Note di Matematica Vol. VII, 311-314(1987)

ON STRONGLY NON-NORMING SUBSPACES

V.B.MOSCATELLI (*)

ABSTRACT. *A Banach space X has a strongly non-norming subspace in its dual if and only if X has infinite codimension in its second dual.*

Let X be a Banach space. A closed subspace M of the dual X' is said to be *total* if it is w*-dense in X'. M is said to be *norming* if its unit ball is w*-dense in some multiple of the unit ball of X'. Clearly, this is equivalent to $M^1=X'$, where M^1 (= the *derived set* of M) is the collection of all limits of w*-convergent and bounded nets in M. In any case, we may define the successive derived sets M^n of M by $M^n = (M^{n-1})^1$ and we shall call M *strongly non-norming* if $M^n \neq X'$ (i.e., $M^n \neq M^{n+1}$) for all n.

In [3] Davis and Lindenstrauss proved, in a constructive and elegant way, that a total non-norming subspace exists in X' if and only if X is non-quasi-reflexive (i.e., dim $X''/X=\infty$).

In [4] S. Dierolf and the author showed that a certain problem in Fréchet-space theory (the so-called Bellenot-Dubinsky problem) was intimately connected with the Banach-space question of the existence of strongly non-norming subspaces and we refer to [4] for a detailed analysis. Now the latter question has a long history which goes back to Banach himself, who already observed in his

(*) Partially supported by the Italian Ministry of Education (MPI).

book [1, p.213] that there are (many!) quotients of c_o with strongly non-norming subspaces.

However, the question of the existence of strongly non-norming subspaces is of interest in its own right and in full generality, and led the author to conjecture (cf. [5], Problem 17) that a strongly non-norming subspace exists in the dual of every non-quasi-reflexive Banach space.

The purpose of this note is to settle the conjecture in the affirmative.

THEOREM. *Let* X *be a Banach space with* $\dim X''/X=\infty$*. Then* X' *contains a strongly non-norming subspace.*

Proof. We accomplish the proof in two steps.

1) First we show inductively that for every n there is a subspace M of X' such that $M^n \neq X'$. For $n=1$ this is already proved in [3].

Now it follows from [2, Corollary 3] and the proof of the theorem in [3] that there are a separable subspace Y in X and an infinite-dimensional subspace Z in X'' such that $\dim Y''/Y=\infty$ and $Z\cap(Y''+X)=\{0\}$. By the induction hypotesis, Y' contains a total subspace V with $V^n \neq Y'$. If $q: X' \to Y'$ is the map which assigns to every $f \in X'$ its restriction to Y, it is easily seen that the subspace $N=q^{-1}(V)$ is total and satisfies $N^n \neq X'$.

Since Y is separable, the w*-closure of the unit ball of V is compact and metrizable in the w*-topology, hence is separable and, consequently, V is w*-separable. Let (y'_k) be a sequence of unit vectors in V whose linear span is w*-dense in V (hence in Y').

As in the proof of Lemma 1 of [3] we define a map $T:Y'' \to Z$ by $Ty'' = \sum_k \varepsilon_k y''(y_k')z_k$, where (z_k) is a normalized basic sequence in Z and the sequence (ε_k) satisfies $\varepsilon_k > 0$, $\sum_k \varepsilon_k \leq 1/2$. Put $U=\{y''+Ty'':y''\in Y''\}$ and $M=U^\circ$ (polar in X'). Clearly $\ker T \subset U$ and hence $M\subset(\ker T)^\circ$ (polar in X')$\subset N$. Now let $f\in M^1$: there exists a bounded net (f_i) in M such that $f=w^*\text{-}\lim_i f_i$. Denote by $p:X' \to Z'$ the map which associates with each $f\in X'$ its restriction to Z and let T_Y be the restriction of T to Y. Then T_Y is compact and hence so is $T_Y':Z' \to Y'$. Since $f_i \in M$ we have $f_i(y) = -f_i(T_Y y)$ for all $y\in Y$, i.e. $q(f_i)=-T_Y' p(f_i)$. Thus, for a subnet (f_j) we have that $q(f_j)$ converges in norm to $q(f)$ and hence weakly: $q(f)(y'') = \lim_j q(f_j)(y'')$ for all $y''\in Y''$. Now take any $y''\in \ker T$. Since $M\subset(\ker T)^\circ$, $q(f_j)(y'')=f_j(y'')=0$ for all j, hence $f(y'') = q(f)(y'') = 0$ and, therefore, $f\in(\ker T)^\circ\subset N$.

We have shown that $M^1 \subset N$, hence $M^{n+1} \subset N^n \neq X'$.

2) By [2, Theorem 2 and Corollary 3] X contains a basic sequence (x_n) for which there is a partition (N_k) of the positive integers into mutually disjoint infinite subsets such that neither $[x_n:n\in N_k]$ nor $Z/[x_n:n\in N_k]$ are quasi-reflexive, where $Z=[x_n]$. If (x_n') is the sequence in Z' of biorthogonal functionals associated to the sequence (x_n), for every k denote by Z_k' the w^*-closure in Z' of span $(x_n':n\in N_k)$. By 1) each Z_k' contains a total subspace M_k with $(M_k)^k \neq Z_k'$. Put $V=\sum_k M_k$. If for some k we had $V^k=Z'$, it would follow $(M_k)^k=Z_k'$ which is impossible. Thus $V^k \neq X'$ for all k and V is strongly non-norming in Z'. But then so is $M=q^{-1}(V)$ in X', if $q:X' \to Z'$ is the canonical quotient map.

REFERENCES

[1] S.BANACH: "*Théorie des opérations linéaires*". Chelsea, New York 1955.

[2] W.J.DAVIS and W.B.JOHNSON: "Basic sequences and norming subspaces in non-quasi-reflexive Banach spaces". *Israel J.Math.* 14, 353-367 (1973).

[3] W.J.DAVIS and J.LINDENSTRAUSS: "On total nonnorming subspaces". *Proc. Amer. Math. Soc.* 31, 109-111(1972).

[4] S.DIEROLF and V.B.MOSCATELLI: "A note on quojections". *Funct. Approx. Comment. Math.* 17, 131-138(1987).

[5] *Open Problems*, Presented at the Ninth Seminar (Poland-GDR) on Operator Ideals and Geometry of Banach Spaces, Georgenthal, April 1986.

Dipartimento di Matematica
Università - C.P. 193
I-73100 LECCE
ITALY