r_k}(\sum_{n\in J\setminus K} x_n)| \leqq c_{k-1}$$

by (3.7). Therefore ( $\delta_k$) yields (3.8). (The sliding hump was over the subset $\{n \in J : s_{k-1} < n \leqq s_k\}$ of J.)

By Lemma 2.5 one has the following immediate consequence of Thenrem 3.1.

3.3. COROLLARY. *If X is a sequentially complete barrelled space with Schauder decomposition* $(X_n)_{n\in\mathbb{N}}$ *whose projectors* $q_J : X \to X$ $(J \subset \mathbb{N}$ **finite;** cf. (2.5)) *form an equicontinuous family, then also the corresponding dense subspaces* M *and* L *are barrelled.*

We apply nnw Thenrem 3.1 to sequence spaces. The fnllnwing is immediate.

3.4. COROLLARY. *Let* $X \subset \mathbb{K}^{\mathbb{N}}$ *be a sequence space which contains* $\bigoplus_{n\in\mathbb{N}} \mathbb{K}e_n$ *and is invariant under the canonica1 projectors* $p_J$ $(J \subset \mathbb{N})$ *of* $\mathbb{K}^{\mathbb{N}}$. *Let X be endowed with a barrelled topology such that* $x = \sum_{n=1}^{\infty} x_n e_n$ *for all* $x = (x_n)_{n\in\mathbb{N}} \in X$. *Thus* $((\mathbb{K}e_n)_{n\in\mathbb{N}}, D)$ *with* $D := \{(x_n e_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in X\}$ *is a projection invariant pseudodecomposition with unique expansions. Then also the corresponding dense subspaces* M *and* L *are barrelled.*

Note that in Cnrnllary 3.4 the pseudndecnmpnsitinn need nnt be a decnmpnsitinn by nur Main Example 2.8. - Obvinusly Köthe's example cited in the Intrnductinn is a special case of Cnrnllary 3.4.

The spaces $l^p$ $(1 \leq p < \infty)$, $c_0$, $\mathbb{K}^{\mathbb{N}}$ with their Schauder decnmpnsition $(\mathbb{K}e_n)_{n\in\mathbb{N}}$ also satisfy the hypotheses of Cornllary 3.4, and hence the cnrrespnnding spaces L and M are barrelled fnr these spaces.

We show now that alsn Webb's Lemma (D) .1 qunted in the Intro- ductinn is cnntained in Cnrollary 3.4. Since X is perfect it cnntains $\varphi := \operatorname{span}\{e_n : n \in \mathbb{N}\}$ and is normal by G.Köthe [7], [8], §30,1.(3) and hence is prnjectinn invariant. The Mackey topnlngy $\tau(X,X^\times)$ is barrelled by assumptinn. Fnr every $x = (x_n)_{n\in\mathbb{N}} \in X$ nne has $x = \sum_{n=1}^{\infty} x_n e_n$ with respect to $\tau(X,X^\times)$ by G.Köthe [7], [8], §30,5.(10). So Cnrnllary 3.4 yields that the relative topology $\tau(X,X^\times)|L$ is alsn barrelled. Hence, by the Remark after Thenrem 3.1, also $\tau(L,X^\times)$ is barrelled.

We note that in normal sequence spaces X cnntaining $\varphi$, $(e_n)_{n\in\mathbb{N}}$ always is a $\tau(X,X^\times)$ - Schauder basis.

If X is an l.c. space with pseudndecnmpnsitinn $((X_n)_{n\in\mathbb{N}}, D)$ with unique expansinns and if there is an $x \in X\setminus L$ such that $(2^{-nt}x_n)_{n\in\mathbb{N}} \in D$ fnr all real $t \geq 0$, then codim L is at least the pnwer of the cnntinuum. Fnr it is straightfnrward to show that any linear combination $\sum_{j=1}^{r} \lambda_j \sum_{n=1}^{\infty} 2^{-nt_j} x_n$ with $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$ and $0 \leq t_1 < t_2 < \ldots < t_r$, which lies in L, is trivial.

This result can be applied to a Fréchet space X with a decompo- sition $(X_n)_{n\in\mathbb{N}}$ such that $X_n \neq \{0\}$ fnr all $n \in \mathbb{N}$: To find a suitable

$x \in X\setminus L$ choose a basis $(U_n)_{n\in\mathbb{N}}$ of neighbourhoods of 0 in X such that Un is circled and satisfies $U_{n+1} + U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. Choose $x_n \in U_n \cap X_n$, $x_n \neq 0$ for all $n \in \mathbb{N}$. Then $x := \sum_{n=1}^{\infty} x_n$ converges, i.e., $(x_n)_{n\in\mathbb{N}} \in E$, and $x \notin L$. Fnr all $t \geq 0$ nne has $2^{-nt} x_n \in U_n$ $(n\in\mathbb{N})$, hence $(2^{-nt} x_n)_{n\in\mathbb{N}} \in E$. and x is as desired. Hence codim L is at least the pnwer of the continuum.

Finally we note that if $((X_n)_{n\in\mathbb{N}}, D)$ is a prnjectinn invariant pseudndecnmpnsition and $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation then also $((X_{\sigma(n)})_{n\in\mathbb{N}}, D^{\sigma})$ with $D^{\sigma} := \{(x_{\sigma(n)})_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in D\}$ is a prnjectinn invar iant pseudndecnmpositinn; but the cnrrespnnding analngues $M^{\sigma}$, $L^{\sigma}$ of the spaces M,L will in general be different frnm M.L.

## 4. TWO EXAMPLES ON THE EQUICONTINUITY CONDITION.

In this sectinn and the fnllnwing one we analyze the sharpness of Theorem 3.1 and of Corollary 3.4 by means of some examples involving the classica1 Banach spaces $l^p$ $(1 \leq p < \infty)$, $c, c_o, bv, bv_o, bs$, CS, and the Fréchet space $\mathbb{K}^{\mathbb{N}}$. Reca11 that c is the space of all cnnvergent sequencesin $\mathbb{K}$ equipped with the supremum-nnrm and $c_o$ is the subspace of all sequences cnnverging to zero. The space bv cnnsists of all sequences $x = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ such that

$$\|x\|_{bv} := |x_1| + \sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty ,$$

whereas $bv_0:=bv\cap c_0$ is equipped with the nnrm

$$\|x\|_{bv_0}:=\sum_{n=1}^{\infty}|x_n-x_{n+1}|$$

(cf. Dunford-Schwartz [2], p.239). The space bs consists of all sequences $x=(x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ with

$$\|x\|_{bs}:=\sup\{|\sum_{k=1}^{n}x_k|: \quad n \quad \in \mathbb{N}\}<\infty,$$

and cs is the subspace of all sequences such that $\sum_{n=1}^{\infty}x_n$ converges. The space $\mathbb{K}^{\mathbb{N}}$ with the metr ic

$$d(x,y) := \sum_{n=1}^{\infty}\frac{|x_n-y_n|}{2^n(1+|x_n-y_n|)}$$

is a Fréchet space. Let $e_0:=(1,1,1,\ldots)$ and let again $e_n:=(\delta_{nk})_{k\in\mathbb{N}}$ fnr $n\geq 1$. Then $(\mathbb{K}e_n)_{n\in\mathbb{N}}$ is a Schauder decnmposition of $l^p(1\leq p<\infty)$, $c_0, bv_0, cs$, and $\mathbb{K}^{\mathbb{N}}$, whereas $(\mathbb{K}e_{n-1})_{n\in\mathbb{N}}$ is a Schauder decnmpnsitinn of c and of bv. These will be referred to as the ***standard Schauder decompositions.*** It is tacitly assumed in the sequel that the above spaces are endowed with their standard Schauder decompositinns.

The following two examples throw light nn the hypnthesis on prnjectinn invariance. Our first example shnws that this hypothesis cannnt be drnpped in Theorem 3.1 and also that the equicnntinuity condition cannnt be nmitted in Cnrnllary 3.3. This proves assertion (a) frnm the Intrnductinn.

4.1. EXAMPLE. Let $X=bv_0$. Clearly the standard Schauder decomposition $(\mathbb{K}e_n)_{n\in\mathbb{N}}$ is not projection invariant since $x:=(\frac{1}{n})_{n\in\mathbb{N}}\in bv_0$

and

$$\left\| \sum_{k=1}^{n} \frac{1}{2k} e_{2k} \right\|_{bv_o} = 2 \sum_{k=1}^{n} \frac{1}{2k} \to \infty \quad \text{for } n \to \infty .$$

This also prnves that the set of projectnrs $q_J$ ($J \subset \mathbb{N}$ finite) is nnt equi contl inunus whereas X meets all nther requirements of Cnrnllary 3.3. Let again $\varphi$ dennte the subspace of $\mathbb{K}^{\mathbb{N}}$ spanned by the unit vectnrs $e_n (n \in \mathbb{N})$. Fnr any $x=(x_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} \setminus \varphi$ we dennte by $h_n(x)$ $(n \in \mathbb{N})$ the index of the n-th non-zero term in the sequence $(x_k)_{k\in\mathbb{N}}$. Then plainly (0) is equivalent to

$$(\Theta') \qquad x \in \varphi \quad \text{or} \lim_{n\to\infty} \frac{n}{h_n(x)} = 0.$$

We shall prove that the dense subspace

$$L := \{ x \in bv_o : x \in \varphi \text{ or } \lim_{n\to\infty} \frac{n}{h_n(x)} = 0 \}$$

of $bv_o$ is non-barrelled. Fnr this end we take $0 < \delta < 1$ and fnrm

$$L_\delta := \mathbf{span}\{x \in bv_o : x \in \varphi \text{ or } \limsup_{n\to\infty} \frac{n}{h_n(x)} \leq \delta\} .$$

Then $L_\delta \supset L \supset M$. *We claim that $L_\delta$ (and hence L and M) is non-barrelled.*

Proof (cf. Greifenegger [4], p.18-20). The dual of $bv_o$ is nnrm-isnmorphic to the space bs, the canonica1 bilinear form fnr the dual pajr $\langle bv_o, bs\rangle$ being

$$<x,y> = \sum_{n=1}^{\infty} x_n\, y_n \quad \text{for } x = (x_n)_{n\in\mathbb{N}} \in bvo.\ y = (y_n)_{n\in\mathbb{N}} \in bs.$$

Now f ix p e $\mathbb{N}$ such that $\frac{p-1}{p} > \frac{1+\delta}{2}$ and define

$$y^{(n)} := \sum_{k=1}^{n} e_{p^k} \in bs \qquad (n \in \mathbb{N}).$$

Then

$$\|y^{(n)}\|_{bs} = n \qquad \text{for all } n \in \mathbb{N};$$

hence $(y^{(n)})_{n\in\mathbb{N}}$ is not equicontinuous and hence not $\sigma(bs,bv_o)$-bnunded. Remembering Lemma 1.1 we prnceed to show that $(y^{(n)})_{n\in\mathbb{N}}$ is $\sigma(bs,L_\delta)$-bounded. Firstly, if $x \in \varphi$, then the sequence $(<x,y^{(n)}>)_{n\in\mathbb{N}}$ is nbvinusly bnunded. Secnndly. assume that

$$x \in bv_o \setminus \varphi \quad \text{and'} \quad \limsup_{n\to\infty} \frac{n}{h_n(x)} \leq \delta .$$

**We** cnnsider the blncks of consecutive non-zero terms of the sequence $x=(x_\nu)_{\nu\in\mathbb{N}}$. Define the strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ of natura1 numbers by the condition that $x_\nu \neq 0$ $(\nu \in\mathbb{N})$ if and nnly if $m_k \leq \nu < n_k$ fnr. some $k \in \mathbb{N}$. Let $l_k \in [m_k,n_k[ \cap \mathbb{N}$ be arbitrary ($k \in$ N). Then

$$|x_{l_k}| = |x_{l_k} - x_{n_k}| \leq \sum_{n=m_k}^{n_k-1} |x_n - x_{n+1}|$$

and hence

(4.1) $$\sum_{k=1}^{\infty} |x_{1_k}| \leq \|x\|_{bv_0} \;.$$

Now choose $r \in \mathbb{N}$ such that

(4.2) $$\frac{m}{h_m(x)} < \frac{1+\delta}{2} \quad \text{for all } m \geq p^r .$$

Assume that $m_k \leq p^\nu < p^{\nu+1} < n_k$ for some $\nu \geq r$ and for some $k \in \mathbb{N}$. Then there would be at least $p^{\nu+1} - p^\nu$ non-zero terms of the sequence $(x_k)_{k\in\mathbb{N}}$ among the first $p^{\nu+1}$ terms, and hence

(4.3) $$\frac{{}^{\nu+1} \qquad {}^{\nu}}{h_{p^{\nu+1}-p^\nu}(x)} \geq \frac{{}^{\nu+1} \qquad {}^{\nu}}{p^{\nu+1}} > \frac{1+\delta}{2}$$

by the choice of p. Since $\nu \geq r$ by assumption, we have $p^{\nu+1} - p^\nu \geq p^r$ and hence (4.3) contradicts (4.2). Thus. for all $k \in \mathbb{N}$, at most one of the numbers $p^\nu$ with $\nu \geq r$ is contained in the k-th block $[m_k, n_k[ \,\cap\, \mathbb{N}$. This yields for all $n \in \mathbb{N}$

$$|\langle x, y^{(n)}\rangle| = |\sum_{\nu=1}^{n} x_{p^\nu}$$

$$\leq \sum_{\nu=1}^{r} |x_{p^\nu} + \sum_{\nu=r+1}^{\infty} |x_{p^\nu}$$

$$\leq \sum_{\nu=1}^{r} |x_{p^\nu}| + \|x\|_{bv_0}$$

(cf. (4.1)) . This proves our assertion.

We add a few further remarks on Example 4.1. Although the space $L_\delta$ is defined by means of a thinness condition, it is easy to see that $L_\delta$ contains plenty of "non-thin" vectnrs since e.g. $l^1 \subset L_\delta$. In particular, $l^1$ is a non-barrelled subspace of $bv_o$. (This fact is nbvinus anyway.) We mentinn withnut prnnf that $L_\delta = L_{\frac{1}{2}}$ fnr all $\delta \in \,]0,1[$. -

Note that $bv_o$ is nnrm-isnmnrphic to $l^1$ by the map $f{:}bv_o \to l^1$, $f((x_n)_{n\in\mathbb{N}}) := (x_n - x_{n+1})_{n\in\mathbb{N}}$ fnr $(x_n)_{n\in\mathbb{N}} \in bv_o$. The inverse map $f^{-1}$ transfers the Schauder basis $(e_n)_{n\in\mathbb{N}}$ of $l^1$ to the Schauder basis $(\sum_{k=1}^{n} e_k)_{n\in\mathbb{N}}$ of $bv_o$. The subspaces L and M for the Schauder decom- position $(\mathbb{K}\sum_{k=1}^{n} e_k)_{n\in\mathbb{N}}$ of $bv_o$ are obviously bnrrelled since their corresponding images in $l^1$ are bnrrelled by Corollary 3.4. *Hence for one and the same Banach space* X *one choice of a Schauder basis may produce dense barrelled subspaces* L *and* M, *whereas another choice of a Schauder basis can lead to dense non-barrelled spaces* L,M.

Our secnnd example shows that there exist Banach spaces with Schauder decnmpnsitinn such that L and even a certain subspace $\tilde{M} \subset L$ are barrelled althnugh prnjectinn invariance dnes nnt hnld. This will prove assertion (b) frnm the Intrnductinn.

4.2. EXAMPLE. The standard Schauder decnmpnsitinn of the Banach space cs is nnt prnjectinn invariant since $((-1)^n/n)_{n\in\mathbb{N}} \in$ cs and

$$(\|\sum_{k=1}^{n} \frac{1}{2k} e_{2k}\|_{cs})_{n\in\mathbb{N}}$$

is unbounded. The space cs also does not satisfy the equicontinuity hypothesis of Corollary 3.3; cs is neither norma1 nor perfect. Nevertheless. **the** space L *defined by* (3.1) ***and even a certain dense subspace*** $\tilde{M} \subset L$ ***are barrelled.***

Proof. The map $f : c \to cs$, $f((x_n)_{n\in\mathbb{N}}) := (x_n - x_{n-1})_{n\in\mathbb{N}}$ for $(x_n)_{n\in\mathbb{N}} \in c$ with $x_0 := 0$ is a norm isomorphism. The standard Schauder decnmpnsitinn $(\mathbb{K}e_{n-1})_{n\in\mathbb{N}}$ of c satisfies the equicnntinuity condition nf Cnrnllary 3.3. To prove this, take any $x=(x_n)_{n\in\mathbb{N}} \in c$ and define $\xi := (\xi_{n-1})_{n\in\mathbb{N}}$ by $\xi_0 := \lim_{n\to\infty} x_n$, $\xi_n := x_n - \xi_0$ fnr $n \geq 1$. Then $x = \sum_{n=0}^{\infty} \xi_n e_n$, and fnr every finite subset $J \subset \mathbb{N} \cup \{0\}$ we have

$$\|q_J(x)\|_c = \|\sum_{n\in J} \xi_n e_n\|_c \leq 2\|x\|_c .$$

Hence the set *of* all prnjectnrs $q_J$ ($J \subset \mathbb{N} \cup \{0\}$ finite) is equicontinunus.

Applying Cnrnllary 3.3 to the 'standard Schauder decnmpnsitinn of c we conclude that the subspace

$$M := \operatorname{span} M_0$$

of c with

$$M_0 := \{x = \sum_{n=0}^{\infty} \xi_n e_n \in c : \{n \in \mathbb{N} : \xi_{n-1} \neq 0\} \subset J \text{ some } J \in \mathcal{J}\}$$

$$=\{(x_n)_{n\in\mathbb{N}}\in c:\{n\in\mathbb{N}: x_{n-1}\neq\lim_{k\to\infty}x_k\}\subset J \text{ for some } J\in J\}$$

is barrelled. We claim that the subspace

$$\tilde{M} := \operatorname{span} M_0 \subset L$$

of cs with

$$\tilde{M}_0 := \{(x_n)_{n\in\mathbb{N}}\in cs: \{n\in\mathbb{N}: x_n\neq 0\}\subset J \text{ for some } J\in\hat{J}\}$$

is barrelled as well. (Remember that $\hat{J}$ was defined after (J.2).)

In order to prove this assertion it suffices to show that $f(M_0)\subset\tilde{M}$. Suppose that $x=(x_n)_{n\in\mathbb{N}}=\sum_{n=0}^{\infty}\xi_n e_n\in M_0$. If only finitely many of the $\xi_\nu$ $(\nu\geq 0)$ are different from zero. then $x_n=\xi_0$ for all but finitely many $n\in\mathbb{N}$ and hence $f(x)\in\varphi\subset\tilde{M}_0$. Assume now that $\xi=(\xi_{\nu-1})_{\nu\in\mathbb{N}}$ satisfies $\xi_{\nu-1}\neq 0$ for infinitely many $\nu\in\mathbb{N}$. Then $\{h_n(\xi): n\in\mathbb{N}\}$ with $h_n(\xi)$ from Example 4.1 is a subset of some set $J\in J$. Choose $m\in\mathbb{N}$ so large that $h_{\nu+1}(\xi)-h_\nu(\xi)\geq 4$ for all $\nu\geq m-1$. Then with $n=h_m(\xi)$, $p=h_{m+1}(\xi)$, $q=h_{m+2}(\xi)$, $r=h_{m+3}(\xi),\ldots$, . . . $x_n\neq\xi_0$, $x_p\neq\xi_0$, $x_q\neq\xi_0$, $x_r\neq\xi_0$ the sequence x looks like

$$x=(*,\ldots,*,\xi_0,\xi_0,\xi_0,x_n,\xi_0,\ldots,\xi_0,x_p,\xi_0,\ldots,\xi_0,x_q,\xi_0,\ldots,\xi_0,x_r,\xi_0,\ldots),$$

where the chains of terms $\xi_0$ contain at least three terms each. Hence with $a := x_n-\xi_0$, $b := x_p-\xi_0$, $c := x_q-\xi_0$, $d := x_r-\xi_0,\ldots$ the sequence f(x) has the form

$f(x) = (*,\dots,*,0,0,a,-a,0,\dots,0,b,-b,0,\dots,0,c,-c,0,\dots,0,d,-d,0,\dots)$.

We now write down the sequences

$y_1 := (\xi_0,\xi_1,\dots,\xi_{n-2}, a, 0, 0,\dots,0,-a,0, 0,\dots,0, c, 0, 0,\dots,0,-c,0,0,\dots)$,

$y_2 := (\xi_0,\xi_1,\dots,\xi_{n-2}, 0,-a, 0,\dots,0, a,0, 0,\dots,0, 0,-c, 0,\dots,0, c,0,0,\dots)$,

$y_3 := (\xi_0,\xi_1,\dots,\xi_{n-2}, -a, 0, 0,\dots,0, b,0, 0,\dots,0,-b, 0, 0,\dots,0, d,0,0,\dots)$,

$y_4 := (\xi_0,\xi_1,\dots,\xi_{n-2}, a, 0, 0,\dots,0,0,-b, 0,\dots,0, b, 0, 0,\dots,0,0,-d,0,\dots)$,

where the dots between the zeros indicate chains of zeros with the same lenght as the dotted chains of zeros in the vector $f(x)$. Then $(h_n(y_j)-h_n(\xi))_{n\in\mathbb{N}}$ is bounded for $j = 1,\dots,4$ and hence $y_1,\dots,y_4 \in M_o$. By definition we have $y_1+y_2+y_3+y_4-f(x)\in\varphi \subset \tilde{M}_o$ and hence $f(x) \in \tilde{M}$ which proves our assertion.

## 5. DISCUSSION OF OTHER THINNESS CONDITIONS.

We analyze in the present section to what extent our thinness conditions are best possible. First we introduce another thinness condition which produces barrelled subspaces in some cases and non-barrelled ones in others. This will corroborate assertion (c) from the Introduction.

**5.1. DEFINITION.** Suppose that X is an l.c. space with decomposition $(X_n)_{n\in\mathbb{N}}$, and for $x \in X \setminus \operatorname{span} \bigcup_{n\in\mathbb{N}} X_n$, $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ $(n\in\mathbb{N})$, let $h_n(x)$ denote the index of the n-th non-zeroterm in the sequence $(x_k)_{k\in\mathbb{N}}$. Let $(d_n)_{n\in\mathbb{N}}$ be a preassigned sequence of positive real numbers such that $\sum_{n=1}^{\infty} d_n$ diverges and put

$$W_o := \{x \in X : x \in \operatorname{span} \bigcup_{n\in\mathbb{N}} X_n \text{ or } \sum_{n=1}^{\infty} d_{h_n(x)} < \infty\},\ W := \operatorname{span} W_o .$$

An elementary check shows that $W_o$ already is a linear space if $(d_n)_{n\in\mathbb{N}}$ is monotonically decreasing. - In general, the thinness condition

$$\sum_{n=1}^{\infty} d_{h_n(x)} < \infty$$

employed in the definition of $W_o$ is too restrictive to produce barrelled subspaces of barrelled spaces as the following theorem shows.

5.2. THEOREM. ***Suppose that $X$ is one of the Banach spaces $l^p (1 < p < \infty)$, $c_o$, $c$, $bv_o$, $bv$, $cs$ equipped with its standard Schauder decomposition. Then $W$ (as in Definition 5.1) is a dense non-barrelled subspace of $X$.***

Proof. We argue by Lemma 1.1.

a) In the case $X = l^p$ $(1 < p < \infty)$ let $q > 1$ be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Put

$$y^{(n)} := \sum_{k=1}^{n} d_k^{1/q} e_k \in l^q \quad (n \in \mathbb{N}).$$

Then $(y^{(n)})_{n\in\mathbb{N}}$ is not bounded in $l^q$, but for all $x \in W$, the sequence $(\langle x, y^{(n)} \rangle)_{n\in\mathbb{N}}$ is bounded: For $x \in \varphi$ this is trivial anyway, and for $x = (x_k)_{k\in\mathbb{N}} \in W_o \setminus \varphi$ and all $n \in \mathbb{N}$ we have by Hölder's inequality

$$|\langle x, y^{(n)}\rangle| = \left|\sum_{\nu: h_\nu(x) \leq n} x_{h_\nu(x)} d_{h_\nu(x)}^{1/q}\right|$$

$$\leq \left(\sum_{\nu=1}^{\infty} d_{h_\nu}(x)\right)^{1/q} \|x\|_p < \infty .$$

b) For $X = c_o$ we identify X' with $l^1$.

The sequence $(y^{(n)})_{n\in\mathbb{N}}$ defined by

$$y^{(n)} := \sum_{k=1}^{n} d_k e_k \qquad (n\in\mathbb{N})$$

is unbounded in $l^1$; but for all $x\in W$, the sequence $(\langle x, y^{(n)}\rangle)_{n\in\mathbb{N}}$ is bounded: For $x\in\varphi$ this is again trivial, and for $x\in W_o\setminus\varphi$ and for all $n\in\mathbb{N}$ we have

$$|\langle x, y^{(n)}\rangle| = \left|\sum_{\nu: h_\nu(x) \leq n} x_{h_\nu(x)} d_{h_\nu}(x)\right|$$

$$\leq \left(\sum_{\nu=1}^{\infty} d_{h_\nu(x)}\right) \|x\|_{c_o} .$$

c) The dual of $X = c$ is norm-isomorphic to the space $l_o^1$ of all sequences $y = (y_\nu)_{\nu > 0}$ in $\mathbb{K}$ such that

$$\|y\|_{l_o^1} = \sum_{\nu=0}^{\infty} |y_\nu| < \infty .$$

The canonical bilinear form for the dual pair $\langle c, l_o^1\rangle$ is

$$\langle x, y\rangle = y_0 \lim_{n\to\infty} x_n + \sum_{n=1}^{\infty} x_n y_n$$

for $x = (x_n)_{n\in\mathbb{N}} \in c$, $y = (y_\nu)_{\nu>0} \in l_0^1$.

Defining now

$$z^{(n)} := (-\sum_{\nu=1}^{n} d_\nu , d_1, \ldots, d_n, 0, \ldots) \in l_0^1 .$$

the argument is essentially the same as in case b).

d) For $X = bv_0$, we let $y^{(n)}$ be as in b). Then $(y^{(n)})_{n\in\mathbb{N}}$ is unbounded in the dual bs of $bv_0$. But the same estimate as in b) yields that $(\langle x,y^{(n)}\rangle)_{n\in\mathbb{N}}$ is bounded for every $x\in W$. (Note that $bv_0 \subset c_0$.)

e) Let $X = bv$ and $x = (x_n)_{n\in\mathbb{N}} \in X$. Put $\xi_0 = \lim_{n\to\infty} x_n$, $\xi_n := x_n - \xi_0$ for $n\geq 1$. Then $x = \sum_{n=0}^{\infty} \xi_n e_n$. The elements $y^{(n)}$ (cf. b)) act as continuous linear forms on bv according to

$$\langle x,y^{(n)}\rangle := \sum_{k=1}^{n} \xi_n d_n .$$

The sequence $(y^{(n)})_{n\in\mathbb{N}}$ is unbounded in $(bv)'$ by d), but for all $x\in W$ the sequence $(\langle x,y^{(n)}\rangle)_{n\in\mathbb{N}}$ is bounded because of d).

f) The case $X=cs$ is settled as follows: It is easy to show by partial summation that $bv_0$ is norm-isomorphic to a subspace of $X'$, the elements $y=(y_n)_{n\in\mathbb{N}}\in bv_0$ acting on cs according to

$$\langle x,y\rangle = \sum_{n=1}^{\infty} x_n y_n \qquad (x=(x_n)_{n\in\mathbb{N}} \in cs).$$

Now the sequence $(y^{(n)})_{n\in\mathbb{N}}$ defined by

$$y^{(n)} := \sum_{k=1}^{n} (-1)^k d_k e_k \qquad (n \in \mathbb{N})$$

is unbounded in $bv_o$. But the sequence $(\langle x, y^{(n)} \rangle)_{n\in\mathbb{N}}$ is bounded for all $x\in\varphi$, and we have for all $x=(x_k)_{k\in\mathbb{N}} \in W_o \setminus \varphi$ and all $n\in\mathbb{N}$

$$|\langle x,y^{(n)} \rangle| = |\sum_{\nu:h_\nu(x)\leq n} (-1)^{h_\nu(x)} d_{h_\nu(x)} x_{h_\nu(x)}|$$

$$\leq (\sum_{\nu=1}^{\infty} d_{h_\nu(x)}) \|x\|_\infty < \infty$$

For $X=l^1$ and for $X = \mathbb{K}^{\mathbb{N}}$ the following result on the barrelledness of W holds:

5.3. PROPOSITION. Suppose *that* $X=l^1$ *or* $X = \mathbb{K}^{\mathbb{N}}$ *with its standard Schauder decomposition. Then* W *(as in Definition* 5.1) *is barrelled if and only if* $(d_n)_{n\in\mathbb{N}}$ *converges to zero.*

**Proof.** Let $X = l^1$ and assume first that $(d_n)_{n\in\mathbb{N}}$ converges to zero. Let $(y^{(n)})_{n\in\mathbb{N}}$ be an unbounded sequence in $l^\infty = X'$. We have to show: There exists an $x\in W$ such that $(\langle x,y^{(n)} \rangle)_{n\in\mathbb{N}}$ is unbounded. To prove this we may assume that there is no $x\in\varphi$ with this property, since $\varphi \subset W$.

Let $y^{(n)} = (y_k^{(n)})_{k\in\mathbb{N}}$ $(n \in \mathbb{N})$. Then for every $p\in\mathbb{N}$ the set

$$\{y_k^{(n)} : n \in \mathbb{N}, \ 1 \leq k \leq p\}$$

is bounded. Since $(\|y^{(n)}\|_\infty)_{n\in\mathbb{N}}$ is unbounded. there exist two strictly increasing sequences $(n_j)_{j\in\mathbb{N}}$ and $(k_j)_{j\in\mathbb{N}}$ of natural numbers such that

$$|y_{k_j}^{(n_j)}| \geq j \quad \text{for all} \quad j \in \mathbb{N}$$

and

$$\sum_{j=1}^{\infty} d_{k_j} < \infty \ .$$

Focusing attention to the spaces $l^1$ and $l^\infty$ over the subsequence $(k_j)_{j\in\mathbb{N}}$, our assertion now becomes obvious. The sequence $(k_j)_{j\in\mathbb{N}}$ and hence the sequence $(h_j(x))_{j\in\mathbb{N}}$ can even be made arbitrarily thin. The same idea of proof works in the case $X = \mathbb{K}^{\mathbb{N}}$, $X' = \varphi$ .

To prove the converse, assume that $x = l^1$ and suppose that $\limsup_{n\to\infty} d_n > 0$ . Choose $\varepsilon > 0$ and a strictly increasing sequence $(m_j)_{j\in\mathbb{N}}$ of natural numbers such that

$$d_{m_j} \geq \varepsilon \text{ for all } j \in \mathbb{N}.$$

Define $y^{(n)} = (y_k^{(n)})_{k\in\mathbb{N}} \in l^\infty$ by

$$y_k^{(n)} := j \text{ for } k = m_j,\ k \leq n,\ j \in \mathbb{N},$$

$$y_k^{(n)} := 0 \text{ otherwise.}$$

Then $(y^{(n)})_{n\in\mathbb{N}}$ is an unbounded sequence in $l^\infty$. Now let $x \in W_o$. Then $x_{m_j} \neq 0$ only for finitely many $j \in \mathbb{N}$. Hence $(\langle x, y^{(n)}\rangle)_{n\in\mathbb{N}}$ is bounded for all $x \in W_o$ and hence also for all $x \in W$. Thus $W$ is not barrelled. The same idea of proof works for $X=\mathbb{K}^{\mathbb{N}}$.

Another class of barrelled subspaces of $l^1$ was constructed by Saxon [10], p. 155-157.

Suppose now that $d_n = \frac{1}{n}$ ($n \in \mathbb{N}$). Then Theorem 5.2 says that for $X = l^p$ ($1 < p < \infty$), $c_o$, $bv_o$, $cs$ the subspace W of X containing all vectors $x \in X$ satisfying the thinness condition

$$(\Theta'') \qquad x \in \varphi \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{h_n(x)} < \infty$$

is non-barrelled. Now remember that for every monotonically decreasing sequence $(c_n)_{n\in\mathbb{N}}$ of positive real numbers with $\sum_{n=1}^{\infty} c_n < \infty$ one has $\lim_{n\to\infty} n\, c_n = 0$. This means that $(\Theta'')$ implies $(\Theta')$ from Example 4.1 whence $W \subset L$, and L is barrelled. On the other hand. for $X = l^1$ or $\mathbb{K}^{\mathbb{N}}$, W is barrelled by Proposition 5.3. This yields assertion (c) from the Introduction. In addition, this observation also lies in the direction of claim (d) from the Introduction. In order to justify assertion (d) more precisely we modify thinness condition $(\Theta')$ and introduce another class of subspaces.

5.4. DEFINITION. Let X, $(X_n)_{n\in\mathbb{N}}$, $(h_n(x))_{n\in\mathbb{N}}$ $(x \in X \setminus \operatorname{span} \bigcup_{n=1}^{\infty} X_n)$ be AS in Definition 5.1. Let $a = (a_n)_{n\in\mathbb{N}}$ be an arbitrary preassigned sequence of positive real numbers and define

$$U_o := \{ x \in X : x \in \operatorname{span} \bigcup_{n\in\mathbb{N}} X_n \text{ or } \lim_{n\to\infty} \frac{a_n}{h_n(x)} = 0 \},$$

$$U := U(a) := \operatorname{span} U_o .$$

Valdivia [13] considers a slight variant of the thinness condition employed in the definition of $U_o$.

5.5. **REMARKS.** (a) If $a = (a_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of positive real numbers, there exists a strictly monotonically increasing sequence $b = (b_n)_{n\in\mathbb{N}}$ of positive real numbers such that

(i) $U(a) = U(b)$ and

(ii) $(\frac{a_n}{n})_{n\in\mathbb{N}}$ is bounded if and only if $(\frac{b_n}{n})_{n\in\mathbb{N}}$ is bounded.

(b) If $a = (a_n)_{n\in\mathbb{N}}$ is a monotonically increasing sequence of positive real numbers such that $(a_{2n}/a_n)_{n\in\mathbb{N}}$ is bounded, then $U_\theta$ is a linear space, i.e. $U = U_o$.

(c) Suppose that $X = l^1$ or $X = \mathbb{K}^{\mathbb{N}}$ with its standard Schauder decomposition. Then $U(a)$ is barrelled (without any restriction on a).

We omit the elementary proofs of the remarks (a) and (b). The proof of remark (c) is similar to that of Proposition 5.3.

For $a_n = n$ $(n \in \mathbb{N})$, we have $U = L$, and this space is barrelled under the hypothesis of Theorem 3.1 or Corollary 3.4. The next theorem shows that for none of the Banach spaces $l^p (1 < p < \infty), c_o, c, cs$ there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers tending to infinity more rapidly than the sequence $(n)_{n\in\mathbb{N}}$ such that U is barrelled. *Hence thinness condition* $(\Theta)$ *is best possible for the construction of barrelled subspaces among all the thinness conditions introduced in Definition* 5.4. This justifies assertion (d) from the Introduction.

5.6. **THEOREM.** *Suppose that* $X$ *is one of the Banach spaces* $l^p$ $(1 < p < \infty)$, $c_0$, $c$, $cs$ *with its standard Schauder decomposition. Then the space* $U(a)$ *(as in Definition 5.4) is barrelled if and only if* $(\frac{a_n}{n})_{n\in\mathbb{N}}$ *is bounded.*

Note that the Banach space $cs$ appearing in this theorem is not projection invariant.

Proof. Assume first that $(\frac{a_n}{n})_{n\in\mathbb{N}}$ is bounded. Then $L \subset U$. Now $L$ is barrelled by Corollary 3.4 (for $X=l^p$ $(1<p<\infty)$ or $X=c_0$ or $X=c$) and by Example 4.2 (for $X=cs$). Hence $U$ is barrelled as well.

Assume now that $(\frac{a_n}{n})_{n\in\mathbb{N}}$ is unbounded. We have to show that $U$ is non-barrelled. Referring to Remark 5.5, (a) we may assume without loss of generality that $(a_n)_{n\in\mathbb{N}}$ is strictly monotonically increasing. We claim that it is sufficient to prove the following assertion (A):

(A) There exists a monotonically decreasing sequence $(d_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\sum_{n=1}^{\infty} d_n$ diverges and such that $\sum_{n=1}^{\infty} d_{h_n}$ converges for every sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ with $\lim_{n\to\infty} \frac{a_n}{h_n} = 0$.

Once we have proved assertion (A), it is obvious that the space $W$ from Definition 5.1 corresponding to the sequence $(d_n)_{n\in\mathbb{N}}$ contains $U$. Since $W$ is non-barrelled by Theorem 5.2, we conclude that $U$ is non-barrelled as well. Hence we are left to prove (A).

Since $(\frac{a_n}{n})_{n\in\mathbb{N}}$ is unbounded, there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\sum_{n=1}^{\infty} c_n$ converges whereas $\sum_{n=1}^{\infty} \frac{[a_n]}{n} c_n$ diverges. Define

$$d_n := \sum_{k:a_k \geq n} \frac{c_k}{k} \qquad (n \in \mathbb{N}).$$

Then $(d_n)_{n\in\mathbb{N}}$ is a monotonically decreasing sequence of positive real numbers. The series $\sum_{n=1}^{\infty} d_n$ diverges since

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \sum_{k:a_k \geq n} \frac{c_k}{k} = \sum_{k=1}^{\infty} \frac{[a_k]}{k} c_k = \infty \ .$$

Now let $(h_n)_{n\in\mathbb{N}}$ be any sequence of natural numbers such that $\lim_{n\to\infty} \frac{a_n}{h_n} = 0$. Then there exists some $p \in \mathbb{N}$ such that $h_n \geq a_n$ for all $n \geq p$. Using the strict monotonicity of $(a_n)_{n\in\mathbb{N}}$ we thus obtain

$$\sum_{n=p}^{\infty} d_{h_n} = \sum_{n=p}^{\infty} \sum_{k:a_k \geq h_n} \frac{c_k}{k}$$

$$\leq \sum_{n=p}^{\infty} \sum_{k:a_k \geq a_n} \frac{c_k}{k}$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{c_k}{k}$$

$$= \sum_{n=1}^{\infty} c_n < \infty.$$

This yields (A) and the proof is complete.

Theorem 5.6 holds analogously with $U$ replaced by $V = \operatorname{span} V_0$ with

$$V_0 := \{x \in X : x \in \operatorname{span} \bigcup_n \mathbb{K} e_n \text{ or } (a_n/h_n(x))_{n\in\mathbb{N}} \text{ is bounded}\}.$$

We remark in conclusion that the methods of this work may be adapted to produce dense barrelled or non-barrelled subspaces of spaces of continuous functions and of the spaces $L^p$ $(1 \leq p < \infty)$. As a sample we mention the following results.

5.7. THEOREM. *Suppose that* $(X, \mathcal{A}, \mu)$ *is a u-finite measure space without atoms, and let* $X = \bigcup_{k=1}^{\infty} E_k$ *be a representation of* $X$ *as a countable disjoint union of measurable sets of positive* finite *measure. Then for* $1 < p < \infty$ *the space*

$$V := \{f \in L^p : \lim_{k\to\infty} \frac{1}{\mu(E_k)} \mu(\{t \in E_k : f(t) \neq 0\}) = 0\}$$

*is a dense barrelled subspace of* $L^p$.

5.8. PROPOSITION. *Suppose that* $(X, \mathcal{A}, \mu)$ *is a a-finite measure space and that* $1 < p < \infty$. *Then the space*

$$W := \{f \in L^p : \mu(\{t \in X : f(t) \neq 0\}) < \infty\}$$

*is barrelled if and only if* $\mu(X)$ is finite, i.e., *if and only if* $W = L^p$. *If* $X$ *is a* $\sigma$*-finite measure space without atoms and* $p = 1$, *then* $W$ *is barrelled.*

PROBLEM. The trigonometrica1 system is a Schauder basis in

$L^p([0,2\pi])$ for $1 < p < \infty$. It is known that the associated Schauder deromposition is not projection invariant for all $p \neq 2$ (see J.Marti [9], p. 51-53, proof of Theorem 8). We do not know if the spaces L or M for this Schauder decomposition are barrelled for some $p \neq 2$, $1 < p < \infty$. We do not know either if the corresponding space W (cf. Definition 5.1) is non-barrelled.

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