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SOME SURJECTIVITY RESULTS FOR A CLASS OF MULTIVALUED MAPS AND APPLICATIONS

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INTRODUCTION. Let X be a Banach space over K ($\mathbb{R}$ or $\mathbb{C}$) and let $F: X \multimap X$ be a multivalued upper semicontinuous (u.s.c.) map with acyclic values. In [MV] Martelli and Vignoli extended to multivalued maps F the definition of a quasinorm of F (notation $|F|$) given by Granas in [Gr] for singlevalued ones. Using this definition they gave some surjectivity results in the context of α-nonexpansive and condensing maps with $|F|<1$.

In the present paper we improve these results in two ways. Firstly, we use the numerical radius of F (notation n(F)) instead of the quasinorm of F (we have $n(F) \leq |F|$ and there exist examples showing that this inequality can be strict). Secondly, we consider the class of admissible maps, which contains the u.s.c. acyclic valued ones.

As a consequence we obtain, in particular, surjectivity results for the sum of two singlevalued maps not necessarily one-to-one. We will see that such results could not be obtained by using u.s.c. acyclic valued maps instead of admissible maps. It seems that only Webb [W] has obtained some surjectivity results of this kind.

Notice that various fixed-point theorems for admissible maps are also obtained in [Go], to which we refere for a nearly exhaustive reference.

We conclude this paper by applying our results to establish existence theorems for an integral equation of Volterra-Hammerstein type in $C([0,1])$ and for a multivalued integral equation in $L^2([0,\infty[)$.

For other relevant papers on this topics see [A], [GR], [GS], [Rz].

Let us remark that our techniques can be applied also to the sum of integral operators of type different than those considered in the above references.

1. NOTATIONS AND DEFINITIONS. Let X,Y be two Banach spaces over $K(\mathbb{R}$ or $\mathbb{C})$ and let F be a multivalued map from X into Y with nonempty and compact values (notation $F : X \multimap Y$). Recall that F is said to be *upper semicontinuous* (*u.s.c.*) at a given point $x \in X$ if, for any open subset $W \subset Y$ with $F(x) \subset W$, there exists an open neighborhood V of x such that $F(V) \subset W$, where $F(V) = \cup\{F(z), z \in V\}$ (see [B]).

F is said to be u.s.c. on X if F is u.s.c. at each point of X. Note that $F : X \multimap Y$ is u.s.c. on X if and only if it sends compact sets into relatively compact sets and its graph is closed (on $X \times Y$).

For a singlevalued map A from X into Y we will use the notation $A : X \to Y$.

A multivalued map $F : X \multimap Y$ is said to be *acyclic valued* if, for any $x \in X$, $F(x)$ has the same homology of a point in the Vietoris-Čech homology with coefficients in Q.

An u.s.c. map $F : X \multimap Y$ is *compact* if it sends bounded sets into relatively compact sets.

Let Z be a bounded subset of X and let $\alpha(Z)$ be the measure of noncompactness of Z (see [K]). An u.s.c. map $F : X \multimap Y$ is said to be *condensing* if $\alpha(F(Z)) < \alpha(Z)$ for every bounded set $Z \subset X$ with $\alpha(Z) \neq 0$. If, for every bounded set $Z \subset X$, $\alpha(F(Z)) \leq \alpha(Z)$, then F is said to be *α-nonexpansive*. Clearly a condensing map is α-nonexpansive. If for some $K>0$ and for every bounded set $Z \subset X$, $\alpha(F(Z)) \leq K\,\alpha(Z)$, then F is said to be *α-Lipschitz* (with constant K).

Let $F : X \multimap Y$ be an u.s.c. map. We define

$$|F| = \limsup_{\|x\| \to \infty} \frac{\Psi(F(x))}{\|x\|}$$

where $\Psi(F(x)) = \{\sup\|y\|, y \in F(x)\}$ (see [M]).

If $|F| < \infty$, then F is said to be *quasibounded* and $|F|$ is called the *quasinorm* of F.

Let X' be the dual of X and denote by $\langle\cdot,\cdot\rangle$ the pairing between X' and X. The *duality map* $J : X \multimap X'$ is defined by

$$Jx = \{x' \in X' : \langle x', x\rangle = \|x'\| \cdot \|x\|,\ \|x'\| = \|x\|\}.$$

Let U and Z be subsets of X' and X respectively. We denote by $\langle U,Z\rangle$ the subset of K defined as follows

$$\langle U,Z\rangle = \{\langle x', x\rangle : x' \in U,\ x \in Z\}.$$

The *asymptotic numerical range* of an u.s.c. map $F : X \multimap X$ is defined as follows

$$N_\infty(F) = \bigcap_{r>0} \overline{\phi(X \setminus B_r)}$$

where $B_r = \{x \in X : \|x\| \leq r\}$ and $\phi : X \setminus \{0\} \multimap K$ is the multivalued map defined by

$$\phi(x) = \frac{\langle Jx, F(x) \rangle}{\|x\|^2} \qquad \text{(see [CDP]).}$$

Following Canavati [C] we say that an u.s.c. map $F : X \multimap X$ is *numerically bounded* if $N_\infty(F)$ is a bounded subset of K. In other words, F is numerically bounded if there exist two real numbers $M, r > 0$ such that, for every $x \in X \setminus B_r$, we have $\frac{\langle Jx, F(x) \rangle}{\|x\|^2} \leq M$.

Let F be numerically bounded. We define the *numerical radius* of F as follows

$$n(F) = \{\sup|\lambda| : \lambda \in N_\infty(F)\}.$$

Obviously a quasibounded map F is numerically bounded and $n(F) \leq |F|$. There are examples of maps F for which $n(F) < |F|$ (see [C]).

Finally we recall some facts about admissible maps. Let $X_o, X_1 \ldots \ldots, X_{n+1}$ be metric spaces and let $G_i : X_i \multimap X_{i+1}$, $i = 0, 1, \ldots, n$ be u.s.c. acyclic valued maps. Then the composite map $F : X_o \multimap X_{n+1}$, $F = G_n \circ G_{n-1} \circ \ldots \circ G_o$ is called *admissible*. Notice that F is an u.s.c. map. The following result is an immediate consequence of Theorem 5.6 in [P].

THEOREM 1.1. *Let* C *be a compact and convex subset of a Banach space* X *and let* $F : C \multimap C$ *be an admissible map. Then* F *has a fixed point.*

2. SURJECTIVITY RESULTS.

THEOREM 2.1. *Let* X *be a Banach space and let* $F : X \multimap X$ *be an admissible map, numerically bounded with* $n(F)<1$. *If* F *is condensing, then* I+F *is onto.*

Proof. Let $y \in X$. Obviously $n(F+y)=n(F)$. We claim that there exists $r \in \mathbb{R}$ such that, for every $x \in S_r = \{x \in X : \|x\| = r\}$,

$$|\langle Jx, F(x) + y\rangle| \leq \|x\|^2.$$

Indeed assume the contrary. Then for any $n \in N$ there exists $x_n \in S_n$ such that

$$|\langle Jx_n, F(x_n) + y\rangle| > \|x_n\|^2.$$

Let $y_n \in F(x_n) + y$, $x_n' \in Jx_n$. We have $\dfrac{\langle x_n', y_n\rangle}{\|x_n\|^2} > 1$.

Hence, being F+y numerically bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \frac{\langle x_{n_k}', y_{n_k}\rangle}{\|x_{n_k}\|^2} \geq 1$$

contradicting the hypothesis $n(F+y) < 1$.

Let $H = -F|B_r + y$. Clearly H is a numerically bounded condensing map from B_r into X. To prove the theorem it is enough to show that H has a fixed point.

Let $r : X \to B_r$ be the radial retraction. Since r is α-nonexpansive (see [M]), then the map $r \circ H : B_r \multimap B_r$ is condensing. From a result of Martelli [M] it follows that there exists a compact subset C of B_r such that $\overline{co}((r \circ H)(C)) = C$, where $\overline{co}((r \circ H)(C))$ denotes the closed convex hull of $(r \circ H)(C)$. By Dugundji's theorem (see [D]) there exists a retraction $r_1 : B_r \to C$. Let $i_C : C \to B_r$ be the inclusion map of C into B_r and consider the map $G = r_1 \circ r \circ H \circ i_C$. Clearly $G : C \multimap C$ is admissible, hence hy Theorem 1.1, there exists $x \in C$ such that $x \in G(x)$.

We want to show that $x \in H(x)$. Since $x \in C$ and $\overline{co}((r \circ H)(C)) = C$, we have that $x \in (r \circ H)(x)$.

Let $z \in H(x)$ such that $x = r(z)$. If $\|z\| > r$, then $x = \frac{rz}{\|z\|}$, hence $\langle x', z \rangle \geq \|x\|^2$, contradicting the hypothesis $n(F) < 1$. Thus $\|z\| \leq r$, so that $x = r(z) = z \in H(x)$.

Since an acyclic u.s.c. map is admissible and $n(F) \leq |F|$, as a consequence of Theorem 2.1 we obtain the following results

COROLLARY 2.1. *Let* $F : X \multimap X$ *be an acyclic valued map. Suppose that* F *is numerically bounded with* $n(F) < 1$. *If* F *is condensing, then* I+F *is onto.*

COROLLARY 2.2 (Martelli, Vignoli [MV]). *Let* $F : X \multimap X$ *be an acyclic valued, condensing map. If* F *is quasibounded and* $|F| < 1$, *then* I-F *is onto.*

Theorem 2.1 is not true if F is α-nonexpansive instead of condensing. To see this, consider the following simple example.

Let c_o be the Banach space of real sequences $x = \{x_n\}$ converging to zero with the norm $\|x\| = \sup_n |x_n|$. Let $a_n = \frac{2^n+1}{2^n+2}$ and $T:B_1 \to B_1$ defined by

$$T(x) = (1, a_1 x_1, \dots, a_n x_n, \dots) \quad \text{(see [Ro])}.$$

Consider the map $F = i \circ T \circ r$, where r is the radial retraction to B_1 and i is the inclusion map of B_1 into c_o. $F : c_o \to c_o$ is α-nonexpansive since $\|T(x) - T(y)\| < \|x-y\|$ $\forall x,y \in B_1$. Moreover $n(F)=0$. Neverthless I-F is not onto since T does not have fixed points.

However we can prove the following result.

THEOREM 2.2. *Let* $F : X \multimap X$ *be an admissible numerically bounded map with* $n(F) < 1$. *If* F *is* α*-nonexpansive, then* $(I+F)(X)$ *is dense on* X.

Proof. Let $y \in X$. For every $n \in \mathbb{N}$ consider the map $F_n = \frac{n}{n+1}F$. Clearly F_n is condensing and $n(F_n) < 1$, hence for every $n \in \mathbb{N}$ there exists x_n such that $y \in x_n + F_n(x_n)$, i.e. $y - x_n \in \frac{n}{n+1} F(x_n)$.

It follows that there exists $K>0$, $0<\delta<1$ and $x_n' \in Jx_n$ such that, for every $x_n \in X$ with $\|x_n\| > K$, we have

$$\frac{n+1}{n} \langle x_n', y - x_n \rangle \geq (\delta - 1)\|x_n\|^2.$$

This implies that

$$\|y\|\|x_n\| \geq \langle x_n', y\rangle \geq \frac{n\delta+1}{n+1}\|x_n\|^2,$$

hence $\|x_n\| \leq \frac{n+1}{n\delta+1}\|y\| \leq \frac{1}{\delta}\|y\|$.

Since F is α-nonexpansive, it sends bounded sets into bounded sets, hence the last relation implies that $\{F(x_n)\}$ is bounded.

Let $y = x_n + \frac{n}{n+1}z_n$, with $z_n \in F(x_n)$. We have $x_n+z_n = y + \frac{1}{n+1}z_n$, so that, being $\{z_n\}$ bounded, $\lim_{n\to\infty}(x_n+z_n)=y$.

COROLLARY 2.3 (Martelli, Vignoli [MV]). *Let* $F : X \multimap X$ *be an* α*-nonexpansive acyclic valued map. If* F *is quasibounded and* $|F| < 1$, *then* $(I-F)(X)$ *is dense on* X.

Using the previous results we will show a surjectivity theorem for the sum of two singlevalued maps.

THEOREM 2.3. *Let* X *and* Y *be two Banach spaces and* $A : D(A) \subset X \to Y$ *a surjective map such that* $A^{-1}: Y \multimap X$ *is u.s.c. with acyclic values. Let* $B : X \to Y$ *be a continuous map. If* $B\circ A^{-1} : Y \multimap Y$ *is condensing and numerically bounded with* $n(B\circ A^{-1}) < 1$, *then* $A+B: D(A) \to Y$ *is onto.*

Proof. The map $B\circ A^{-1} : Y \multimap Y$ satisfies the hypotheses of Theorem 2.1, hence for every $y \in Y$ there exists $z \in Y$ such that $y \in z+B\circ A^{-1}(z)$. Then we can find $z_1 \in B\circ A^{-1}(z)$ such that $y=z+z_1$.

On the other hand there exists $x \in A^{-1}(z)$ such that $B(x)=z_1$, hence $y = z + B(x)$ with $x \in A^{-1}(z)$. This implies $y=A(x)+B(x)$.

Observe that it is possible to obtain this result since Theorem 2.1 has been proved for admissible maps. Namely if A^{-1} is an acyclic valued map and B is a continuous map, then $B\circ A^{-1}$ is not necessarily an acyclic valued map.

On the other hand we cannot carry the study of the surjectivity of A+B to that of $I+A^{-1}B$. We can apply this method only if A is linear (see e.g. [GT]).

Notice that Theorem 2.3 does not need A to be injective. It seems that only Webb [W] has shown some surjectivity results of this kind.

Also recent fixed point theorems for the sum of two maps were obtained under the hypothesis that A is injective (see e.g. [H] and [IZ]).

On the other hand we obtain a result of [W] as consequence of our Theorem 2.3.

COROLLARY 2.4 (see [W]). *Let* X, Y *be two Banach spaces and let* $T : D(T) \subset X \to Y$ *be a surjective map with closed graph and convex preimages, such that* $\|Tx\|\to\infty$ *as* $\|x\|\to\infty$. *Assume moreover that for some* $c>0$ *and* $\Omega \subset D(T)$ *we have that* $\alpha(T(\Omega))\geq c\alpha(\Omega)$. *Let* $S:X\to Y$ *be any continuous* α*-Lipschitz map with constant less than* c *and such that for some* $b<1$, $d>0$, $\|S(x)\|\leq b\|Tx\|+d$; *then the map* T-S *is surjective.*

Proof. Let us see that under above assumptions the hypotheses of Theorem 2.3 hold. Since T is surjective with convex preimages it follows that the multivalued map T^{-1} is acyclic valued. Moreover, from the assumptions on T it follows that, for every $p\in Y$, $\alpha(T^{-1}(p))\leq$ $\leq \frac{1}{c}\,\alpha(\{p\})$ and hence $\alpha(T^{-1}(p))=0$, thus $T^{-1}(p)$ being closed) is also compact. Analogously we can show that T^{-1} sen as compact sets into relatively compact sets.

From these facts and the closedness of the graph of T=graph T^{-1} it follows also that T^{-1} is upper semicontinuous.

Clearly $S\circ T^{-1}$ is condensing since, for every bounded subset Ω of Y, we have $\alpha((S\circ T^{-1})(\Omega))<c\,\alpha(T^{-1}(\Omega))\leq\alpha(\Omega)$.

Finally $S\circ T^{-1}$ is quasibounded of quasinorm less than 1; in fact

$$\phi((S\circ T^{-1})\ (y))\leq b\|y\| + d.$$

The following result gives a sufficient condition for A^{-1} to be an u.s.c. acyclic valued map.

THEOREM 2.4. *Let* X,Y *be Banach spaces and let* $A:X\to Y$ *be a continuous proper map* (i.e. the inverse image of each compact set is a compact set). *Suppose that for any* $y_0\in Y$ *there exist a subset* Z *of* X *and a sequence* $\{A_n\}$ *of continuous proper maps from* Z *into* Y *converging uniformly to* A *on* Z *such that for all* y *in a neighborhood* $N(y_0)$ *of* y_0 *in* Y *, there exists exactly one solution* x_n *of the equation* $A_n(x)=y$. *Let* A *be onto. Then* A^{-1} *is an u.s.c. acyclic valued map.*

Proof. From Theorem 7 of [BG] it follows that A^{-1} is acyclic valued. Moreover since A is continuous and proper, A^{-1} is u.s.c. on Y.

Note that if A is condensing, then I+A is proper.

3. APPLICATIONS. In this section we obtain some existence theorems for an integral equation of Volterra-Hammerstein type in C([0,1])

and for a multivalued integral equation in $L^2([0,\infty[)$. For the sake of simplicity we do not attempt to prove the best possible results.

We want to note that in the example 3.1 first of all we are interested in finding a map A such that A^{-1} is a multivalued u.s.c. acyclic valued map.

In the second example our main purpose is to calculate the numerical radius of a multivalued (non compact) map.

3.1. Consider the following integral equation of Volterra-Hammerstein type in $C([0,1])$.

$$y(t)=x(t)+\int_0^t k(t,s)f(s,x(s))ds+\int_0^1 h(t,s)g(s,x(s))ds \qquad (3.1)$$

THEOREM 3.1. *Assume that* k *and* h *are continuous functions from* $[0,1]^2$ *into* $\mathbb{R}$. *Suppose that* f *is a continuous function from* $[0,1]\times\mathbb{R}$ *into* $\mathbb{R}$ *such that* $|f(s,x)| \leq b+d|x|$, b *and* $d \geq 0$, $s\in[0,1]$, $x\in\mathbb{R}$. *Moreover assume that* g *is a continuous function from* $[0,1]\times\mathbb{R}$ *into* $\mathbb{R}$ *such that there exists a continuous positive function* $\gamma:[0,1]\to\mathbb{R}$ *for which* $|g(s,x)| \leq \gamma(s)$, $s\in[0,1]$, $x\in\mathbb{R}$. *Then equation (3.1) has solutions.*

Proof. Let F,G be the Nemytskii's operators generated by f and g respectively. From the hypotheses it follows that F and G are continuous and bounded operators from $C([0,1])$ into itself.

Define $K,H : C([0,1]) \to C([0,1])$ as follows

$$Hx(t) = \int_0^1 h(t,s)x(s)ds, \quad Kx(t) = \int_0^t k(t,s)x(s)ds.$$

K and H are compact operators, hence so are the operators $D=K\circ F$ and $B = H\circ G$. Furthermore $B(C([0,1]))$ is relatively compact; in particular $|B| = 0$.

Fix $y_0\in C([0,1])$. For any $n\in N$ let $f_n : [0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that

i) $|f_n(s,x)-f_n(s,y)| \leq L_n|x-y|$ for any $x,y\in\mathbb{R}$, $s\in[0,1]$.

ii) $\sup_{s\in[0,1]} |f_n(s,x)-f(s,x)| < \frac{1}{n}$ for any $x \in\mathbb{R}$ such that

$|x|\leq\beta<(\|y_0\|+1+M+Mb)\exp(Md)$, $M=\sup\{|k(t,s)|:t,s\in[0,1]\}$.

Let $Z = \{x\in C([0,1]) : \|x\| \leq \beta\}$. For any $n\in N$ define

$$D_n : C([0,1])\to C([0,1]) \text{ by } D_nx(t) = \int_0^t k(t,s)\, f_n(s,x(s))ds.$$

Clearly D_n is a compact operator, hence $A_n=I+D_n$ is proper. Moreover condition ii) insures that $\{A_n\}$ converges to $A=I+D$ uniformly on Z. By a standard technique we obtain that for any $\bar{y}\in C([0,1])$ the equation $\bar{y}=x+D_nx$ has exactly one solution x_n. We want to show that if $\bar{y}\in N(y_0,1) = \{y\in C([0,1]):\|y-y_0\|<1\}$ then $x_n\in Z$. We have

$$|x_n(t)| \leq |\bar{y}(t)-y_0(t)|+|y_0(t)|+\int_0^t k(t,s)f_n(s,x_n(s))ds \leq$$

$$\leq 1 + \|y_0\| + M\frac{1}{n} + Mb + Md\int_0^t|x_n(s)|ds.$$

From Gronwall's Lemma it follows that

$$|x_n(t)| \leq (1+\|y_o\|+M\frac{1}{n} + mb)\exp(Md).$$

It remains to show that A is onto.

Let $y = \lambda x+D(x)$. By our hypotheses we have that

$$|\lambda||x(t)| \leq \|y\| + Mb + Md\int_0^t |x(s)|ds,$$

hence by Gronwall's Lemma we have

$$|\lambda||x(t)| \leq (\|y\|+Mb)\exp(Md).$$

A simple application of Schaefer's Theorem gives the assertion. Hence, according to Theorem 2.4, A is onto and A^{-1} is an u.s.c. acyclic valued *map*. Moreover $B\circ A^{-1}$ is compact and $|B\circ A^{-1}| = 0$. Hence by Theorem 2.3 we have that equation (3.1) has solutions.

3.2. We give now an application of Corollary 2.1 to the existence of solutions for a multivalued integral equation in $L^2([0,\infty[)$.

First we recall the definition of integral of a multivalued map. Let I be an interval of $\mathbb{R}$ and let $G : I \multimap \mathbb{R}$ be a multivalued map. Let ζ be the set of all integrable (over I) selections of F. Then

$$\int_I G(s)ds = \{\int_I \mu(s)ds : \mu\varepsilon\zeta\} .$$

Consider the following multivalued integral equation in $L^2([0,\infty[)$.

$$y(t) \in x(t) + \int_0^t k(t,s)f(s,x(s))ds + \\ + \int_0^t v(t)u(s)g(s,x\ (s))ds, \tag{3.2}$$

where f and g are multivalued closed and convex-valued, maps from $[0,\infty[\times \mathbb{R}$ into $\mathbb{R}$.

Assume that

a) k is a nonnegative real function defined on $[0,\infty[\times[0,\infty[$ satisfying

$\int_0^\infty \int_0^\infty k^2(t,s)dt\ ds < \infty.$

It is known that under this hypothesis the operator

$K : L^2([0,\infty[) \to L^2([0,\infty[)$ defined by

$$Kx(t) = \int_0^t k(t,s)x(s)ds$$

is a compact operator.

Furthermore assume that

b) $v \in L^2([0,\infty[)$;

c) $u \in L^2([0,b[)$ for all $b > 0$;

d) for every $s \in [0,\infty[$, $g(s,\cdot)$ is u.s.c. on $\mathbb{R}$ and, for every $x \in \mathbb{R}$, $g(\cdot,x)$ is measurable on $[0,\infty[$ (i.e., for every $z \in \mathbb{R}$, $\text{dist}(z,g(\cdot,x))$ is a measurable function on $[0,\infty[$);

e) there exists $c > 0$ such that $\Psi(g(s,x)-g(s,y)) \le c|x-y|$, for all $x,y \in \mathbb{R}$, $s \in [0,\infty[$, with $\Psi(g(s,0)) \le \delta(s)$, $\delta \in L^2([0,\infty[)$;

f) for every $s \in [0,\infty[$, $f(s,\cdot)$ is u.s.c. on $\mathbb{R}$ and, for every $x \in \mathbb{R}$, $f(\cdot,x)$ is measurable on $[0,\infty[$;

g) there exists a real function $\sigma \in L^2([0,\infty[)$ such that $\Psi(f(t,x)) \leq \sigma(t) + a|x|$, with $a > 0$;

h) $2c \sup_{t \geq 0} J + a\lambda_o < 1$, where $\lambda_o = \sup_{\|x\|=1} |<Kx,x>|$ and J is defined as follows (see [S])

$$J = (\int_0^t u^2(s)ds)^{\frac{1}{2}} (\int_t^\infty v^2(s)ds)^{\frac{1}{2}}.$$

THEOREM 3.2. *Assume that the hypotheses* a) - h) *are satisfied. Then the integral equation* (3.2) *has solutions.*

Proof. Given $x \in L^2([0,\infty[)$, denote by $\mathscr{F}(x)(\mathscr{G}(x))$ the set of all measurable functions $z : [0,\infty[\to \mathbb{R}$ such that $z(s) \in f(s,x(s))$ $(z(s) \in g(s,x(s)))$. By hypothesis e) it follows that $\Psi(g(s,x)) \leq c|x| + \delta(s)$, hence by hypotheses d) - g) we have that the multivalued maps $\mathscr{F}$ and $\mathscr{G}$ are bounded maps from $L^2([0,\infty[)$ into itself with nonempty, closed and convex values (see Theorems 1 and 2 of [LO]). Remark that Theorems 1 and 2 of [LO] has not been proved under our hypotheses, but we can see that they hold in these cases too (see also [AZ]).

Let T be the operator

$$Tx(t) = \int_0^t v(t)u(s)x(s)ds.$$

From hypotheses b), c) and h) it follows that the operator T is an α-*Lipschitz* operator from $L^2([0,\infty[)$ into itself (see [S]). Moreover from hypotheses e) and h) it follows that $T \circ \mathscr{G}$ is a multivaluded condensing map with convex values. Hence so is the map $K \circ \mathscr{F} + T \circ \mathscr{G}$ (note that $K \circ \mathscr{F}$ is a compact multivalued operator).

Now we want compute $n(K\circ\mathcal{F}+T\circ\mathcal{G})$. By $\|T\| \leq 2 \sup_{t\geq 0} J$ (see [S]), we have $n(T\circ\mathcal{G}) \leq |T\circ\mathcal{G}| \leq \|T\|\,|\mathcal{G}| \leq 2c \sup_{t\geq 0} J$. Take $y \in \mathcal{F}(x)$. We have

$$\frac{\left|\int_0^\infty\left(\int_0^t k(t,s)\; y(s)ds\right)x(t)dt\right|}{\|x\|^2} \leq \frac{\int_0^\infty\left(\int_0^\infty k(t,s)\;\sigma(s)ds\right)|x(t)|dt}{\|x\|^2} +$$

$$+\frac{\left|\int_0^\infty\left(\int_0^t a\; k(t,s)\;|x(s)|ds\right)\;|x(t)|dt\right)|}{\|x\|^2} \leq \frac{\int_0^\infty\left(\int_0^\infty k^2(t,s)ds\right)^{\frac{1}{2}}\;|x(t)|dt}{\|x\|^2}\|\sigma\|+$$

$$+\; a\lambda_o \leq \|\sigma\|\;\frac{\left(\int_0^\infty\int_0^\infty k^2(t,s)dt\;ds\right)^{\frac{1}{2}}}{\|x\|} + a\lambda_o .$$

Hence, for $0<\varepsilon<\frac{1}{\lambda_o}(1 - 2c \sup_{t\geq 0} J)-a$, we have

$$\frac{\left|\int_0^\infty\left(\int_0^t k(t,s)y(s)ds\right)x(t)dt\right|}{\|x\|^2} \leq 1 - 2c \sup_{t\geq 0} J - \varepsilon\lambda_o +$$

$$\|\sigma\|\;\frac{\left(\int_0^\infty\int_0^\infty k^2(t,s)dt\;ds\right)^{\frac{1}{2}}}{\|x\|} .$$

This implies that $n(K\circ\mathcal{F}) \leq 1 - 2c \sup_{t\geq 0} J - \varepsilon\lambda_o$. So follows that $n(K\circ\mathcal{F}+T\circ\mathcal{G}) \leq n(K\circ\mathcal{F})+n(T\circ\mathcal{G})<1$, then the statement is a consequence of Corollary 2.1.

REMARK. If k is a symmetric real function (not necessarilly nonnegative), then K is a selfadjoint operator.

Then $\|K\| = \lambda_o$, where λ_o is the greatest eigenvalue in absolute value of the operator K.

In this case the proof of the last part is more simple. In fact

$$n(T\circ\mathscr{G} + K\circ\mathscr{F}) \leq |T\circ\mathscr{G}|+|K\circ\mathscr{F}| \leq$$

$$\leq 2c \sup_{t\geq 0} J + \|K\| \; |\mathscr{F}| = 2c \sup_{t\geq 0} J + \lambda_o a < 1 \; .$$

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