No di Matematica Vol.VII, 211-229(1987)

APPROXIMATION OF FINITELY DEFINED OPERATORS IN FUNCTION SPACES

F. ALTOMARE

Sunto. - *Si studiano delle condizioni necessarie e sufficienti circa la convergenza di reti di operatori positivi verso operatori finitamente definiti operanti su spazi di funzioni continue definite su uno spazio localmente compatto e separato. Si presentano inoltre diversi esempi ed applicazioni.*

INTRODUCTION. - In this paper we establish some Korovkin - type theorems for finitely defined operators of order n ($n \in \mathbb{N}$, $n \geq 1$) in the context of locally convex function spaces defined on a locally compact Hausdorff space.

These operators have been studied by many authors (see [7], [12], [14],[15], [16]) in connection with the Korovkin approximation theory. However all the result which they obtain, concern with spaces of continuous functions defined on a compact space.

Our results are sufficiently general to be applied in many other spaces which are useful in Functional Analysis such as spaces of continuous functions which vanish at infinity, adapted spaces of continuous functions, weighted function spaces and so on.

In a preliminary section we study the convergence of nets of positive linear operators and of nets of positive contractions to an arbitrary positive linear operator acting from an ordered normed space to a $\mathscr{C}_0(X,\mathbb{R})$ space.

In Section 2 we completely characterize the Korovkin spaces for finitely defined operators of order n both for positive linear operators and for positive contractions (cf. Th.2.1 and Th.2.2).

This characterization seems to be new also for n=1 and for the identity operator.

Furthermore other sufficient conditions are stated which ensure that a subspace is a Korovkin space for finitely defined operators.

Finally Examples and applications are indicated in various function spaces.

1. CONVERGENCE OF POSITIVE CONTRACTIONS. - In what follows we shall denote by X an arbitrary locally compact Hausdorff space. $\mathcal{M}^+(X)$ will be the set of all positive Radon measures on X.

We shall denote by $\mathcal{F}(X,\mathbb{R})$ the space of all real functions on X, endowed with the natural order and with either the topology of the pointwise convergence on X or the topology of the locally uniform convergence on X (i.e. the uniform convergence on the compact subsets of X). We shall denote by $\mathcal{K}(X,\mathbb{R})$ the space of all real continuous functions on X having compact support.

The space $\mathcal{C}_0(X,\mathbb{R})$ of all real continuous functions on X which vanish at infinity, will be endowed with the natural order and the sup-norm.

A linear subspace E of the space $\mathcal{C}(X,\mathbb{R})$ of all real continuous functions on X, endowed with a locally convex topology which is compatible with the natural order induced on E by $\mathcal{C}(X,\mathbb{R})$ and which is finer than the topology of the pointwise convergence on

X, will be simply said a *function space on* X.

Let E and F be two ordered locally convex spaces, $T:E \to F$ a continuous positive linear operator and H a linear subspace of E. We say that H *is a T-Korovkin space in* E *for positive linear operators* if for every equicontinuous net $(L_i)^{\leq}_{i\in I}$ of positive linear operators from E to F such that $\lim_{i\in I}^{\leq} L_i(h)=T(h)$ for all $h\in H$, we also have $\lim_{i\in I}^{\leq} L_i(f)=T(f)$ for all $f\in E$.

If E and F are ordered normed spaces and T is a contraction ($\|T\| \leq 1$), then H is called a T-Korovkin *space in* E *for positive contractions* if a similar property is satisfied by considering only nets of positive contractions $(L_i)^{\leq}_{i\in I}$.

If E is an ordered locally convex space, we shall denote by E'_+ the cone of all continuous positive linear forms on E. Moreover, if F is a function space on some locally compact Hausdorff space Y and $T:E \to F$ is a continuous positive linear operator, then for all $y\in Y$ we shall denote by $\mu^T_y \in E'_+$ the linear functional defined by putting $\mu^T_y(f)(y) = T(f)(y)$ for all $f \in E$.

Finally, if $\mu \in E'_+$ (resp. $\mu \in E'_+$ and $\|\mu\| \leq 1$, whenever E is an ordered normed space), we put

$$U_+(H,\mu) = \{f\in E \mid \nu(f) = \mu(f) \text{ for all } \nu\in E'_+ \text{ such that } \nu=\mu \text{ on } H\}$$

(resp. $U^1_+(H,\mu) = \{ f\in E \mid \nu(f) = \mu(f)$ for all $\nu\in E'_+$, $\|Y\|\leq 1$, such that $\nu = \mu$ on $H \}$.

In [17] (Th.4.2 and Coroll. 2.1) (see [9], [11], n.3, [13], Sect. 1.2, too) the following result has been proved:

THEOREM 1.1. *Let* E *be an ordered locally convex space,* Y *a locally compact Hausdorff space,* $T:E\to\mathcal{F}(Y,\mathbb{R})$ *a continuous positive linear operator and* H *a subspace of* E. *The following propositions are equivalent:*

a) H *is a* T-*Korovkin space in* E *for positive linears operators;*

b) *For all* $y \in Y$ $U_+(H,\mu_y^T) = E$.

Our next goal is to characterize the T-Korovkin spaces (also for positive contractions) in the contest of spaces of type $\mathcal{C}_0(Y,\mathbb{R})$.

In this context some result has been obtained (with different methods) only when T is the identity operator ([5], [6], [10]) or for other types of Korovkin spaces ([1]).

THEOREM 1.2. *Let* E *be an ordered normed space,* Y *a locally compact Hausdorff space,* $T : E\to \mathcal{C}_0(Y,\mathbb{R})$ *a positive linear contraction and* H *a subspace of* E.

The following propositions are equivalent:

a) H *is a* T-*Korovkin space in* E *for positive contractions.*

b) 1) *for every* $y\in Y$ $U_+^1(H,\mu_y^T) = E$;

2) *if* $\mu \in E'_+$ *and if* $\mu=0$ *on* H, *then* $\mu = 0$.

Proof. a) $\Longrightarrow$ b). Let us consider $y\in Y$ and $\nu\in E'_+$, $\|\nu\| \leq 1$ such that $\nu = \mu_y^T$ on H. Let $(V_i)_{i\in I}$ be a neighbourhood base of y consisting of open relatively compact subsets; we consider on I the order relation $\leq$ defined by putting $i \leq j$ if $V_j \subset V_i$. For every $i\in I$ let $g_i\in \mathcal{C}_0(Y,\mathbb{R})$ be a function such that $0\leq g_i\leq 1$, $g_i(y)=1$ and $g_i(z) = 0$ for all $z\in Y\setminus V_i$. Let us consider the linear positive operator

$L_i : E \to \mathscr{C}_0(Y,\mathbb{R})$ defined by putting for all $f \in E$

$L_i(f) = \nu(f)g_i + T(f)(1-g_i)$. The operator L_i is a contraction since $|L_i(f)| \le |\nu(f)|g_i + |T(f)|\ (1-g_i) \le \|f\|g_i + \|f\|(1-g_i) = \|f\|$ for every $f \in E$. Moreover by a standard argument it is easy to show that $\lim_{\substack{\le \\ i\in I}} L_i(h) = T(h)$ in $\mathscr{C}_0(Y,\mathbb{R})$ for all $h \in H$ (see [1], p.425). Consequently for all $f \in E$ $\lim_{\substack{\le \\ i\in I}} L_i(f) = T(f)$ uniformly on Y. In particular $\lim_{\substack{\le \\ i\in I}} L_i(f)(y) = T(f)(y)$ and so $\nu(f) = T(f)(y) = \mu_y^T(f)$.

In order to show property 2), let us fix $\mu \in E'_+$ such that $\mu = 0$ on H. Suppose that $\mu \ne 0$.

As in [2], p. 237, we can construct a net $(f_i)^{\le}_{i\in I}$ in $\mathscr{C}_0(Y,\mathbb{R})$ such that $0 \le f_i \le 1$, $\|f_i\| = 1$ and $f_i g \to 0$ uniformly for all $g \in \mathscr{C}_0(Y,\mathbb{R})$. So for all $i \in I$ let us consider the linear positive contraction $L_i : E \to \mathscr{C}_0(Y,R)$ defined by putting for all $f \in E$ $L_i(f) = \frac{\mu}{\|\mu\|}(f)\cdot f_i + T(f)(1-f_i)$. Then for all $h \in H$ $L_i(h) \to T(h)$ uniformly and, hence, for all $f \in E$ $L_i(f) \to T(f)$ uniformly. Therefore $\frac{\mu}{\|\mu\|}(f)\ f_i = L_i(f) - T(f) + T(f)\ f_i \to 0$ uniformly and so $|\mu(f)| = \|\mu\|\cdot$ $\cdot\|\frac{\mu}{\|\mu\|}(f) f_i\| \to 0$. Thus $\mu = 0$ and this is a contraddition.

b) $\Longrightarrow$ a). Let $(L_i)^{\le}_{i\in I}$ be a net of linear positive contractions from E to $\mathscr{C}_0(Y,\mathbb{R})$ such that $\lim_{\substack{\le \\ i\in I}} L_i(h) = T(h)$ uniformly for all $h \in H$. Let us suppose that there exists $f \in E$ such that $(L_i(f))^{\le}_{i\in I}$ does not converge to Tf in $\mathscr{C}_0(Y,\mathbb{R})$. Then there exists $\varepsilon > 0$ and for all $i \in I$ there exist $\alpha(i) \in I$, $\alpha(i) \ge i$, and $y_i \in Y$ such that

(*) $$|L_{\alpha(i)}(f)(y_i) - T(f)(y_i)| \geq \varepsilon.$$

Let ω be the point at infinity of Y and let us suppose that $y_i \to \omega$. Then for all $g \in \mathscr{C}_0(Y,\mathbb{R})$ $g(y_i) \to 0$. For every $i\in I$ let us consider the positive linear contraction μ_i on E defined by putting $\mu_i(g) = L_{\alpha(i)}(g)(y_i)$. By the Alaoglu-Bourbaki theorem there exists $\mu\in E'_+$ with $\|\mu\| \leq 1$, and there exists a filter $\mathscr{F}$ on I finer then the filter of sections on I such that $w^*\text{-}\lim_{i\in I^{\mathscr{F}}} \mu_i = \mu$ in E'.
Then, if $h \in H$, for all $i \in I$

$$|\mu_i(h)| \leq |L_{\alpha(i)}\, h(y_i)-Th(y_i)|+|Th(y_i| \leq \|L_{\alpha(i)}(h)-T(h)\|+|Th(y_i)|,$$

hence $\mu_i(h) \to 0$ and so $\mu(h) = 0$. By virtue of the hypothesis 2), in particular we have $\mu(f) = 0$; therefore $\lim_{i\in I^{\mathscr{F}}} \mu_i(f)=0$ and so $\lim_{i\in I^{\mathscr{F}}} L_{\alpha(i)}f(y_i)-Tf(y_i) = 0$ in contradiction to (*).

If $(y_i)^{\leq}_{i\in I}$ does not converge to ω, then there exists a compact subset K_0 of Y and for all $i\in I$ there exists $\beta(i)\in I$, $\beta(i) \geq i$, such that $y_{\beta(i)} \in K_0$. Let us consider the positive linear contraction μ_i on E defined by putting $\mu_i(g) = L_{\alpha(i)}\, g(y_{\beta(i)})$ for all $g\in E$.

Again according to the Alaoglu-Bourbaki theorem there exist $\mu\in E'_+$ with $\|\mu\| \leq 1$ and a filter $\mathscr{F}_1$ on I finer than the filter of sections on I such that $\lim_{i\in I^{\mathscr{F}_1}} \mu_i = \mu$ weakly; moreover, since K_0 is compact, there exists another filter $\mathscr{F}_2$ on I finer than $\mathscr{F}_1$ and there exists $y\in K_0$ such that $\lim_{i\in I^{\mathscr{F}_2}} y_i = y$. Now fix $h\in H$; since

$$\mu_i(h) = [L_{\alpha(i)}h(y_{\beta(i)}) - Th(y_{\beta(i)})] + Th(y_{\beta(i)})$$

and

$$|L_{\alpha(i)}h(y_{\beta(i)}) - Th(y_{\beta(i)})| \leq \|L_{\alpha(i)}(h) - T(h)\| \to 0,$$

then we have $\lim_{i\in I^{\mathcal{F}_2}} \mu_i(h) = Th(y) = \mu_y^T(h)$, that is $\mu(h) = \mu_y^T(h)$. Then the hypothesis 1) implies $\mu(f) = \mu_y^T(f) = Tf(y)$ and hence $\lim_{i\in I^{\mathcal{F}_2}} L_{\alpha(i)}f(y_{\beta(i)}) - Tf(y_{\beta(i)}) = 0$ in contradiction to (*).

REMARK: As the above proof shows, if Y is compact (and hence $\mathscr{C}_0(Y,\mathbb{R}) = \mathscr{C}(Y,R)$), then the condition 2) of Th.1.2 may be dropped.

We are able to give a different formulation of the property 2) of part b) of Th.1.2 provided E is a normed function space on a locally compact Hausdorff space X which verifies the following condition:

(A) For all $\mu \in E'_+$ there exists a positive Radon measure $\tilde{\mu}$ on X such that $E \subset \mathscr{L}_1(X,\tilde{\mu})$ and $\int_X f d\tilde{\mu} = \mu(f)$ for all $f \in E$.

PROPOSITION 1.3. *Let* E *be a normed function space on a locally compact Hausdorff space* X *which is countable at infinity. Moreover let us suppose that* E *is a lattice, contains* $\mathscr{K}(X,\mathbb{R})$ *and satisfies* (A). *Then for all linear subspace* H *of* E *the following propositions are equivalent:*

a) *If* $\mu \in E'_+$ *and* $\mu = 0$ *on* H, *then* $\mu = 0$;

b) *For every* $\varepsilon > 0$ *and for every compact subset* K *of* X *there exist* $h \in H$ *and* $u \in E_+$ *such that* $0 \leq h+u$ *on* X, $1 \leq h+u$ *on* K *and* $\|u\| < \varepsilon$.

Proof. a) $\Longrightarrow$ b). Given an $\varepsilon > 0$ and a compact subset K of X, we can choose $f \in \mathcal{K}(X,\mathbb{R}) \subset E$, $f \geq 0$, such that f=1 on K. From a) and from a result established in [3] (Th. 1.6 and Rem. 1.7) we deduce that there exist $h \in H$ and $u \in E_+$ such that $f \leq h+u$ and $\|u\| < \varepsilon$ and so the result follows.

b) $\Longrightarrow$ a). Let $\mu \in E'_+$ and suppose that $\mu = 0$ on H. Let us consider $\tilde{\mu} \in \mathcal{M}^+(X)$ such that $E \subset \mathcal{L}_1(X,\tilde{\mu})$ and $\mu(f) = \int_X f\,d\tilde{\mu}$ for all $f \in E$.

We shall show that $\tilde{\mu} = 0$ and to this end, since $\tilde{\mu}$ is regular (because X is countable at infinity), it suffices to prove that $\tilde{\mu}(K) = 0$ for every compact subset K of X. Infact if K is a such subset, then for all $\varepsilon > 0$ there exist $h \in H$ and $\mu \in E_+$ satisfying the properties of part b) and so,

$$0 \leq \tilde{\mu}(K) \leq \int_K (h+u)\,d\tilde{\mu} \leq \int_X (h+u)\,d\tilde{\mu} = \mu(h)+\mu(u) = \mu(u) \leq \|\mu\|\,\varepsilon \quad .$$

Since $\varepsilon > 0$ is arbitrary we have $\tilde{\mu}(K) = 0$.

REMARK: The implication b) $\Longrightarrow$ a) remains true even if E is not a lattice or it does not contains $\mathcal{K}(X,\mathbb{R})$.

THEOREM 1.4. *Let* E *be a normed vector lattice,* Y *a locally compact Hausdorff space* $T : E \to \mathcal{C}_0(Y,\mathbb{R})$ *a continuous positive linear operator and* H *a subspace of* E*. The following propositions are equivalent:*

a) H *is a* T*-Korovkin space in* E *for positive linear operators;*

b) *For all* $y \in Y$ $\quad U_+(H,\mu_y^T) = E$.

Proof. By imitating the same proof of Th. 1.2 one may show that H is a T-Korovkin space in E if and only if 1) $U_+(H,\mu_y^T)=E$ for all $y\in Y$ and 2) for all $\mu\in E'_+$ such that $\mu=0$ on H, one has $\mu=0$.

Now the desired result will follow if we prove that property 1) implies property 2). Infact let $\mu\in E'_+$ and suppose that $\mu=0$ on H. By applying Th.1.6, (ii), and Lemma 1.2 of [3] to an arbitrary functional of the form μ_y^T, we have that for all $f\in E$ and for all $\varepsilon>0$ there exist h, $K\in H$ and u, $v\in E_+$ such that $k-v\leq f\leq h+u$ and $\|u\|<\varepsilon$ and $\|v\|<\varepsilon$. Consequently $-\mu(v)\leq\mu(f)\leq\mu(u)$ and hence, since $\mu(u)\leq\|\mu\|\|\mu\|\leq\varepsilon\|\mu\|$ and $\mu(v)\leq\varepsilon\|\mu\|$, $|\mu(f)|\leq\varepsilon\|\mu\|$. So $\mu(f)=0$ since ε is arbitrary.

2. APPROXIMATION OF FINITELY DEFINED OPERATORS.

As an application of the above results we shall study the T-Korovkin spaces (both for positive linear operators and for positive contractions) when the operator T is finitely defined of order n $(n\geq 1)$.

Let E and F be two function spaces defined on some locally compact Hausdorff spaces X and Y. Following A.S.Cavaretta ([7]) we say that a (positive) linear operator $T:E\to F$ is finitely defined of order n $(n\in\mathbb{N},\ n\geq 1)$ if there exist finitely many continuous functions $\varphi_i:Y\to X$ and $\psi_i\in\mathscr{C}^+(Y,\mathbb{R})$ such that

$$(1)\qquad T(f)=\sum_{i=1}^{n}\psi_i(f\circ\varphi_i)\qquad\text{for all } f\in E$$

(of course, it is assumed that the right-hand side belongs to F).

The set of all finitely defined operators of order n will be denoted by $\mathscr{F}_n(E,F)$. Finally we shall denote by $\mathscr{F}_n^1(E,F)$ the set of all operators T in $\mathscr{F}_n(E,F)$ which have a representation as (1) with $\sum_{i=1}^{n} \psi_i = 1$.

In the next results it will be necessary to impose the condition (A) which makes sense also for locally convex function spaces.

The following theorem generalizes the main result of [14] (Th.3):

THEOREM 2.1. *Let* E *be a normed function space on a locally compact Hausdorff space* X. *Let us suppose that* E *is a lattice, contains* $\mathscr{K}(X,R)$ *and satisfies* (A) . *For every linear subspace* H *of* E *and for every* $n \in \mathbb{N}$, $n \geq 1$, *the following propositions are equivalent:*

a) *For every locally compact Hausdorff space* Y *and for every* $T \in \mathscr{F}_n(E,F)$, *where* $F = \mathscr{F}(Y,\mathbb{R})$ *or* $F = \mathscr{C}_0(Y,\mathbb{R})$, H *is* T-*Korovkin space in* E *for positive linear operators;*

b) *For every choice of different points* $x_1,\dots,x_n \in X$, *for every compact subset* K *of* X *such that* $K \cap \{x_1,\dots,x_n\} = \emptyset$ *and for every* $\varepsilon > 0$ *there exists* $h \in H$ *and* $u \in E_+$ *such that* $\|u\| < \varepsilon$, $0 \leq h+u$ *on* X, $1 \leq h+u$ *on* K *and* $h(x_i)+u(x_i) < \varepsilon$ *for all* $i=1,\dots,n$.

Proof. a) $\Longrightarrow$ b). We shall use Th. 2.1 of [3]. Let $x_1,\dots,x_n \in X$ be. Let us consider $\alpha_1,\dots,\alpha_n \in \mathbb{R}_+$ and put $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ (where δ_{x_i} denotes the valuation functional at x_i defined on E). Then $\mu \in \mathscr{F}_n(E,F)$ where $F = \mathbb{R} = \mathscr{F}(\{1\},\mathbb{R})$ (it suffices to consider the functions ψ_i: $\{1\}\to\mathbb{R}$ and φ_i : $\{1\}\to X$ defined by putting $\psi_i(1)=\alpha_i$ and $\varphi_i(1) = x_i$ for all $i=1,\ldots,n$). Hence H is a μ-Korovkin space in E. From all this and from Th.2.1 of [3] the result directly follows.

b) $\Longrightarrow$ a). Let $T\in \mathscr{F}_n(E,F)$ be given according to a) and suppose that for all $f\in E$

$$T(f) = \sum_{i=1}^{n} \psi_i(f\circ\varphi_i)$$

(cf. (1)).

By virtue of Th. 1.1 or of Th. 1.4 it sufficies to show that for all $y\in Y$ $U_+(H,\mu_y^T) = E$ where, in this case, $\mu_y^T= \sum_{i=1}^{n}\psi_i(y)\delta_{\varphi_i(y)}$

But this last equality is a consequence of b), taking into account Th. 2.1 and Prop. 2.3 of [3].

By reasoning as in the above proof and by using the above Th.1.2 and the Th.2.2 of [3] it is easy to prove the further result:

THEOREM 2.2. *Let* E *be a normed function space on a locally compact Hausdorff space* X *which satisfies* (A). *Moreover let us suppose that* E *is a lattice and contains* $\mathscr{K}(X,\mathbb{R})$. *Let* Y *be a locally compact Hausdorff space and* $T\in\mathscr{F}_n^1(E,\mathscr{C}_0(Y,\mathbb{R}))$ *with a rappresentation, as* (1) *of the form* $T(f) = \sum_{i=1}^{n}\psi_i(f\circ\varphi_i)$ *for all* $f\in E$ $(n\geq 1)$.

Let us consider a linear subspace H *of* E *such that for all* $y\in Y$ *there exist* $h_1,\ldots,h_n\in H$ *such that* $\det(h_i(\varphi_j(y)))\neq 0$.

Then the following propositions are equivalent:

a) H *is a* T- *Korovkin space in* E *for positive contractions;*

b) 1) *For every* $\varepsilon>0$ *for every* $y\in Y$ *and for every compact subset* K *of* X *such that* $y \notin \bigcap_{i=1}^{n} \varphi_i^{-1}(K)$, *there exist* $h\in H$ *and* $u \in E_+$ *such that* $0 \leq h+u$ *on* X, $1\leq h+u$ *on* K, $h(\varphi_i(y))+u(\varphi_i(y))<\varepsilon$ *for all* $i=1,\dots,n$ *and* $\|u\| < \varepsilon + \sum_{i=1}^{n} \psi_i(y)u(\varphi_i(y))$.

2) *if* $\mu \in E'_+$ *and* $\mu = 0$ *on* H, *then* $\mu = 0$.

When the function space E is not a lattice or does not contain $\mathcal{K}(X,\mathbb{R})$, we can indicate only sufficient conditions in order that a subspace of E is a Korovkin space for finitely defined operators.

THEOREM 2.3. *Let* E *be a function space on a locally compact Hausdorff space* X *which is countable at infinity and let us suppose that* E *satisfies* (A) .

Let H *be a linear subspace of* E *and* $n \in \mathbb{N}$, $n\geq 1$, *such that for every choice of* $n+1$ *different points* $x_1,\dots,x_{n+1}\in X$ *there exists* $h \in H$, $h \geq 0$, *such that* $h(x_{n+1}) > 0$ *and* $h(x_1) = \dots = h(x_n) = 0$. *Then for every locally compact Hausdorff space* Y *and for every* $T \in \mathcal{F}_n(E,F)$, *where* $F = \mathcal{F}(Y, \mathbb{R})$ (*or* $F = \mathcal{C}_0(Y,\mathbb{R})$ *provided* E *is a normed vector lattice*), *the subspace* H *is a* T*-Korovkin space in* E *for positive linear operators.*

Proof. Let us fix $T \in \mathcal{F}_n(E,F)$ and let us suppose that T has a representation

$$T(f) = \sum_{i=1}^{n} \psi_i(f\circ\varphi_i)$$

as in (1). By virtue of Th.1.1 (resp. of Th.1.4) it suffices to show that for all $y \in Y$ $U_+(H,\mu_y^T) = E$ where, in this case, $\mu_y^T = \sum_{i=1}^{n} \psi_i(y)\,\delta_{\varphi_i(y)}$. Now from the hypothesis it follows that for $i=1,\dots,n$ there exists $h_i \in H$, $h_i \geq 0$ such that $h_i(\varphi_j(y)) = \delta_{ij}$ for all $j=1,\dots,n$ and hence $\det(h_i(\varphi_j(y))) = 1 \neq 0$.

Moreover, if $x \in X$ and $x \notin \{\varphi_1(y),\dots,\varphi_n(y)\}$, then there exists $h \in H$, $h \geq 0$ such that $h(x) > 0$ and $h(\varphi_1(y)) = \dots = h(\varphi_n(y)) = 0$. From all this and from Th. 2.4 of [3] it follows that $U_+(H,\mu_y^T)=E$.

THEOREM 2.4. *Let* E *be a normed function space on a locally compact Hausdorff space* X *which is countable at infinity and let us suppose that* E *is a vector lattice and satysfies* (A). *Let* $n \in \mathbb{N}$ *be,* $n \geq 1$*, and let us consider a subspace* H *of* E *satisfying the following conditions:*

1) *for all* $x_1,\dots,x_n \in X$ *there exist* $h_1,\dots,h_n \in H$ *such that* $\det(h_i(x_i)) \neq 0$;
2) *for all* $x_1,\dots,x_{n+1} \in X$ *there exists* $h \in H + \mathbb{R}_+$ *such that* $h \geq 0$, $h(x_{n+1}) > 0$ *and* $h(x_1) = \dots = h(x_n) = 0$;
3) *for all* $x \in X$ *there exists* $h \in H$ *such that* $h \geq 0$ *and* $h(x) > 0$.

Then H *is a* T-*Korovkin space in* E *for positive contractions for all* $T \in \mathscr{F}_n^1(E,F)$ *where* $F = \mathscr{C}_0(Y,\mathbb{R})$, Y *being an arbitrary locally compact Hausdorff space.*

Proof. Let $T \in \mathscr{F}_n^1(E,F)$ be; in order to show the result we have to verify the conditions 1) and 2) of Th.1.2. The condition 1) is fullfilled by virtue of the hypotheses 1) and 2), of Th.2.4

of [3] and of the fact that in this case $\mu_y^T = \sum_{i=1}^{n} \psi_i(y)\delta_{\varphi_i(y)}$ for all $y \in Y$. As regards condition 2), according to Prop. 1.3 and the relative Remark we shall prove part b) of Prop. 1.3. Infact, if K is a compact subset of X, by using hypothesis 3) and a compactness argument, it is easy to show that there exists $h \in H$, $h \geq 0$ such that $h \geq 1$ on K and so part b) of Prop. 1.3 obviously follows by putting $u = 0$.

COROLLARY 2.5. *Let* E *be a function space on a locally compact Hausdorff space* X *which is countable at infinity. Let us suppose that* E *satysfies* (A) *and contains the constant function* 1. *Let* $n \in \mathbb{N}$ *be,* $n \geq 1$ *, and let us consider a subset* S *of* E *satisfying the following conditions:*

1) *for all* $p = 2,3,\ldots,2n$ $S^p = \{f^p | f \in S\} \subset E$;

2) *for all* $x_1,\ldots,x_{n+1} \in X$ *there exists* $f \in S$ *such that* $f(x_{n+1}) \neq f(x_i)$ *for all* $i = 1,\ldots,n$.

Then the subspace H *generated by* $\{1\} \cup S \cup S^2 \ldots \cup S^{2n}$ *is a* T*-Korovkin space in* E *for positive linear operators for all* $T \in \mathcal{F}_n(E,F)$; *where* $F = \mathcal{F}(Y,\mathbb{R})$ *or* $F = \mathcal{C}_0(Y,\mathbb{R})$ *if* E *is a normed vector lattice* (Y *being an arbitrary locally compact Hausdorff space*).

Proof. It is a direct consequence of Th.2.3 because, if $x_1,\ldots,x_{n+1} \in X$, there exists $f \in S$ such that $f(x_{n+1}) \neq f(x_i)$ for all $i=1,\ldots,n$. Hence we may choose a polynomial P on $\mathbb{R}$ of degree $\leq n$ such that $P(f(x_{n+1})) = 1$ and $P(f(x_i))=0$ for all $i=1,\ldots,n$. Consequently $P^2(f) \in H$, $P^2(f) \geq 0$, $P^2(f)(x_{n+1})=1$ and $P^2(f(x_i))=0$

for every $i=1,\dots,n$.

COROLLARY 2.6. *Let* E *be a normed function space on a locally compact Hausdorff space* X *which is countable at infinity and let us suppose that* E *is a vector lattice which satisfies* (A). *Let* $n\in\mathbb{N}$ *be,* $n\geq 1$ *, and let us consider a subset* S *of* E *satisfying the following conditions:*

1) *for all* $p=2,3,\dots,2n$ $S^p\subset E$;
2) *for all* $x_1,\dots,x_{n+1}\in X$ *there exists* $f\in S$ *such that* $f(x_{n+1})\neq f(x_i)$ *for every* $i=1,\dots,n$;
3) *for all* $x_1,\dots,x_n\in X$ *there exists* $f\in S$ *such that* $f(x_i)\neq 0$ *for every* $i=1,\dots,n$ *and (only if* $n>1$) $f(x_i)\neq f(x_j)$ *for all* $i,j\in\{1,\dots,n\}$, $i\neq j$.

Then the subspace H *generated by* $S\cup S^2\cup S^{2n}$ *is a* T*-Korovkin space in* E *for positive contractions for all* $T\in\mathcal{F}_n^1(E,F)$ *where* $F=\mathcal{C}_0(Y,\mathbb{R})$, Y *being an arbitrary locally compact Hausdorff spaces.*

Proof. We shall use Th.2.4. In fact from 3) we deduce that, if $x\in X$, there exists $f\in S$ such that $f(x)\neq 0$ and so $f^2(x)>0$. Hence the conditions 3) of Th. 2.4 is satisfied. Again by virtue of 3), if we consider $x_1,\dots,x_n\in X$ then by choosing a function $f\in S$ which satisfies 3) and by using a Vandermonde determinant, we have $\det(f^p(x_i))_{\substack{1\leq i\leq n\\ 1\leq p\leq n}} = f(x_1)\dots f(x_n)\prod_{1\leq i<j\leq n}(f(x_j)-f(x_i))\neq 0$ and

so the condition 1) of Th.2.4 is fulfilled too. As regards condition 2) of Th.2.4 it suffices to argue as in the proof of Coroll.2.5

(in this case infact $P^2(f) \in H + \mathbb{R}_+$).

EXAMPLES 2.7. 1. *Let* E *be a function space on* $\mathbb{R}^p$ $(p\geq 1)$ *satisfying the condition* (A) *and let* $n \in \mathbb{N}$ *be*, $n \geq 1$. *Let us suppose that for every* $h_1,\dots h_p \in \mathbb{N}$ *with* $h_1+\dots+h_p \leq 2n$ *the functions*

$$(2) \quad x_1^{h_1}\dots x_p^{h_p} \exp(-\|x\|^2) \qquad\qquad x = (x_1,\dots,x_p) \in \mathbb{R}^p,$$

belongs to E (*for example* $E = \mathscr{C}_0(\mathbb{R}^p,\mathbb{R})$ *or* $E = \mathscr{S}(\mathbb{R}^p)$, *the space of rapidly decreasing, infinitely differentiable real functions on* $\mathbb{R}^p$, etc.).

Then the subspace H *generated by the functions of the form* (2) *is a* T-*Korovkin subspace in* E *for positive linear operators for all* $T \in \mathscr{F}_n(E,F)$ *where* $F=\mathscr{F}(Y,\mathbb{R})$ *or* $F=\mathscr{C}_0(Y,\mathbb{R})$ *provided* E *is a normed vector lattice.*

In fact, if we consider n+1 distinct points $x_1,\dots,x_{n+1} \in \mathbb{R}^p$, then the function $h(x) = \prod_{i=1}^{n} \|x-x_i\|^2 \exp(-\|x\|^2)$ belongs to H, is positive and vanishes only on $x_1,\dots,x_n$. So the result follows from Th.2.3.

2. *Let* E *be a function space on a locally compact Hausdorff space* X *which is countable at infinity and let us suppose that* E *satisfies* (A) *and* $1 \in E$. *If* $f \in E$ *is an injective function, then Coroll.* 2.5 *applies to the subspace generated by* $\{1,f,f^2,\dots,f^{2n}\}$ *for all* $n \in \mathbb{N}$ *such that* $f^2,f^3,\dots,f^{2n} \in E$.

This is the case, for instance, of the function $f(x)=x$ in $\mathscr{C}(X,\mathbb{R})$ where X is an interval of $\mathbb{R}$ and $\mathscr{C}(X,\mathbb{R})$ is endowed with the topology of the pointwise convergence on X or the topology of the locally uniform convergence on X.

Another example may be constructed ad follows. Let X be a locally compact Hausdorff space which is countable at infinity and let $f \in \mathscr{C}(X,\mathbb{R})$ be an injective function such that there exists $\lim_{x\to\omega} f(x) \in \mathbb{R} \cup \{-\infty,+\infty\}$ (where ω is the point at infinity of X). Then, for all $n \in \mathbb{N}$, $n \geq 1$, the conclusion of the Ex.2 is valid for the space $E_n = \{ g \in \mathscr{C}(X,\mathbb{R}) \mid \text{There exists } \lim_{x\to\omega} \frac{g(x)}{1+f(x)^{2n}} \in \mathbb{R} \}$ endowed with the norm $\|g\| = \sup_{x\in X} \frac{|g(x)|}{1+f(x)^{2n}}$.

3. *Let* E *be a normed function space on a locally compact Hausdorff space* X *which is countable at infinity and let us suppose that* E *is a lattice and satisfies* (A). *If* $f \in E$ *is an injective function which never vanishes on* X, *then Coroll.*2.6 *applies to the subspace generated by* $\{f, f^2, \ldots, f^{2n}\}$ *for all* $n \in \mathbb{N}$ *such that* $f^2, \ldots, f^{2n} \in E$.

This is the case, for example, for $f(x) = x$ in $\mathscr{C}_0(]0,1[)$, $f(x) = x^{-1}$ in $\mathscr{C}_0([1,\to[)$, $f(x) = \exp(-\alpha x)$ in $\mathscr{C}_0([0,\to[)$ $(\alpha > 0)$.

REFERENCES

[1] F.ALTOMARE: Teoremi di approssimazione di tipo Korovkin in spazi di funzioni, *Rend. Mat.* (3)(1980) 13 Serie VI, 409-429, MR 82h: 41029.

[2] F.ALTOMARE: On the universal convergence sets, *Ann.Mat.Pura Appl.* (IV)(1984), Vol. CXXXVIII, 223-243.

[3] F.ALTOMARE: Positive linear forms and their determining subspaces, to appear.

[4] H.BAUER: Theorems of Korovkin type for adapted spaces, *Ann.Inst.* Fourier (Grenoble) 23(1973), 245-260, MR 50#10643.

[5] H.BAUER-K.DONNER: Korovkin approximation in $\mathscr{C}_0(X)$, *Math. Ann.* 236(1978) 225-237, MR. 58 # 12115.

[6] H.BAUER-K.DONNER: *Korovkin clorures in* $\mathscr{C}_0(X)$, *in*: Aspects of Mathematics and its applications, North Holland Math. Library, 34, 1986.

[7] A.S.CAVARETTA: *A Korovkin theorem for finitely defined operators, in*: Approximation theory, Proc. Intern. Symp.Univ.Texas, Austin, Texas, 1973, 299-305, MR 48 # 11872.

[8] G.CHOQUET: *Lectures on Analysis*, II, W.A.Benjamin, Inc. London, 1969.

[9] L.B.O.FERGUSON-M.D.RUSK: Korovkin sets for an operator on a space of continuous functions, *Pacific J.Math.* 65 (2)(1976), 337-345, MR 54 # 8117.

[10] H.O.FLÖSSER: Sequences of positive contractions on AM-spaces, *J. Approx. Theory* 31 (1981), (2), 118-137, MR 82h: 41043.

[11] M.W.GROSSMAN: Korovkin theorems for adapted spaces with respect to a positive operator, *Math. Ann.* 220(1976), 253-262, MR 53 # 1124.

[12] C.A.MICCHELLI: Chebyshev subspaces and convergence of positive linear operators, *Proc. Amer.Math.Soc.* 40,(2)(1973), 448-452, MR 48 # 6787.

[13] C.A.MICCHELLI: Convergence of positive linear operators on $\mathscr{C}(X)$, *J.Approx. Theory* 13(1975), 305-315 MR 52 # 3819.

[14] I.RASA: Determining sets for finitely defined operators, *Anal. Numér. Théor. Approx.* 10(1981), (7), 55-62 MR 84i:41033.

[15] M.D.RUSK: Determining sets and Korovkin sets on the circle, *J.Approx. Theory* 20, 278-283, 1977, MR 56 # 6240.

[16] Yu.A.ŠAŠKIN: Finitely defined linear operators in spaces of continuous functions (Russian) *Uspeki Mat. Nauk.* 20(1965),(6) (126), 175-180, MR 33 # 1715.

[17] H.WATANABE: Theorems of Korovkin type in an ordered vector space with a locally convex topology, *Natur. Sci. Rep.* Ochanomizu Univ. 30(1979), (2), 37-46, MR 81h: 41028.

Ricevuto il 23/3/1987

Istituto di Matematica
Università della Basilicata
Via N.Sauro, 85
85100 POTENZA