ON REGULAR r-PACKINGS

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Abstract. This article is concerned with the connection between regular 2-packings in $\text{PG}(2r-1,q)$ and translation planes of order $q^{2r}$ whose components are defined by a set of rational Desarguesian nets coordinatized by quadratic field extensions of a given field of order $q$. This work is a natural extension of the studies of Walker [17] and Lunardon [14].

INTRODUCTION. Prohaska and Walker [15], Walker [17] and Lunardon [14] have, independently, shown a connection between regular 2-packings in $\text{PG}(3,q)$ and certain translation planes of order $q^4$ and kernel $\text{GF}(q)$. These planes are of particular interest as they admit a regulus $\mathcal{R}$ (of $1+q$ components) and the components are defined by $1+q+q^2$ derivable nets $\mathcal{D}_i$, $i=1,\ldots,1+q+q^2$ such that $\mathcal{R} \subseteq \mathcal{D}_i$ and $\mathcal{R} = \mathcal{D}_i \cap \mathcal{D}_j$, $i \neq j$, $i,j = 1,\ldots,1+q+q^2$.

In [8], the authors show how to connect 2-packings (or parallelisms) in $\text{PG}(3,q)$ with general translation planes of order $q^4$ admitting $\text{SL}(2,q)$ as a collineation group.

In this note, we give the natural extensions of the work of Walker [17], Lunardon [14], and the authors [8] to 2-packings in $\text{PG}(2r-1,q)$ related to translation planes of order $q^{2r}$ and kernel $F \cong \text{GF}(q)$ with a regulus $\mathcal{R}$ (of $1+q$ components) and whose components are defined by $\frac{q^{2r-1}-1}{q-1}$ nets $\mathcal{D}_i$, $i=1,\ldots,\frac{q^{2r-1}-1}{q-1}$ such that
is a rational Desarguesian net coordinatized by a quadratic
field extension of $F$, $\mathcal{D}_i \supseteq \mathbb{R}$ and $\mathcal{D}_i \cap \mathcal{D}_j = \mathbb{R}$ for all $i \neq j$;
$i,j=1,\ldots,q^{2r-1}-1$.

The arguments supporting the results are quite similar or natura1
extensions of those of Prohaska and Walker [15] and Jha-Johnson [8].
However, we try to give direct proofs in order to make this article
more or less self-contained.

We require the following results:

(1.1). THEOREM (Jha[5], LEMMA 2).

Let $V$ be an elementary abelian group of order $p^{sr} = q^r > q^2$
and suppose $U$ is any non-trivial group of order $u^t$ for $t \geq 1$ in
$\text{Aut}(V,+)$ where $u$ is a prime $p$-primitive divisor of $q^{(r-1)}-1$.

Then

(a) $|\text{Fix } U| = q$

(b) $U$ is semi regular on $V/\text{Fix}(U)$

(c) $U$ is cyclic

(d) If $r > 2$ then $V = \text{Fix } U \oplus C_U$ where $C_U$ is the unique $U$-sub
module of $V$ which is disjoint from $\text{Fix}(U)$.

(e) If $r > 2$ and $W$ is a $U$-submodule of $V$ then either $W \subseteq \text{Fix}(U)$
or $|W| \geq q^{r-1}$.

(1.2). THEOREM (Johnson [11]).

Let $\pi$ be a translation plane of order $p^{2kr}$ which admits $\mathcal{D}_{\sim} \text{SL}(2,p^r)$
as a collineation group in the translation complement. Assume
the $p$-elements are elations and $\mathcal{N}$ denotes the elation net.
(1) There is a rational Desarguesian net $\mathcal{D}$ containing $\mathcal{N}$ (coordinatized by a field $\cong \mathbb{F}^2(p^{2r})$ which is fixed by $\mathcal{D}$.

(2) $(\mathcal{D} - \mathcal{N}) \cap l_\infty$ is an orbit under $\mathcal{D}$ and an orbit of $\mathcal{D}$ of length $p^{2r} - p^r$ defines a rational Desarguesian net containing $\mathcal{N}$.

(3) If $\mathcal{N}$ is coordinatized by the field $\mathbb{K} \cong \mathbb{F}(q^r)$ then each such orbit net $\mathcal{D}$ may be coordinatized by an extension field $\mathbb{K}[t] \cong \mathbb{F}(p^{2r})$ (where $\mathbb{K}[t]$ depends on $\mathcal{D}$).

2. REGULAR t-PACKINGS AND TRANSLATION PLANES.

(2.1) Definition. Let $V$ be a vector space of dimension $k$ over a field $F \cong \mathbb{F}(q)$ for $q = p^r$, $p$ a prime, $r$ an integer. A partial $t$-spread $\mathcal{P}$ of $V$ is a set of mutually disjoint $t$-dimensional subspaces. A $t$-spread of $U$ is a partial $t$-spread which covers the vectors of $V$. (In this case, $t|r$).

A Desarguesian or regular partial $t$-spread is a partial $t$-spread $\mathcal{P}$ such that there is a field extension $K$ of $F$ and the elements of $\mathcal{P}$ are 1-dimensional subspaces over $K$ (note that $K$ is isomorphic to $\mathbb{F}(q^r)$).

(2.2) Definition. Let $V$ be a vector space of dimension $2k$ over a field $F \cong \mathbb{F}(q)$. Let $\mathcal{N}$ be a partial $k$-spread and let $\mathcal{P}$ be a partial $2t$-spread of $V$. We shall say that $V$ is $t$-transversal to $\mathcal{P}$ if and only if $L \in \mathcal{N}$ and $c \in \mathcal{P}$ then $L \cap c$ is a $t$-subspace of $c$. We also shall say that $c$ and $L$ are $t$-transversal to each other.

We initially follow Prohaska and Walker [15].

(2.3) Proposition. Let $\mathcal{P}$ be a partial $k$-spread of a vector space of dimension $2k$ over $F \cong \mathbb{F}(q)$. Let $\mathcal{F}$ denote the set of all
2t-spaces t-transversal to \( \mathcal{P} \). Let \( f \in \mathcal{F} \) and let \((f)_{\mathcal{P}} = \{ L \cap f | L \in \mathcal{P} \} \).

If \((f)_{\mathcal{P}} \) is a t-spread of \( f \) then for every element \( g \in \mathcal{F} \), \((g)_{\mathcal{P}} \) is a t-spread and \( \mathcal{F} \) is a partial 2t-spread.

**Proof.** (We follow the argument of Prohaska and Walker.) If \((f)_{\mathcal{P}} \) is a t-spread then \((f)_{\mathcal{P}} \) is a translation plane of order \( q^t \) and \( |\mathcal{P}| = 1+q^t \). Hence, \((g)_{\mathcal{P}} \) is a partial t-spread with \( 1+q^t \) elements. That is, \( \mathcal{L} \cap g \neq \mathcal{M} \cap g \) and \((L \cap g) \cap (M \cap g) \subseteq \mathcal{L} \cap \mathcal{M} = \emptyset \). So \((g)_{\mathcal{P}} \) is a t-spread. It remains to show that \( \mathcal{F} \) is a partial 2t-spread. So, let \( j, k \in \mathcal{F} \) and \( j \cap k \neq \emptyset \). Let \( P \in \mathcal{F} \cap \mathcal{F} \) be a vector \( \neq \emptyset \). There exists a unique element \( \mathcal{M} \) of \( \mathcal{P} \) which contains \( P \). Given \( \mathcal{N}, \mathcal{L} \in \mathcal{P} - \{ \mathcal{M} \} \) by projection, there is a unique 2-dimensional subspace \( U \) (line of projective space) which contains \( P \) and which intersects \( \mathcal{N} \) and \( \mathcal{L} \) (as \( \mathcal{N} \cap \mathcal{L} = \emptyset \)). But, similarly, there is a unique 2-space \( U \) of \( j \) containing \( P \) and which intersects \( j \cap \mathcal{N} \) and \( j \cap \mathcal{L} \) (in a 1-space of \( j \)). That is, \( U = \emptyset \) and \( U \subseteq j \) and similarly, \( U \subseteq k \) so \( U \subseteq j \cap k \). Now suppose \( Q \) is any 2-space containing \( P \) such that \( Q \subseteq j \). If \( Q \subseteq \mathcal{M} \) then since \((f)_{\mathcal{P}} \) is a t-spread of \( f \), it must be that \( Q \) intersects at least two elements of \( \mathcal{P} \) (in \((f)_{\mathcal{P}} \)). But, this means that \( 0 \subseteq j \cap k \). Choose any vector in \( j \cap k \) and together with \( P \) form a 2-dimensional subspace \( T \). As above \( T \subseteq j \cap k \). Hence, \( j \cap k \subseteq j \cap k \) and similarly \( k \cap k \subseteq j \cap k \). And, \( j \cap k \subseteq j \cap k \). So, \( j = k \) so that \( \mathcal{F} \) is a partial spread.

(2.4) **PROPOSITION.** Let \( A \) and \( B \) be mutually disjoint \( k \)-spaces of a \( 2k\)-dimension vector space \( V \). Let \( \mathcal{A} \) be a t-spread of \( A \), \( \mathcal{B} \) a t-spread of \( B \) and \( f \) a linear bijection of \( V \) from \( \mathcal{A} \) onto \( \mathcal{B} \). Then \( \mathcal{F} = (X \cap f | X \in \mathcal{A}) \) is a partial 2t-spread with \( A, B \) t-transversal to \( \mathcal{P} \). Furthermore, \( A_{\mathcal{P}} = \mathcal{A} \), \( B_{\mathcal{P}} = \mathcal{B} \).
Proof. We must show that $\mathcal{P}$ is a partial $2t$-spread. Let $\mathcal{A}, \mathcal{T}, \mathcal{P}$ and let $P \cap T$. Let $\mathcal{A} = \tilde{x} \oplus \tilde{x}^f$, $T = \tilde{y} \oplus \tilde{y}^f$ for $\tilde{x}, \tilde{y}$ $t$-spaces in $\mathcal{A}$. If $P$ (or $P \cap B$) then $\tilde{x} = \tilde{y}$ because $\mathcal{A}$ is a (partial) $t$-spread so that $R = T$. Assume $P \not\in A$ and $P \not\in B$. There is a $k$-space $C$ containing $P$ and mutually disjoint to $A$ and $B$. There is a unique $2$-dim space $L$ on $P$ which nontrivially intersects $A$ and $B$. $R = \tilde{x} \oplus \tilde{x}^f$ is a $2t$-space and $P \not\in x$ or $x^f$ so as in the previous argument there is a unique $2$-space $L$ of $x \oplus x^f$ which contains $P$ and which intersects $R \cap A$ and $R \cap B$. That is, $L = L$. Hence, $L$ is in $R \cap T$. But $T$ intersects $A$ (and $B$). Hence $R$ and $T$ have a (vector) point in common on $A$ and because $\mathcal{A}$ is a (partial) $t$-spread, $R = T$.

(2.5) Proposition. Suppose $A, B, C$ are mutually disjoint $k$-subspaces (of $V$ a $2k$-dimension vector space) and let $\mathcal{A}$ be a $t$-spread of $A$. Then there exists precisely one partial $2t$-spread $\mathcal{P}$ $t$-transversal to $A, B, C$ and with $(A) = \mathcal{A}$. Further, the regular $\mathcal{A}(A, B, C)$ is contained in the set of all $t$-transversal to $\mathcal{P}$.

Proof. There is a unique involution $i_C$ of $V$ which fixes $C$ pointwise and interchanges $A$ and $B$ (i.e., $A = (x = 0), (y = 0) = B$. $C$ is $(y = x)$ then $i_C$ is

\[
\begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix}
\]

Consider $\mathcal{P} = \{ \tilde{x} \oplus \tilde{x}^iC | \tilde{x} \in C \}$. By (2.4), $\mathcal{P}$ is certainly a partial $2t$-spread and $(A) = \mathcal{A}$. Suppose $\mathcal{P}$ is a partial $2t$-spread $t$-transversal to $A, B, C$ with $(\mathcal{P}) = \mathcal{A}$. Then consider $\tilde{x} \in \mathcal{A}$. There is a $2t$-space $U$ (of $\mathcal{P}$) containing $\tilde{x}$ and transversal to $B$ and $C$. Hence, $\tilde{x} \in U \cap (\tilde{x} \oplus \tilde{x}^iC)$. We now again use the argument of (2.3).

We repeat part of the argument for $U$ and $\tilde{x} \oplus \tilde{x}^iC = T$. 


Note that we may take the partial k-spread \( \{A,B,C\} \) and \( \mathcal{F} \) the set of all 2t-spaces transversal to \( \mathcal{F} \) such that \( \mathcal{F}|A = \mathcal{A}, U \) and \( T \) are transversal to \( B \) and \( C \). Consider a point \( P \in \bar{x} \). \( \bar{x} \oplus \bar{x}^{ic} = T \) is a 2t-space and there exists a unique 2-space which contains \( P \) and intersects \( B \) and \( C \). That is, \( L \) is also in \( \bar{x} \oplus \bar{x}^{ic} \) and in \( U \). Thus, \( U \) and \( T \) intersect on \( B \). Moreover, this is true for every point \( P \) of \( \bar{x} \). Let \( \bar{p}, \bar{p} \) be distinct 1-spaces on \( \bar{x} \) and \( L, L \) the unique 2-spaces \( \mathcal{P}eL, \mathcal{P}eL \) such that \( L, L \) intersect \( B, C \) non-trivially. Then \( L, L \in \bar{x} \oplus \bar{x}^{ic} \) and \( L, L \subseteq U \). Can \( L \cap L \neq \emptyset \)?

Then

\[
L = \langle (\bar{x}_1 \ldots \bar{x}_t, \emptyset \ldots \emptyset), (\emptyset \ldots \emptyset, \bar{x}_1 \ldots \bar{x}_t) \rangle
\]

\[
L = \langle (x^*_1 \ldots x^*_t, \emptyset \ldots \emptyset), (\emptyset \ldots \emptyset, x^*_1 \ldots x^*_t) \rangle
\]

Now

\[
s = (((\bar{x}_1 \ldots \bar{x}_t) \alpha, (\bar{x}_1 \ldots \bar{x}_t) \beta))
\]

\[
= (((x^*_1 \ldots x^*_t) \delta, (x^*_1 \ldots x^*_t) \gamma))
\]

\[\Rightarrow \bar{x}^*_1 \alpha = x^*_1 \delta \Rightarrow \bar{x}_1 = x^*_1 \delta \alpha^{-1} \text{ if } \alpha \neq 0\]

and

\[\Rightarrow \bar{x}^*_1 \beta = x^*_1 \gamma \Rightarrow \bar{x}_1 = x^*_1 \gamma \beta^{-1} \text{ if } \beta = 0.\]

Hence if \( \alpha \) or \( \beta \neq 0 \) \( \Rightarrow L = L \). If \( \alpha = 0 \) then \( \delta = 0 \) so \( \beta \neq 0 \). Hence \( L \cap L = \emptyset \) or \( L \).

So, this means \( L, L \in \bar{x} \oplus \bar{x}^{ic} \) and \( U \), so that \( \bar{x} \oplus \bar{x}^{ic} = U \).

So, we have shown that the space \( \mathcal{F} \) of all 2t-spaces which are t-trasversal to \( A, B, C \) and which when restricted to \( A \) give \( \emptyset \) is precisely \( \mathcal{F} = \{ \bar{x} \in \bar{x}^{ic} | \bar{x} \in \emptyset \} \) (for \( \mathcal{F} \) is a partial 2t-spread). Now consider the regulus \( R(A,B,C) \) and \( D \in R(A,B,C) \). The regulus
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is covered by 2-spaces (little Desarguesian planes). So, if U is a 2t-spaces t-trasversal to A,B,C and considering U as a union of its 2-spaces (transversal to A,B,C) we see, by the above argument, that there exist disjoint 2-dim spaces of U which intersect D-one 2-space for each 1-space on \( A \cap U \). Hence, there are \( \geq \frac{q^{t-1}}{q-1} \) 1-spaces of U on \( \mathcal{D} \).

Hence dim \( U \cap D \geq t \). But, also dim \( U \cap A = t \) and \( A \cap D = \emptyset \). Hence, dim \( U \cap D = t \). Thus, every 2t-space t-transversal to A,B,C is also transversal to the elements of \( \mathcal{A}(A,B,C) \).

We want to investigate the situation when there are precisely \( 1+q^t \) k-spaces which are t-transversal to \( \mathcal{F} = \{ \bar{x} \oplus \bar{x}^{i}C | \bar{x} \in \mathcal{A} \} \). In this case the set \( \mathcal{I} \) of t-transversal k-spaces is exactly covered by \( \mathcal{F} \). Or, another way of saying this is that if \( \mathcal{I} \) is the partial spread of t-transversal k-spaces to \( \mathcal{F} \), i.e., each element of \( \mathcal{I} \) is t-transversal to \( \mathcal{F} \), then \( f(y) \) is a 2t-spread of \( f e \mathcal{F} \).

By Foulser's covering theorem [12] when this happens the t-spread of \( \mathcal{A} \) is Desarguesian. Conversely, consider a Desarguesian t-spread \( \mathcal{A} \) of A. Then there is a field extension K of F such that \( \mathcal{F} = \{ \bar{x} \oplus \bar{x}^{i}C | \bar{x} \in \mathcal{A} \} \) is a partial 2-spread over K. And, we may consider A,B,C as subspaces over K and V a vector space over K \( \supseteq F \). Now applying the previous result the regulus \( \mathcal{A}(A,B,C) \) over K is contained in the set of 1-transversals to \( \mathcal{F} \) over K. But, this mens \( \mathcal{A}_K(A,B,C) \) is the set of 1-transversal to \( \mathcal{F} \) over K. Hence,

(2.6) THEOREM (See Prohaska and Walker [15] when t=2)

There is all 1-1 correspondence between Desarguesian t-spreads
\( \mathcal{A} \) of a \( k \)-space \( A \) of \( \mathcal{A}(A,B,C)(\text{regulus over } F \text{ generated by } A,B,C) \) and partial spreads \( \mathcal{D} \) of degree \( 1+q^t \) of \( t \)-transversal \( k \)-spaces to \( A,B,C \) such that surface \( \mathcal{D} = \text{surface of } \mathcal{F}(\overline{x}e^{-iC}|\overline{x} \in \mathcal{A}) \) which contain the regulus \( \mathcal{A}(A,B,C) \).

That is, there is a 1-1 correspondence between rational Desarguesian nets of degree \( 1+q^t \) containing a regulus \( \mathcal{A} \) and Desarguesian \( t \)-spreads of a component of the regulus.

(2.7) PROPOSITION. Given a regulus \( \mathcal{A}(A,B,C) \) and \( Q \in \text{surf} \mathcal{A} \), there is a unique 4-space containing \( Q \) which is 2-transversal to \( A,B \) and \( C \) and thus to \( \mathcal{A}(A,B,C) \).

Proof. Again we argue as in Prohaska and Walker [15](3). If \( \{\overline{x},\overline{y}\} \subseteq \{A,B,C\} \), then there is a unique 2-space transversal to \( \overline{x} \) and \( \overline{y} \) and containing \( Q \). So there are three 2-spaces \( U_{A,B}U_{A,C}U_{B,C} \) containing \( Q \) and transversal to \( (A,B),(A,C),(B,C) \) respectively. Then suppose two are equal. Then there is a 2-space which hits \( A,B,C \) and thus lies on the regulus. But, \( Q \notin \text{regulus} \) so that these three are completely distinct spaces. \( (U_{A,B}U_{A,C}U_{B,C}) \) is a 4-dimensional space which contains \( Q \) and is the unique such 4-space which is 2-transversal.

Now suppose \( \mathcal{A}_1, \mathcal{A}_2 \) are two distinct regular 2-spreads. Then consider the rational Desarguesian nets \( \mathcal{D}_1, \mathcal{D}_2 \) so constructed. Then suppose \( Q \in \mathcal{D}_1 \cap \mathcal{D}_2 - \mathcal{A}(A,B,C) \). Then there is a unique 4-space \( \mathcal{L}_Q \) containing \( Q \) and 2-transversal to \( \mathcal{A} \). But, this 4-space is simultaneously then a 2-space over two fields \( K_1,K_2 \simeq GF(q^2) \). So, \( \mathcal{L}_Q | A \) is a 1-space over \( K_1 \) and over \( K_2 \). That is, if we obtain a partial spread we must have that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) do not share a 1-space and conversely.
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We have:

(2.8) THEOREM (See Prohaska and Walker [15], Walker [17] and Lunardon [14] for order \( q^4 \).)

Let \( V \) be a vector space of dimension \( 4k \) over \( F \cong GF(q) \). Let \( \mathcal{A} \) be a regulus of \( V \). Let \( \Gamma \) be a set of rational Desarguesian nets isomorphic to \( GF(q^2) \) containing \( \mathcal{A} \). Then \( U(\Gamma - \mathcal{A}) \cup \mathcal{A} \) is a \( q \)-ical spread \( \iff \exists (A)_{\mathcal{A}} \mid \mathcal{A} \in \Gamma \) is a partial 2-parallelism of \( \mathcal{A} \) which is a component of \( \mathcal{A} \).

(2.9) Notes

i) Theorem (2.8) is also noted by Walker in [17]. However, our proof generally follows and extends Prohaska and Walker's unpublished notes.

ii) Stinson and Vanstone [16] have determined a great number of \( 2 \)-packings in \( PG(5,2) \). It is not clear if any are regular but such regular \( 2 \)-packings would correspond to translation planes of order \( 2^6 \) and kernel \( GF(2) \) which contain a regulus of \( 1+2+3 \) lines and whose components consist of \( \frac{25-1}{2-1} = 31 \) rational nets each of which may be coordinatized by a field isomorphic to \( GF(4) \) containing the same prime field.

iii) If \( \pi \) is a translation plane of order \( q^{2r} \) constructed as in (2.8) then \( GL(2,q) \) is a collineation group of \( \pi \).

Proof. The regulus \( \mathcal{A} \) admits \( GL(2,q) \) and since each rational Desarguesian net \( \mathcal{D}_1 \) is defined by an extension field of the field defining the regulus \( \mathcal{A} \), \( GL(2,q) \) is also a collineation group of \( \mathcal{D}_1 \). Hence, since \( \pi = (\mathcal{D}_1 - \mathcal{A}) \cup \mathcal{A} \), it follows that \( GL(2,q) \) is
a collineation group of the plane \( \pi \).

(2.10) TRANSLATION PLANES AND PARTIAL t-PACKINGS.

Let \( \pi \) be a translation plane of order \( q^{ts} \) and kernel \( GF(q) \) admitting a regulus \( \mathcal{R} \) (of \( 1+q \) components). Suppose the components consist of \( \frac{q^{ts}-q}{q^{t}-q} = \frac{q^{ts-1}-1}{q^{t-1}-1} \) (where \( t-1|ts-1 \)), rational Desarguesian nets isomorphic to \( GF(q^t) \). Then on any component \( \mathcal{L} \) of \( \mathcal{R} \), considering \( \mathcal{L} \) as \( PG(ts-1,q) \), there is an associative partial t-packing.

(sketch) If on \( \mathcal{L} \) two t-space \( \bar{x}_1 = \bar{x}_2 \) are equal (one from two different t-spreads) then \( \bar{x}_1 \oplus \bar{x}_1^c = \bar{x}_2 \oplus \bar{x}_2^c \) so that the associated nets are equal.

3. TRANSLATION PLANES OF ORDER \( q^{2r} \) ADMITTING SL(2,q).

In [8], the authors show how to obtain regular parallelisms in \( PG(3,q) \) (2-packings) directly from an associated translation plane of order \( q^4 \). In this section, it is noted that the same theorems are valid for 2-packings in \( PG(2r-1,q) \).

That is, we prove:

(3.1) THEOREM. (See (2.4)[8])

Let \( \pi \) be a translation plane of order \( q^{2r} \), \( q=p^s \), \( q \) a prime, \( s \) an integer which admits a collineation group \( \mathcal{G} \) isomorphic to \( SL(2,q) \) in the translation complement.

(i) If the \( p \)-elements are elations and \( \mathcal{G} \) is 1/2-transitive on \( l_{\infty} \cap \mathcal{N} \cap l_{\infty} \) where \( \mathcal{N} \) denotes the net of elation axes then the kernel of \( \mathcal{G} \) is \( GF(q) \), each orbit union \( \mathcal{N} \) is a rational Desarguesian
net coordinatized by a field isomorphic to $\text{GF}(q^2)$. If $\mathcal{L}$ is an elation axis then $\mathcal{L}$ is thought of as $\text{PG}(2r-1,q)$ admits a regular 2-packing.

(ii) Conversely, if $\mathcal{L}$ is a 2r-space over a field $F\cong\text{GF}(q)$ and admits a regular 2-packing as $\text{PG}(2r-1,q)$ then there is a corresponding translation plane which admits a collineation group $\mathcal{D}\cong \text{SL}(2,q)$, $q=p^S$, where the $p$-elements are elations and such that $\mathcal{D}$ acts 1/2-transitively on $l_\infty-N\cap l_\infty$ where $N$ denotes the net of elation axes.

Proof. (ii). By (2.8) there is a corresponding translation plane $\pi$. Since each net may be coordinatized by an extension of a field $K\cong \text{GF}(q)$, clearly the group $\mathcal{D}$ generated by the elations of the regulus $\mathcal{R}$ is isomorphic to $\text{SL}(2,q)$ (see (2.9)(ii) and is a collineation group of $\pi$. Clearly, $\mathcal{D}$ acts 1/2-transitively on $l_\infty-\mathcal{R}\cap l_\infty$ because for each rational Desarguesian net $\mathcal{D}\cong \mathcal{R}$, $\mathcal{D}-\mathcal{R}$ is an orbit under $\mathcal{D}$.

(i) Suppose $\mathcal{D}$ is 1/2-transitive. By (1.2), there is at least one rational Desarguesian net $\mathcal{D}$ coordinatized by a field extension $K[t]$ of the field $K$ defining the net $N$ of elation. And, $\mathcal{D}-N$ is an orbit. Hence, there exist $\frac{q^{2r-q}}{q^2-q} = \frac{q^{2r-1-1}}{q-1}$ such orbits and by (1.2)(3), each such orbit defines another rational Desarguesian net containing $N$. Thus, by (2.8),(i) is proved.

We now consider translation planes of order $q^{2r}$ that admit $\text{SL}(2,q) \times Z \frac{q^{2r-1-1}}{q-1}$ as a collineation group in the translation
complement. Note that the known regular 2-packings define translation planes that admit such groups (see Jha-Johnson [8]).

We prove

(3.2) THEOREM (COMPARE WITH JHA-JOHNSON [8] (2.5)).

Let \( \pi \) be a translation plane of order \( p^{2rs} = q^{2r} \) that admits a collineation group \( \mathcal{D} \) isomorphic to \( \text{SL}(2, q) \times \mathbb{Z} \frac{2r-1}{q-1} \) in the translation complement. Then, the kernel is \( \text{GF}(q) \), the \( p \)-elements are elations and for any elation axis \( \mathcal{L} \) considered as \( \text{PG}(2r-1, q) \), \( \mathcal{L} \) admits a regular 2-packing.

Proof. We structure the proof as in Jha-Johnson [8] (2.5).

We first assume the \( p \)-elements are elations. By Jha-Johnson [8] (2.5), we may assume \( 2r > 4 \) in any case.

Let the elation net be denoted by \( \mathcal{N} \). By (1.2), there is at least one rational Desarguesian net \( \mathcal{D} \supseteq \mathcal{N} \), of degree \( 1 + q^2 \).

Suppose \( g \in \mathbb{Z} \frac{2r-1}{q-1} = \mathbb{Z} \) fixes \( \mathcal{D} \). Let \( h \in \mathbb{Z} \) such that \( |h| \frac{q^{2r-1-1}}{q-1} \) is a prime \( p \)-primitive divisor of \( q^{2r-1-1} \). Since \( 2r > 4 \), there always exists such an element since \( |h| \frac{q^{2r-1-1}}{q-1} \) (note \( 4^3-1 \) is a possible exception and the argument is taken separately in [8]).

Since \( h \) fixes each elation axis and fixes points on each (as of \( |h| \frac{q^{2r-1}}{q-1} \) and \( |q^{2r-1-1} \) then \( |h| \frac{q^{(2r, 2r-1)-1}}{q-1} \) which cannot be the case).

By (1.1). Fix \( h \) is a subplane of order \( q \). \( g \) acts on \( \text{Fix} \ h \) so
that if \( g \) does not fix points of \( \text{Fix} \ h \) then there exists an integer \( j \) such that \( g^j \neq 1 \) and \( |g^j|/q-1 \). Then consider \( q^j \) with \( |q^j|/q-1 \) and fixing \( \mathcal{D} \). Since \( |g_j|/q^{2r-1}-1 \) and \( |g^j|/q-1 \) then \( g^j \) fixes affine points and since \( |\mathcal{D}-\mathcal{N}| = q^2-q \), some power \( g^{jh} \) fixes infinite points of \( \mathcal{D}-\mathcal{N} \). That is, \( g^{jk} \) fixes a subplane of order \( \geq q^2 \) point-wise. And, there is a subplane \( \pi_0 \) of order \( q^2 \) of \( \mathcal{D} \) such that \( \text{Fix} \ h \subseteq \pi_0 \subseteq \text{Fix} \ g^{jk} \). However, \( \pi_0 \) is Desarguesian so that \( |h|/q^2 \) and \( q \) must be odd. But then \( |h|/q-1 \) which cannot be the case.

Hence, if \( g \) fixes \( \mathcal{D} \) then \( |g^j|/q-1 \) and \( |g^j|/q^{2r-1}-1 \).

\[
(q-1+t+q^2+\ldots+q^{2r-2})
\]

\[
= (q-1,(q-1)+(q^2-1)+\ldots+(q^{2r-2}-1)+(2r-1))
\]

So the GCD equals \( (q-1,2r-1) \). So there are at least \( t_1 \) rational Desarguesian nets \( \mathcal{D}_i \supseteq \mathcal{N} \) such that \( \mathcal{D}_i \cap \mathcal{D}_j = \mathcal{N} \) for \( i,j=1,\ldots,t_1 \) since \( \text{SL}(2,q) \) has \( \mathcal{D}_i-\mathcal{N} \) as an orbit for all \( i=1,\ldots,t_1 \).

But, let \( \sigma \in \mathcal{D}_-\text{SL}(2,q) \) be an element such that \( |\sigma|/q^2-1 \), but \( |\sigma|^k-1 \) for \( k \leq 2s \ (g=p^s) \) (see e.g. Johnson \([11]\)). \( \sigma \) permutes the remaining points on \( \ell_\infty \cup_{i=1}^{t_1} \mathcal{D}_i \).

So,

\[
|\ell_\infty \cup_{i=1}^{t_1} \mathcal{D}_i| = (q^{2r-2}-1) - \frac{(q^{2r-1}-1)(q^2-q)}{(q-1)(q-1,2r-1)}
\]

Let \( (q-1,2r-1) = s \).

More generally, suppose there are \( \frac{T}{s}(q^{2r-1}-1) \) rational Desarguesian nets. Then
\[ |\mathcal{L}_{\infty} - \bigcup_{1=1}^{t^2} \mathcal{D}_1 | = (q^{2r-1} - q) - \frac{T_s}{s} q^{2r-1-1} \frac{q^2 - q}{q^2 - q} \]
\[ q(q^{2r-1-1})(1 - \frac{T_s}{s}) \]
\[ = q(q^{2r-1-1})(\frac{s - T_s}{s}). \]

Then if

\[ |\sigma| |q(q^{2r-1-1})(\frac{s - T_s}{s}) | \]

then

\[ |\sigma|(q^{2r-1-1}, q^2 - 1) = q(2r-1,2) - 1 = q - 1. \]

Hence, \( \sigma \) fixes additional points on \( \mathcal{L}_{\infty} \). Now apply the previous argument inductively. That is, remove another set of at least

\[ \frac{q^{2r-1-1}}{(q-1)s} \]

rational Desarguesian nets. Obtain another set of

cardinality \( q(q^{2r-1-1})(\frac{s-2}{s}) \). By (1.2) and induction, there are

\[ \frac{q^{2r-1-1}}{q-1} \]

rational Desarguesian nets \( \mathcal{D}_i \supseteq \mathcal{N} \) such that \( \mathcal{D}_i \cap \mathcal{D}_j = \mathcal{N}. \)

Now apply (2.8).

Now assume the \( p \)-elements in \( SL(2,q) \) are planar. Note the proof in [8] extends directly.) Let \( \pi_0 \) be a subplane of order \( p^k = \text{Fix} \sigma | \sigma | = p \), \( \sigma \in \text{SL}(2,q) \). \( Z_{q^{2r-1-1}} = Z \) must leave \( \pi_0 \) invariant and if \( g \in Z \)

\[ \frac{q^{2r-1-1}}{q-1} \]

has order a prime \( p \)-primitive divisor of \( q(2r-1-1) \) (recall \( 2r > 4 \)) then we assert that \( g \) must fix a component of \( \pi_0 \). That is, by Foulser's Dimension Theorem (e.g., see Jha [7]) \( k \) must divide
2rs if \( q = p^s \). However, if \( |g|1+p^k \) then \( |g| (p^{2k-1}, p((2r-1)s-1) = (p(2k,(2r-1)s)-1) \). Hence we have a contradiction unless
\((2r-1)s|2k|4rs\) so \((2r-1)s|4rs\) or \(2r-1|4r\). But since \(2r-1\) is odd,
\(2r-1|r\) so that \(r=1\). But then the order is \(q^2\) and the group is
\(\text{SL}(2,q)\) and the planes are determined in Foulser-Johnson ([5],[6]).

Now let \(g\) fix \(p^t\) points on a fixed component \(\mathcal{L}\) of \(\pi_0\). \(\Rightarrow\) \(g\)
is completely reducible on \(\mathcal{L} \cap \pi_0\), \(\mathcal{L} \cap \pi_0 = (\text{Fix } g \ 	ext{on } \mathcal{L} \cap \pi_0) \oplus W\)
where \(|W| = \frac{p^k}{p^t}\). Hence \(|g|\) \((p^{k-t},(2r-1)s)-1)\). However, this cannot
be the case unless \((2r-1)s|k-t\). But \(k|2rs\) so \(k-2rs-t\) and \(k \leq rs\).
Then \((2r-1)s|2rs-t\) so \(s|t\) and \(k-t \leq rs-s\). Hence, \((2r-1)s|2rs-ls\)
so that \(l = 1\) and \(s=t\). But, if \((2r-1)s|k-s\) then \((2r-1)s \leq rs-s\)
which obviously cannot be.

Thus, it must be that \(k=t\) so that \(g\) fixes \(\pi_0\) pointwise. By
(1.1)(a) Fix \(g\) on each fixed component has order \(q\). So \(\pi_0 \subseteq \text{Fix } g\)
and Fix \(g\) is a subplane of order \(q\).

Let \(\mathcal{L}\) be a component of \(\pi_0\) then \(\mathcal{L} = (\text{Fix } g) \oplus C_{g,\mathcal{L}}\) where
\(C_{g,\mathcal{L}}\) is the unique \(g\)-submodule on \(\mathcal{L}\) which is disjoint from
\((\text{Fix } g) \oplus \mathcal{L})\). But, \(g\) fixes \(\mathcal{L}\) and therefore must fix the module
\(C_{g,\mathcal{L}}\) since \(g\) permutes the \(g\)-submodules on \(\mathcal{L}\). However, this implies
that \(g\) fixes additional points on \(\mathcal{L}\).

Hence, the \(p\)-elements cannot be planar.

Now let \(\sigma\) be a \(p\)-element and assume \(\text{Fix } \sigma\) lies in a component
\(\mathcal{L}\). The previous argument shows that \(\text{Fix } \sigma \subseteq \text{Fix } g\), \(\text{Fix } g\) has order
\(q\) on \(\mathcal{L}\) and \(\sigma\) must fix the complement \(C_{g,\mathcal{L}}\) of \(g\) on \(\mathcal{L}\). That is,
again \(\sigma\) must fix additional points of \(\mathcal{L}\) (since \(|\sigma| = p\).
This proves (3.2).
REFERENCES


Ricevuto il 31/1/1986

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