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ANALYTIC FUNCTIONALS

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1. **INTRODUCTION.** Let E be a complex, Hausdorff locally convex space and U an open subset of E. We denote by $\mathscr{H}(U)$ the vector space of all analytic functions in U. The *compact-open* topology τ_o defined on $\mathscr{H}(U)$ is the most natural topology but, in the case dim $E = \infty$, $[\mathscr{H}(U), \tau_o]$ may have undesirable properties (for instance, if $E = \mathbb{C}^{\mathbb{N}}$ then $[\mathscr{H}(E), \tau_o]$ is not infrabarreled).
So, in the infinite-dimensional case we weed to consider finer topologies on $\mathscr{H}(U)$ in order to obtain nicer topological properties.

The τ_w-topology, introduced by Nachbin [45], is defined by the semi-norms on $\mathscr{H}(U)$ which are ported by the compact subsets of U. A semi-norm p on $\mathscr{H}(U)$ is *ported* by the compact subset K of U if (and only if) for every open neighbourhood V of K, $V \subset U$, there exists a constant $C(V) > 0$ such that the following inequality

$$p(f) \leq C(V) \|f\|_V$$

holds for every $f \in \mathscr{H}(U)$. (Here $\|f\|_V = \sup_{x \in V} |f(x)|$.)

It is clear that $\tau_o \preceq \tau_w$. This topology was motivated by the results of A.Martineau [39] on analytic functionals in several complex variables. The τ_w topology has better properties than τ_o, but in the case $E = \mathbb{C}^N$, $\tau_o = \tau_w$ and so τ_w may also have unsuitable properties. The τ_w-topology has also some further difficulties. For instance, we cannot in general describe a directed set of semi-norms which generates τ_w. The topology on $\mathscr{H}(U)$ which has the strogest properties is the τ_δ-topology, introduced by Coeuré [12] and Nachbin [46]. A semi-norm p on $\mathscr{H}(U)$ is τ_δ-*continuous* if and only if for each increasing countable open cover of U, $(V_n)_{n=1}^{\infty}$, there exist a positive integer n_o and $C > 0$ such that the inequality

$$p(f) \leq C\|f\|_{V_{n_o}}$$

holds for every f in $\mathscr{H}(U)$. The τ_δ topology on $\mathscr{H}(U)$ is the locally convex topology generated by the τ_δ-continuous semi-norms.

We always have $\tau_o \leq \tau_w \leq \tau_\delta$; τ_δ is barreled and hornological but, unhappily, we cannot in general describe a directed set of τ_δ-continuous seminorms which generates the τ_δ-topology. Of course, the best situation is when $\tau_o = \tau_\delta$. This is, actually, the case when E is the strong dual of a Fréchet-Montel space (Dineen [15]). On the other hand, $\tau_o \neq \tau_w$ when $E = \mathbb{C}^I$, I an uncountable set and $\tau_w = \tau_\delta$ on $\mathscr{H}(\mathbb{C}^I)$ if and only if I is finite (Barroso [3]). Locally convex topologies on the space of analytic functions defined on open subsets of infinite-dimensional locally convex spaces have been extensively studied by several authors and good references on this aspect of the theory are the works of Dineen [17] and Noverraz [51].

In studying the questions: $\tau_o = \tau_w$, $\tau_o = \tau_\delta$, $\tau_w = \tau_\delta$ on $\mathscr{H}(U)$ one approach consists in characterizing the dual space $[\mathscr{H}(U),\tau]'$

(where $\tau = \tau_o, \tau_w, \tau_\delta$). In the one-dimensional case this problem led to the study of Fantappié's analytic functionals. For instance, if U is an open disc of the complex plane, Fantappié's formula for the analytic functional $T \in [\mathscr{H}(U), \tau_o]'$ is given by

$$(*) \qquad T(f) = \frac{1}{2\pi i} \int_{C_{n+1}} u(t) f(t) dt \qquad \text{for every } f \in \mathscr{H}(U),$$

where C_{n+1} is the boundary of the disc V_{n+1}, the increasing sequence of open discs $(V_n)_{n=1}^{\infty}$ covers U and $\bar{V}_n \subset V_{n+1}$ for all n and $u(\lambda)$ is the Fantappié's indicatrix in $\bar{D}_n = \bar{\mathbb{C}} - V_n$, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Fantappié's indicatrix $u(\lambda)$ is a locally analytic function on $\bar{D}_n$. If $K = \bar{\mathbb{C}} - U \subset D_n$, it is easy to see that every locally analytic function is defined on an open neighbourhood of K and the right member of (*) defines a continuous linear functional on $\mathscr{H}(U)$. If $u_1(\lambda)$ and $u_2(\lambda)$ are two locally analytic functions, defined respectively in the open neighbourhoods W_1 and W_2 of K, such that $u_1|W \equiv u_2|W$ for some open neighbourhood W of K, $W \subset W_1 \cap W_2$, then it is easy to show that

$$\frac{1}{2\pi i} \int_{C_{n+1}} u_1(t) f(t) dt = \frac{1}{2\pi i} \int_{C_{n+1}} u_2(t) f(t) dt$$

and so $u_1(t)$ and $u_2(t)$ define the same linear functional.

If we consider on the set G of all locally analytic functions defined on an open neighbourhood of K the equivalence relation given by $u_1 \sim u_2$ if and only if $u_1|W \equiv u_2|W$ for some open neighbourhood W of K with $W \subset W_1 \cap W_2$, where W_1 and W_2 are respectively the open neighbourhoods of K where u_1 and u_2 are locally analytic, then we get the vector space $\mathscr{H}(K)=G/\sim$ of locally analytic functions on K. The analytic functional $T\in[\mathscr{H}(U),\tau_0]'$ is given by the indicatrix $u\in\mathscr{H}(K)$ and, conversely, each $u\in\mathscr{H}(K)$ defines a $T\in[\mathscr{H}(U),\tau_0]'$. The correspondence $u\to T$ between $\mathscr{H}(K)$ and $[\mathscr{H}(U),\tau_0]'$ is linear and bijective and so these spaces are algebraically isomorphic.

If we consider on $\mathscr{H}(K)$ the topology of uniform convergence on the bounded subsets of $[\mathscr{H}(U),\tau_0]$, we have that $\mathscr{H}(K)$ is reflexive and so is a barreled space. By the Mackey-Arens theorem, the topology of $\mathscr{H}(K)$ is the finest locally convex topology such that the mappings

$$\rho_n : \mathscr{H}^{\infty}(\bar{D}_n) \to \mathscr{H}(K)$$

are continuous for all n, ($\mathscr{H}^{\infty}(\bar{D}_n)$ is the Banach space of all locally analytic functions on D_n with continuous extensions to $\bar{D}_n$), i.e., the topology of $\mathscr{H}(K)$ is the inductive limit topology of the spaces $\mathscr{H}^{\infty}(\bar{D}_n)$, $n\in\mathbb{N}$. We denote this by

$$\mathscr{H}(K) = \lim_{\substack{\to \\ n}}[\mathscr{H}^{\infty}(\bar{D}_n);\| \ \|_{D_n}].$$

We have that the dual of $\mathscr{H}(K)$ is $[\mathscr{H}(U),\tau_0]$.

An "analytic functional space" was considered for the first time by Pincherle [54] in 1901. The concept of locally analytic linear functional and the concept of indicatrix were introduced by Fantappié [20] in 1930. The formula (*), which gives the functional through its indicatrix, appeared in Fantappié's work in 1930. A short proof of this formula appears in Fantappié [21] in 1932. A good reference on analytic functionals can be found in Fantappié [23].

The duality for $\mathscr{H}(U)$, in the case U is a disc of $\mathbb{C}$, appears in 1949 with Toeplits [73]. The general case and subsequent generalizations appear in Sebastião e Silva ([58] to [60]), Silva Dias [61],[62], Köthe [32],[33], Grothendieck [26],[27] and Tillmann [69] to [72]. In 1950, Sebastião e Silva [58] presented a new systematic account of the theory using the concepts of "Analyse Generale". From these historical remarks the following definitions seem to be natural.

DEFINITION 1.1. Let K be a non-void compact subset of E. Let G be the collection of pairs (U,f), where U is an open neighbourhood of K and $f \in \mathscr{H}(U)$. Define on G the following equivalence relation:

$$(U,f) \sim (V,g) \Leftrightarrow f|_W \equiv g|_W$$

for some open neighbourhood W of K with $W \subset U \cap V$. An *analytic germ* on K is the equivalence class of an element of G through this equivalence relation. Let $\mathscr{H}(K)$ be the collection of analytic germ on K. $\mathscr{H}(K)$ is a vector space through the canonical mappings

$$\rho_U : \mathscr{H}(U) \to \mathscr{H}(K).$$

The definition of $\mathscr{H}(K)$ does not change if we restrict ourselves to the family $\mathscr{F}$ of open neighbourhoods U of K such that each connected component has a non void intersection with K. For such U's the mappings ρ_U are injective and each $\mathscr{H}(U)$

may be identified with a subspace of $\mathscr{H}(K)$. With this identification we may write

$$\mathscr{H}(K) = \bigcup_{U \supset K} \mathscr{H}(U).$$

DEFINITION 1.2. The topology on $\mathscr{H}(K)$ is the topology τ defined by

$$[\mathscr{H}(K), \tau] = \varinjlim_{U \in \mathscr{F}} (\mathscr{H}^{\infty}(U); \| \|_U),$$

where $(\mathscr{H}^{\infty}(U), \| \|_U)$ is the Banach space of bounded analytic functions on U endowed with the supremum-norm topology, i.e. τ is the finest locally convex topology rendering the applications $\rho_U | \mathscr{H}^{\infty}(U)$ continuous for every $U \in \mathscr{F}$.

The topology τ can be defined also by

$$[\mathscr{H}(K), \tau] = \varinjlim_{U \in \mathscr{F}} [\mathscr{H}(U), \tau_W],$$

i.e. the finest locally convex topology rendering the applications ρ_U continuous for every $U \in \mathscr{F}$.

Since $[\mathscr{H}(K), \tau]$ is an inductive limit of Banach spaces, it is bornological and barreled. It is not difficult to show that τ is a Hausdorff topology. But, even in the case of a metrizable locally convex space E, $\mathscr{H}(K)$ is not a strict inductive limit (for $V \supset U \supset K$, V and U open subset of E the norm on $\mathscr{H}^{\infty}(U)$ induces a norm on $\mathscr{H}^{\infty}(V)$ which may be strictly coarser than the norm on $\mathscr{H}^{\infty}(V)$).

So the main problems concerning $\mathscr{H}(K)$ are the following:

PROBLEM 1. *Describe the continuous semi-norms on* $\mathscr{H}(K)$.

PROBLEM 2. *Characterize the bounded subsets of* $\mathscr{H}(K)$.

PROBLEM 3. *Is* $\mathscr{H}(K)$ *complete?*

These problems are not independent. It is the aim of this survey article to present a sketch of the techniques which allowed to advance into these problems.

in the closure [in $\mathscr{H}(K)$] of a bounded subset of some $\mathscr{H}^{\infty}(K+V_{n_o})$. But, since the closed unit ball in $\mathscr{H}^{\infty}(K+V_{n_o})$ is closed in $\mathscr{H}(K)$, F is contained in $\mathscr{H}^{\infty}(K+V_{n_o})$ and is bounded there. Hence we have the following

THEOREM 2.1. *Let* K *be a compact subset of a metrizable locally convex space* E. *Then a subset* F *of* $\mathscr{H}(K)$ *is bounded if and only if* F *is contained and bounded in* $\mathscr{H}^{\infty}(U)$ *for some open subset* U *containing* K.

This result motivated the following

DEFINITION 2.2. The inductive limit $\mathscr{H}(K) = \varinjlim_{U\in\mathscr{F}}[\mathscr{H}^{\infty}(U);\|\|_U]$ is called ***regular*** if each bounded subset of $\mathscr{H}(K)$ is contained and bounded in some $\mathscr{H}^{\infty}(U)$.

The characterization of the bounded subsets of $\mathscr{H}(K)$ through a family of continuous semi-norms on $\mathscr{H}(K)$ was first investigated by Hirschowitz [29],[30] and Chae [11]. Their approach (which was undertaken before Theorem 2.1 was proved) was to describe families of continuous semi-norms on $\mathscr{H}(K)$ generating a coarser τ_1-topology on $\mathscr{H}(K)$ but having the same bounded subsets. A natural family is the family on (p_1) seminorms on $\mathscr{H}(K)$ defined by

$$(*)\qquad p_1(f) = \sum_{n=0}^{\infty} \sup_{x\in K} p\left[\frac{\hat{d}^m f(x)}{m!}\right], \qquad f \in \mathscr{H}(K),$$

where p runs over the family of continuous semi-norms on $\mathscr{H}(0)$. This family of semi-norms does not, however, yield the same bounded sets, as the following example shows.

EXAMPLE 2.3. Let $E = \mathbb{C}$. For each positive integer n, let $U_n = \{z\in\mathbb{C};\ \text{Re } z < \frac{1}{n+1/2}\}$ and $V_n = \{z\in\mathbb{C};\ \text{Re } z > \frac{1}{n+1/2}\}$.

We define $f_n \in \mathscr{H}(U_n \cup V_n)$ as follows: $f_n \equiv 0$ in V_n and $f_n \equiv 1$ in U_n. If $K = \{1/n\}_{n=1}^{\infty} \cup \{0\}$ then K is a compact subset of $\mathbb{C}$ and $\{f_n\}_{n=1}^{\infty}$ is a τ_1-bounded subset of $\mathscr{H}(K)$ which is not bounded in the τ-topology of $\mathscr{H}(K)$.

The above example motivated the following question:

When are the τ_1-bounded subsets of $\mathscr{H}(K)$ τ-bounded in $\mathscr{H}(K)$? This question was studied by A.Baerstein [2] who gave a positive answer in the case of locally connected compact subsets of $\mathbb{C}$. Zame [76] also gave a positive answer in the case of compact subset of $E = \mathbb{C}^n$ satisfying a very weak connectedness assumption. Zame's method was generalized by Soraggi [64] to infinite dimensional locally convex spaces. Mujica [40], generalized Baerstein's method for metrizable locally convex spaces. For the case of a compact subset of $\mathbb{C}$ having only a finite number of connected components, Rogers and Zame [55] showed that the topologies τ_1 and τ coincide and so we have an internal description of the bounded subsets of $\mathscr{H}(K)$ and also an answer to Problem 1. This result does not extend to $\mathbb{C}^n$, $n \geq 2$. Recently, Grzybowski [28], in using a geometric approach to the examination of the extension property, obtained a result which unified all the above results for compact subsets of $\mathbb{C}^n$.

We now discuss the completeness of $\mathscr{H}(K)$. Recall that a quasi-complete DF-space is complete (see Grothendieck [25]).

So, in the metrizable case it is sufficient to study the completeness of closed bounded subsets. Using Theorem 2.1, Mujica [41] showed that $\mathcal{H}(K)$ is complete for compact subsets of certain metrizable locally convex spaces. Subsequently, Bierstedt and Meise [5] proved the same result for compact subsets of Fréchet-Schwartz spaces. Aviles and Mujica [1] extended this to quasi-normable locally convex spaces. To show completeness of $\mathcal{H}(K)$ for compact subsets of an arbitrary metrizable locally convex space another approach was necessary. This problem remained open for many years and was solved by Dineen [16]. Dineen's approach consists in returning to the problem of describing continuous semi-norms on $\mathcal{H}(K)$, to define continuous semi-norms on $\mathcal{H}(K)$ of the type introduced by Hirschowitz in a correction to his article [29].

Let $(x_n)_{n=1}^{\infty}$ and $(x_n')_{n=1}^{\infty}$ be convergent sequences in K. Let $(y_n)_{n=1}^{\infty}$ and $(y_n')_{n=1}^{\infty}$ be null sequences in E such that $x_n+y_n = x_n'+y_n'$ for all n. Let $(k_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. Define the following continous semi-norm on $\mathcal{H}(K)$:

$$(**) \qquad p_2(f) = \sum_{n=0}^{\infty} 2^{k_n} \left| \sum_{m=0}^{k_n} \frac{\hat{d}^m f(x_n)}{m!}(y_n) - \sum_{m=0}^{k_n} \frac{\hat{d}^m f(x_n')}{m!}(y_n') \right|$$

for all $f \in \mathcal{H}(K)$.

Let τ_2 be the locally convex topology on $\mathcal{H}(K)$ generated by the semi-norms of types (*) and (**). Dineen [16] showed the following

THEOREM 2.4. *Let* K *be a compact subset of a metrizable locally convex space* E. *A subset* B *of* $\mathcal{H}(K)$ *is bounded if and only if* B *is* τ_2*-bounded.*

Proof. The semi-norms of type (**) are continuous, since $\mathcal{H}(K)$ endowed with its natural topology τ is a barreled space.

Conversely, if B is a τ_2-bounded subset of $\mathcal{H}(K)$, then using semi-norms of type (*), we have that the set

$$\{ \frac{\hat{d}^n f(x)}{n}, \quad n \in \mathbb{N}, \quad x \in K, \quad f \in B\}$$

is a bounded subset of $\mathcal{H}(0)$. Since $\mathcal{H}(0)$ is regular, there exist an open neighbourhood W of zero in E and a constant $M > 0$ such that

$$\text{(A)} \qquad \left\|\frac{\hat{d}^n f(x)}{n!}\right\|_W \leq \frac{M}{4^n}$$

for all $f \in B$, $x \in K$ and $n \in \mathbb{N}$.

If $f \in B$, $x \in K$, $y \in W$, let

$$f(x)(y) = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(x)}{n!}(y).$$

The proof will be complete if we show that: "there exists an open neighbourhood U of zero in E, $U \subset W$, such that

$$f(x)(y) = f(x')(y')$$

for all $x,x' \in K$, $y,y' \in U$, whenever $x+y = x'+y'$ and $f \in B$." (Claim (B)).

If it is not possible to find such an U, then there exist sequences $(x_n)_{n=1}^{\infty}$ and $(x'_n)_{n=1}^{\infty}$ in K, null sequences $(y_n)_{n=1}^{\infty}$

and $(y'_n)_{n=1}^{\infty}$ in E, and a sequence $(f_n)_{n=1}^{\infty}$ in B such that $x_n+y_n = x'_n+y'_n$ for all n and

$$|f_n(x_n)(y_n) - f_n(x'_n)(y'_n)| = \delta_n \neq 0.$$

Choose, inductively, a strictly increasing sequence of positive integers $(k_n)_{n=1}^{\infty}$ such that $2^{k_n} \delta_n > n+2M$ for all n. We have

$$p_2(f_n) \geq 2^{k_n} \left| \sum_{m=0}^{k_n} \frac{\hat{d}^m f_n(x_n)}{m!}(y_n) - \sum_{m=0}^{k_n} \frac{\hat{d}^m f_n(x'_n)}{m!}(y'_n) \right|$$

$$\geq 2^{k_n} | \delta_n - 2 \sum_{j=k_n+1}^{\infty} \frac{M}{4^j} | \geq 2^{k_n} \delta_n - 2M > n.$$

Since $(f_n)_{n=1}^{\infty} \subset B$ we must have $\sup_{n\geq 1} p_2(f_n) < \infty$. This contradiction proves claim (B) and the proof is complete.

On examining the proof of Theorem 2.4, we see that the boundedness of semi-norms of type (*) on a subset B of $\mathscr{H}(K)$ implies the existence of uniform Cauchy estimates on K for the elements of B (inequality (A)). On the other hand, the boundedness of semi-norms of type (**) on a subset B of $\mathscr{H}(K)$ implies that there is coherence in the Taylor series development of the elements of B (Claim (B)).

Using Theorem 2.4, Dineen [16] showed the following

THEOREM 2.5. *If K is a compact subset of a metrizable locally convex space, then $\mathscr{H}(K)$ is a complete space.*

To proof Theorem 2.5 Dineen first showed that $[\mathcal{H}(K), \tau_2]$ is a quasi-complete space. Then he considered the barreled topology τ_t on $\mathcal{H}(K)$ associated to τ_2. (See Komùra [35] or Schmets [57]). We have that $[\mathcal{H}(K), \tau_t]$ is quasi-complete (see Noureddine [49] or Noureddine and Schmets [50]). Since $\tau_2 \leq \leq \tau_t \leq \tau$, by Theorem 2.4 τ_t and τ have the same bounded subsets. Since $[\mathcal{H}(K), \tau]$ is a DF-space, $[\mathcal{H}(K), \tau_t]$ is also a DF-space and hence $\mathcal{H}(K)$ is τ_t-complete. The proof is concluded by showing that $\tau = \tau_t$.

Concerning Problem 1, Rusek [56] solved the problem of describing a family of continuous semi-norms on $\mathcal{H}(K)$ which generates the topology of $\mathcal{H}(K)$ for compacts subsets of $\mathbb{C}^n$ satisfying a very weak connectedness assumption.

Nicodemi [48] considered on $\mathcal{H}(K)$ a new topology, i.e.

$$[\mathcal{H}(K), \tau_o] = \lim_{\substack{\to \\ U \in \mathcal{F}}} [\mathcal{H}(U), \tau_o].$$

Considering the topology τ_o on $\mathcal{H}(K)$, Mujica [43] showed the following

THEOREM 2.6. *Let* E *be a Fréchet space and* K *a compact subset of* E. *We have:*

a) $[\mathcal{H}(\dot{K}), \tau_o]$ *is an inductive limit (in the category of topological spaces) of an increasing sequence of compact topological spaces. In particular* $[\mathcal{H}(K), \tau_o]$ *is a* K-*space.*

b) *The topology* τ_o *on* $\mathcal{H}(K)$ *is generated by the semi-norms of the following types:*

(*) $$p_1(f) = \sum_n \sup_{n\in K} \| \frac{\hat{d}^n f(x)}{n!} \|_{L_n} ,$$

where (L_n) *is a sequence of compact subsets of* E, *tending to zero, that is, for each neighbourhood* V *of zero in* E *there exists* $n_o \in \mathbb{N}$ *such that* $L_n \subset V$ *for all* $n \geq n_o$;

or

(**) $$p_2(f) = \sup_n 2^{k_n} \left| \sum_{n=0}^{k_n} \frac{\hat{d}^n f(x_n)}{n!}(y_n) - \sum_{n=0}^{k_n} \frac{\hat{d}^n f(x'_n)}{n!}(y'_n) \right|$$

where $(x_n)_{n=1}^{\infty}, (x'_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (y'_n)_{n=1}^{\infty}$ *and* $(k_n)_{n=1}^{\infty}$ *are as before (in defining the* τ_2*-topology).*

It follows from Theorem 2.6 a) that $[\mathscr{H}(K), \tau_o]$ is always the dual of a Fréchet space endowed with the compact-open topology. It also follows from Theorem 2.6 a) and a result of Bierstedt-Meise [5] that the τ_o and τ topologies coincide on $\mathscr{H}(K)$ when K is a compact subset of a Fréchet-Schwartz space. Consequently, we have $\tau_o = \tau_w$ on $\mathscr{H}(U)$ when U is a balanced open subset of a Fréchet-Schwartz space.

If E is a locally convex space, let E'_i be the dual of E endowed with the inductive topology, i.e. the locally convex topology defined by $E'_i = \varinjlim (E')_{V^o}$ when V runs the family of convex, balanced neighbourhoods of E (see Berezanskii [4] or Floret [24]). This topology is the topology induced on E' by $[\mathscr{H}(E), \tau_w]$ and is finer than the strong topology β of E'.

Using the inductive topology of Berezanskii [4] and a characterization of the dual of a Banach space due to Dixmier [19] and Ng [47], Mujica [44] showed the following

THEOREM 2.7. *Let* $E = \varinjlim E_n$ *be an inductive limit of a sequence of Banach spaces. Suppose there exists a locally convex topology* τ *on* E *such that the closed unit ball* B_n *of each* E_n *is* τ*-compact. Then* $E = F_i'$ *for some Fréchet space* F. *In particular,* E *is complete.*

Now, if $(V_n)_{n=1}^{\infty}$ is a fundamental sequence of open subsets of a Fréchet space E containing the compact subset K of E then, by Ascoli's Theorem, the closed unity ball B_n of $\mathscr{H}^{\infty}(V_n)$ is τ_0-compact in $\mathscr{H}(U)$ and hence τ_0-compact in $\mathscr{H}(K)$ and therefore, Theorem 2.7 has as corollary the following

THEOREM 2.8. *Let* K *be a compact subset of* E. *Then* $\mathscr{H}(K) = Y_i'$ *for some Fréchet space* Y. *In particular,* $\mathscr{H}(K)$ *is complete.*

So, the space of analytic germs on a compact subset of a metrizable locally convex space has been quite well studied.

On the other hand, Krée ([36] to [38]), in his investigations on the mathematical foundations of quantum field theory with infinitely many degrees of freedoom, used the nuclearity of $[\mathscr{H}(U), \tau_0]$ for open subsets of a quasi-complete space E whose strong dual is a nuclear space (a result due to P.Boland [7] and L.Waelbroeck [75]). Krée also wished to know if $[\mathscr{H}(s), \tau_0]$-s the space of rapidly decreasing sequences - was a bornological space. This, together with the problem of whether $[\mathscr{H}(E), \tau_0]$ had a basis - E a nuclear space with a Schauder basis - and some questions related to $\mathscr{H}(\mathscr{D})$ - $\mathscr{D}$ the space of test functions of L.Schwartz-and the possibility of a kernel theorem for analytic functionals in infinitely many variables, motivated an intense research on analytic functions defined on nuclear spaces. First we give some definitions. Our basic references are Boland and Dineen [8] and Dineen [17].

DEFINITION 2.9. *A fully nuclear space* is a locally convex space E such that E and the strong dual E'_β are infrabarreled complete nuclear spaces.

A fully nuclear space is always barreled. Every nuclear Fréchet space is fully nuclear. The space $\mathcal{H}$ is fully nuclear.

DEFINITION 2.10 *A fully nuclear space with basis* is a fully nuclear space with a Schauder basis (hence an equicontinuous, Schauder and absolute basis).

DEFINITION 2.11. (Köthe [34]). Let P be a collection of sequences of non-negative real numbers such that for each positive integer $m > 0$ there exists $\alpha = (\alpha_n)_{n=1}^\infty \in P$ with $\alpha_m > 0$. The *sequence space* $\Lambda(P)$ is defined as the set of all sequences of complex numbers $(z_n)_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty \alpha_n |z_n| < \infty \qquad \text{for all } \alpha = (\alpha_n)_{n=1}^\infty \in P.$$

We consider $\Lambda(P)$ endowed with the topology generated by the family of semi-norms p_α , $\alpha = (\alpha_n)_{n=1}^\infty \in P$, defined by

$$p_\alpha[(z_n)_{n=1}^\infty] = \sum_{n=1}^\infty \alpha_n |z_n| \qquad \text{for all } (z_n)_{n=1}^\infty \in \Lambda(P).$$

Let E be a Hausdorff locally convex space with an absolute basis $(e_n)_{n=1}^{\infty}$. If $P = \{(p(e_n))_{n=1}^{\infty}$, when p runs over the family cs(E) of continuous semi-norms on E$\}$, we define a natural mapping from E into $\Lambda(P)$ as

$$\sum_{n=1}^{\infty} z_n e_n \in E \mapsto (z_n)_{n=1}^{\infty} \in \Lambda(P).$$

E is algebraically isomorphic to its image in $\Lambda(P)$. E is isomorphic to $\Lambda(P)$ if and only if E is complete.

It follows from the Grothendieck-Pietsch criterion (see [53]) that a fully nuclear space E with a basis is a sequence space and so is E'_β hence we can define polydiscs in E and E'_β.

DEFINITION 2.12. A set $U \subset \Lambda(P)$ is a *polydisc* if U is a subset of $\Lambda(P)$ of one of the following types:

(*) $$\{(z_n)_{n=1}^{\infty} \in \Lambda(P); \quad \sup_n |z_n \beta_n| < 1\},$$

or

(**) $$\{(z_n)_{n=1}^{\infty} \in \Lambda(P); \sup_n |z_n \beta_n| \leq 1\},$$

where $\beta_n \in [0,+\infty]$ for all n, $[a\cdot(+\infty) = +\infty$ if $a > 0$ and $0\cdot(+\infty)=0]$.

A polydisc of type (*) is open if and only if $(\beta_n)_n \in P$;
A polydisc of type (**) is always closed.

DEFINITION 2.13. Let $E \simeq \Lambda(P)$ be a fully nuclear space with a basis and let U be a subset of E. We define the *multiplicative polar* U^M of U by

$$U^M = \{(w_n)_{n=1}^{\infty} \in E'_\beta \simeq \Lambda(P'); \sup_n |w_n z_n| \leq 1 \text{ for all } (z_n)_{n=1}^{\infty} \in U\}.$$

If U is an open polydisc in E then the multiplicative polar

U^M is a compact polydisc in E'_β .

Using the fact that $[\mathscr{H}(U), \tau_0]$ has a basis, Boland and Dineen [8] showed the following

THEOREM 2.14. *Let* U *be an open polydisc in a fully nuclear space with a basis. Then* $[\mathscr{H}(U), \tau_0]'$ *is topologically isomorphic to the space* $\mathscr{H}(U^M)$ *of analytic germs on the compact polydisc* U^M *. Under this isomorphism, equicontinuous subsets of* $[\mathscr{H}(U), \tau_0]'$ *correspond to subsets of* $\mathscr{H}(U^M)$ *which are uniformly bounded on some open neighbourhood of* U^M .

For open polydisc U in fully nuclear spaces with a basis, Theorem 2.14 has the following corollary:

$[\mathscr{H}(U), \tau_0]$ *is infrabarreled* $\Leftrightarrow$ $[\mathscr{H}(U^M),]$ *is a regular inductive limit* $\Leftrightarrow$ $[\mathscr{H}(U), \tau_0]$ *is bornological.*

This result motivated the study of regularity of $\mathscr{H}(K)$ for compact subsets K of nonmetrizable locally convex spaces. For a certain category of fully nuclear spaces with basis, regularity and completeness are equivalent when U=E (Boland-Dineen [9], Proposition 10).

Unfortunately, Mujica's method does not apply in this case. But the arguments used to prove Theorem 2.4 can be adapted to study regularity of $\mathscr{H}(K)$ when K is a metrizable compact subset of certain non-metrizable locally convex spaces.

R. Aron showed that for $E = \mathbb{C}^{[\mathbb{N}]}$ the space of germs at the origin - $\mathscr{H}(0)$ - is not regular. In 1978, Boland and Dineen [9] showed that if E is the strong dual of a Fréchet nuclear space with basis and E'_β does not have a continuous norm, then $\mathcal{H}(0_E)$ - the space of germs at the origin of E - is not regular. On the other hand using Theorem 2.14 Boland and Dineen [10] showed that $\mathcal{H}(0)$ - the origin in the space of distributions $\mathcal{D}'$ (respectively $\mathcal{D}$) - is regular (is not regular).

So regularity in spaces of analytic germs on compact subsets of non-metrizable locally convex spaces may be a very delicate question. Even for the strong dual of a Fréchet space, $\mathcal{H}(0)$ may be regular as for instance $0 \in E = s'$ or $\mathcal{H}(0)$ may fail to be regular as for instance $E = \mathbb{C}^{[\mathbb{N}]}$.

The first problem was to study regularity of $\mathcal{H}(0)$. This problem is equivalent to the existence of uniform Cauchy estimates for the elements of a bounded subset B of $\mathcal{H}(0)$, i.e., does there exist $\lambda > 1$, $M > 0$ and a neighbourhood W of the origin of E such that

$$\left\| \frac{1}{m!} \hat{d}^m f(0) \right\|_W \leq \frac{M}{\lambda^m} \qquad \text{for all } m \in \mathbb{N} \text{ and all } f \in B?$$

In the nuclear case, Dineen ([17], Theorem 5.42) gives new examples where $\mathcal{H}(0)$ is regular. Further examples where $\mathcal{H}(0)$ is regular are given by Soraggi [66].

Now, let B be a bounded subset of $\mathcal{H}(K)$. By using semi-norms of (*)-type, we have that

$$\left\{ \frac{\hat{d}^n f(x)}{n!}; \quad n \in \mathbb{N}, \quad x \in K, \quad f \in B \right\}$$

is bounded in $\mathcal{H}(0)$.

If $\mathcal{H}(0)$ is regular, then, as in the proof of Theorem 2.4, the existence of uniform Cauchy estimates for the elements of B implies that for each f in B, x in K and y in a neighbourhood W of the origin of E, we can define

$$f(x)(y) = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(x)}{n!}(y).$$

However the Taylor series expansions may fail to be coherent, i.e.,

QUESTION 2.15. *Is there an open neighbourhood of the origin* $V \subset W$ *such that*

$$\tilde{f}(x)(y) = \tilde{f}(x')(y') \qquad \text{for all } x,\ x' \text{ in } K \text{ and } y, y' \text{ in } V,$$

whenever $x+y = x'+y'$ and $f \in B$?

If the question has a positive answer, then we see easily that B is contained and bounded in $\mathcal{H}^{\infty}(K+V)$ and so regularity of $\mathcal{H}(K)$ will follow. Hence the study of the regularity of $\mathcal{H}(K)$ leads to the following

QUESTION 2.16. *When is regularity of* $\mathcal{H}(0)$ *sufficient to imply regularity of* $\mathcal{H}(K)$?

Two different approaches were considered. The first approach consists in identifying those elements in K where there is no coherence in the Taylor series expansions and to endow the correspondent quotient space with the structure of an analytic space. For this we need to impose very weak connectedness assumptions on the compact subset K in order to show that some functions defined in this analytic space are analytic. This was done by Zame [76] in the finite-dimensional case and by Soraggi [65] in the infinite-dimensional case, where a positive answer to the question can be found. The second approach consists in using semi-norms and type (**) and to adapt the arguments used to prove Theorem 2.4. Soraggi [66] showed that Question 2.16 has always a positive answer when K is metrizable and E satisfies a very weak technical condition which holds in most, if not all, spaces for which $\mathcal{H}(0)$ is regular.

DEFINITION 2.17. A locally convex space E *satisfies condition* P if for each convex, balanced open subset U of E and for each non-trivial (i.e. $f_n \neq 0$ for all n) sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}(U)$, there exist a subsequence $(f_{n_j})_{j=0}^{\infty}$ and a bounded sequence $(y_j)_{j=0}^{\infty}$ in U such that $f_{n_j}(y_j) \neq 0$ for all $j\in\mathbb{N}$.

Baire spaces, metrizable locally convex spaces, products of metrizable locally convex spaces, C(X) - the space of complex-valued continuous functions defined on a completely regular Hausdorff space, endowed with the compact-open topology - are examples of locally convex spaces satisfying condition P.

A sequence $(x_n)_{n=1}^{\infty}$ in a locally convex space E is *very strongly convergent*, if, for each sequence of scalars $(\lambda_n)_{n=1}^{\infty}$, $(\lambda_n x_n)_{n=1}^{\infty}$ is a null sequence in E.

In studying $\mathcal{H}(K)$ there is no loss of generality if we restrict ourselves to complete locally convex spaces. In this case, Soraggi [66] showed that E satisfies condition P if and only if there is no non-trivial very strongly convergent sequence in $\mathcal{H}(E)$ endowed with the compact-open topology τ_0.

Very strongly convergent sequence were introduced by Dineen [14] in studying holomorphic completions and they were applied by Dineen [13] in other areas of infinite-dimensional holomorphy. If E is the strong dual of a reflexive Fréchet space, non existence of a non-trivial, very strongly convergent sequence in $F=E'_\beta \subset [\mathcal{H}(E),\tau_0]$ is equivalent to E satisfying condition P and so the latter is equivalent to F admitting a continuous norm (see Dineen [13]).

Using condition P, Soraggi [66] showed the following

THEOREM 2.18. *Let* E *be a locally convex space satisfying condition* P *and let* K *be a metrizable compact subset of* E. *If* $\mathcal{H}(0)$ *is regular, then* $\mathcal{H}(K)$ *is regular.*

Sketch of proof. Let B be a bounded subset of $\mathcal{H}(K)$ and suppose that there is no coherence in the Taylor series expansions of the elements of B, i.e., for each open neighbourhood of the origin $V_\alpha \subset W$ (where W is given by the uniform Cauchy estimate) we can find

$$x_\alpha, x'_\alpha \text{ in } K,\ y_\alpha,\ y'_\alpha \text{ in } V_\alpha,\ \ x_\alpha + y_\alpha = x'_\alpha + y'_\alpha,\ f_\alpha \in B$$

such that

$$f_\alpha(x_\alpha)(y_\alpha) \neq f_\alpha(x'_\alpha)(y'_\alpha).$$

Ordering $(V_\alpha)_\alpha$ by set-theoretical inclusion, we have $y_\alpha \to 0$, $y'_\alpha \to 0$ and so $x_\alpha - x'_\alpha \to 0$. Since K-K is metrizable, we can find a sequence $x_{\alpha_n} - x'_{\alpha_n} \to 0$. Denote by $x_n - x'_n$ such a sequence, by f_n the corresponding functions and by y_n, y'_n the elements in $V_{\alpha_n} = V_n$.

Let

$$h_n(y) = f_n(x_n)(y) - f_n(x_n')(x_n - x_n' + y), \qquad y \in W$$

Since $h_n(y_n) \neq 0$ for all n, we can find a subsequence $(h_{n_j})_{j=0}^{\infty}$ and a bounded sequence $(z_j)_{j=0}^{\infty}$ in V such that $h_{n_j}(z_j) \neq 0$ for all j. Choose $(\lambda_j)_{j=0}^{\infty}$ a null sequence of scalars such that $h_j(\lambda_j z_j) \neq 0$. For all $j \geq 0$ let $|h_j(\lambda_j z_j)| = \delta_j > 0$. Choose inductively a strictly increasing sequence of positive integers $(k_j)_{j=0}^{\infty}$ such that, for all j,

1) $\left|\sum_{m=0}^{k_j} \frac{\hat{d}^m f_j(x_j)}{m!}(\lambda_j z_j) - \sum_{m=0}^{k_j} \frac{\hat{d}^m f_j(x_j')}{m!}(x_j - x_j' + \lambda_j z_j)\right| \geq \frac{\delta_j}{2}$

2) $2^{k_j} \frac{\delta_j}{2} \geq j$.

Since $(\lambda_j z_j)_{j=0}^{\infty}$ and $(x_j - x_j' + \lambda_j z_j)_{j=0}^{\infty}$ are null sequences in E, the semi-norm defined on $\mathscr{H}(K)$ by

$$p(f) = \sum_{j=0}^{\infty} 2^{k_j} \left|\sum_{m=0}^{k_j} \frac{\hat{d}^m f(x_j)}{m!}(\lambda_j z_j) - \sum_{m=0}^{k_j} \frac{\hat{d}^m f(x_j')}{m!}(x_j - x_j' + \lambda_j z_j)\right|$$

is continuous. Hence $\sup_j p(f_j) < \infty$ since $(f_j) \subset B$.

On the other hand we have, for all j,

$$p(f_j) \geq 2^{k_j} \frac{\delta_j}{2} \geq j.$$

This contradiction shows the coherence of the Taylor series expansions.

Observe that $E = \mathbb{C}^{[\mathbb{N}]}$ does not satisfy condition P, since

$E'_\beta = \mathbb{C}^{\mathbb{N}}$ does not admit a continuous norm. On the other hand, $\mathcal{H}(0)$ - $0 \in \mathbb{C}^{[\mathbb{N}]}$ - is not regular.

The following question naturally suggests itself:

QUESTION 2.19. *What is the relationship between regularity of* $\mathcal{H}(0)$ *and condition* P?

If E is the strong dual of a reflexive Fréchet space and E does not satisfy condition P, we know (Soraggi [66]) that E'_β is a Fréchet space without continuous norm. By a result of Dineen ([13]), E has a quotient space isomorphic to $\mathbb{C}^{[\mathbb{N}]}$, i.e., there exists an open, surjective, continuous mapping $\pi : E \to \mathbb{C}^{[\mathbb{N}]}$. Now, by Soraggi's result ([66], Proposition 1.10) $\mathcal{H}(0)$, $0 \in E$, cannot be regular. We have the following

PROPOSITION 2.20. *Let* E *be the strong dual of a reflexive Fréchet space. If* $\mathcal{H}(0)$, $0 \in E$, *is regular, then* E *satisfies condition* P.

Since compact subsets of DF-spaces are metrizable (Pfister [52] or Valdivia [74] and DF-spaces are quasi-normable (Kats [31]) and in quasi-normable spaces regularity of $\mathcal{H}(K)$ implies that $\mathcal{H}(K)$ is quasi-complete (Bierstadt and Meise [6]), we have the following

THEOREM 2.21. *If* E *is the strong dual of a reflexive Fréchet space and* $\mathcal{H}(0)$ - $0 \in E$- *is regular, then* $\mathcal{H}(K)$ *is regular and quasi-complete for every compact subset* K *of* E.

In studying analytic functions of nuclear type we showed in [67] that for certain nuclear spaces E with basis, regularity of $\mathscr{H}(0)$ implies that E satisfies property P. On the other hand, by using Taylor series expansions it is easy to show that for a complete space E condition P is equivalent to $\mathscr{P}(E)$ - the space of continuous polynomials on E endowed with the compact-open topology - having no non-trivial very strongly convergent sequence. We showed in [66] that E'_β having no non-trivial very strongly convergent sequence is equivalent to E having property P when E is the strong dual of a reflexive Fréchet space, i.e., in this case we need only consider in the definition of property P a sequence of non-zero continuous linear functions on E. Hence we have the following open question.

QUESTION 2.22. *Let E be a complete Montel space. (Hence $\tau_0 = \beta$ on E'). $E'_\beta \subset [\mathscr{P}(E), \tau_0]$. When are the following statements equivalent?*

1) *In E'_β there is no non-trivial very strongly convergent sequence (linear condition* P).

2) *In $\mathscr{P}(E)$ there is no non-trivial very strongly convergent sequence (polynomial condition* P).

It is worth noting that, if we get an answer to the above question, we also get an answer to Question 2.16. In fact, if E does not have linear condition P, i.e., there is a non-trivial very strongly convergent sequence in E'_β, then by a result of Simões [63] E must have a quotient space topologically isomorphic to $\mathbb{C}^{[\mathbb{N}]}$. Again applying Soraggi's result in [66], we have that $\mathscr{H}(0)$, $0 \in E$, is not regular. These results strengthen our conjecture that regularity of $\mathscr{H}(0)$ always implies regularity of $\mathscr{H}(K)$ for metrizable compact subsets of any Hausdorff locally convex space E.

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