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A NOTE ON A FAMILY OF DISTRIBUTIONAL PRODUCTS IMPORTANT IN THE APPLICATIONS

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ABSTRACT; - *We define a family of products of a distribution* $T\in \mathcal{D}'$ *by a distribution* $S\in C^{\infty}\oplus \mathcal{D}'_n$ *where* $\mathcal{D}'_n$ *means the space of distributions with support nowhere dense. Each product depends on the choice of a group* G *of unimodular transformations and a function* $\alpha\in\mathcal{D}$ *with* $\int\alpha = 1$ *which is* G*-invariant. These products are consistent with the usual product of a distribution by a* C^{∞}*-function, their outcome distributive, and verify also the usual law of the derivative of a product together with being invariant by translation and all transformations in* G . *A sufficient condition for associativity is given. Simple physical interpretations of the products* $H\delta$ *and* $\delta\delta$, *where* H *is the Heaviside function and* δ *is the Dirac's measure, are considered. In particular we discuss certain shock wave solutions of the differential equation*

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0 .$$

1. PRELIMINAIRES, MAIN OPERATIONS.

In the sequel we denote by C^{∞} (resp. $\mathcal{D}$) the algebra of indefitely differentiable complex functions (resp. with bounded support) defined on $\mathbb{R}^N$ with the usual topology, and by $L(\mathcal{D})$ the algebra of all continuous linear operators $\phi:\ \mathcal{D}\to\mathcal{D}$ where the usual composition product will be indicated by a dot.

Consider the natural representation $\rho : C^{\infty} \to L(\mathcal{D})$ that maps $\beta \in C^{\infty}$ onto $\rho(\beta) \in L(\mathcal{D})$ defined by $[\rho(\beta)](x) = \beta x$ for all $x \in \mathcal{D}$.

An operator $\phi \in L(\mathcal{D})$ is said to vanish in an open set Ω iff $\phi(x) = 0$ for all x whose support is contained in Ω. We denote by $\text{supp}\,\phi$, the support of the operator ϕ, the complement of the largest open set in which ϕ vanishes. Notice that $\text{supp}(\phi\cdot\psi) \subset \text{supp}\,\psi$ (but not in general $\text{supp}(\phi\cdot\psi) \subset \text{supp}\,\phi$) for all ϕ, $\psi \in L(\mathcal{D})$.

Let h be a C^{∞}-diffeomorphism of $\mathbb{R}^N$ and $S_h \in L(\mathcal{D})$ be defined by $S_h(x) = x \circ h$ for all $x \in \mathcal{D}$. Then, $\phi \Theta h = S_h . \phi . S_{h^{-1}} \in L(\mathcal{D})$ is said the operator that results from ϕ through the change of variable h. In particular, $\bar{\tau}_a \phi = \phi \Theta h$ with $h : \mathbb{R}^N \to \mathbb{R}^N$ defined by $h(t) = t - a$ and $a \in \mathbb{R}^N$, is said the a-translated of the operator $\phi \in L(\mathcal{D})$.

We call partial derivative of an operator $\phi \in L(\mathcal{D})$ in order to the variable t_k, $1 \leq k \leq N$, the operator $\bar{D}_k \phi = [D_k, \phi] = D_k \phi - \phi D_k \in L(\mathcal{D})$ where D_k is the ordinary partial derivative operator on C^{∞} in order to t_k. Note that this derivative is an inner derivation in $L(\mathcal{D})$ so that satisfies Leibnitz formula $\bar{D}_k(\phi . \psi) = (\bar{D}_k \phi) . \psi + \phi . (\bar{D}_k \psi)$ and $\text{supp}\,\bar{D}_k \phi \subset \text{supp}\,\phi$ for all $\phi, \psi \in L(\mathcal{D})$. The definition of directional derivative of an operator $\phi \in L(\mathcal{D})$ is as for functions. We must note that all concepts defined in $L(\mathcal{D})$ are consistent with the natural representation ρ, for example $\bar{D}_k \rho(\beta) = \rho(D_k \beta)$ if $\beta \in C^{\infty}$.

2. AN EPIMORPHISM ONTO $\mathcal{D}'$ AND α-REPRESENTATION OF AN OPERATOR $\phi \in L(\mathcal{D})$.

Let us consider the surjection $\tilde{\zeta}: L(\mathcal{D}) \to \mathcal{D}'$ defined by $<\tilde{\zeta}(\phi), x> = \int \phi(x)$ for all $\phi \in L(\mathcal{D})$ and all $x \in \mathcal{D}$ where the integral is taken on $\mathbb{R}^N$ in the usual sense. It can proved that $\tilde{\zeta}$ is an epimorphism for the structure defined by the operations on $L(\mathcal{D})$ and the correspondent ones on $\mathcal{D}'$:

a) Addition: $(\phi,\psi)\to\phi+\psi$ of $L(\mathcal{D})\times L(\mathcal{D})$ to $L(\mathcal{D})$

b) Right product induced by the natural representation $\psi\in\rho(C^\infty)$: $\phi\to\phi.\psi$ of $L(\mathcal{D})$ into $L(\mathcal{D})$.

c) Directional derivation in the direction of $u\in\mathbb{R}^N$: $\phi\to\bar{D}_u\phi$ of $L(\mathcal{D})$ into $L(\mathcal{D})$.

d) Translation defined by $a\in\mathbb{R}^N$: $\phi\mapsto\bar{\tau}_a\phi$ of $L(\mathcal{D})$ onto $L(\mathcal{D})$

e) Change of variable defined by an unimodular transformation h: $\phi\mapsto\phi\Theta h$ of $L(\mathcal{D})$ onto $L(\mathcal{D})$ (we call unimodular transformation, a linear map $h:\mathbb{R}^N\to\mathbb{R}^N$ such that $|\det h| = 1$).

The consistence with the above operations is immediate from the definitions, the fact that $\tilde{\zeta}$ is onto can be verified by observing that if $T\in\mathcal{D}'$ we have $\tilde{\zeta}(\phi)=T$ where ϕ is the operator $x\to\alpha<T,x>$, $x\in\mathcal{D}$ with $\alpha\in\mathcal{D}$ such that $\int\alpha=1$.

Given $\alpha\in\mathcal{D}$ with $\int\alpha=1$ we define the α-representation of an operator $\phi\in L(\mathcal{D})$ as the operator $\psi\in L(\mathcal{D})$ such that

$$[\psi(x)](y) = \int\phi_t[\alpha(y-t)x(t)]dt \quad \text{for all } x\in\mathcal{D} \text{ and } y\in\mathbb{R}^N$$

where ϕ_t denotes the operator ϕ when it acts on functions of t in $\mathcal{D}$. We shall write $\psi = s_\alpha(\phi)$.

The operation s_α on $L(\mathcal{D})$ is an a) b) c) d) - endomorphism and also an e) endomorphism if h is such that $\alpha\circ h=\alpha$. Moreover, s_α is a projector ($s_\alpha\circ s_\alpha = s_\alpha$) and we have $\tilde{\zeta}\circ s_\alpha = \tilde{\zeta}$,

$\mathrm{Ker}\,\tilde{\zeta} = \mathrm{Ker}\, s_\alpha$.

We call the operator $\phi \varepsilon L(\mathcal{D})$ a representation of $T \varepsilon \mathcal{D}'$ if $\phi \varepsilon \tilde{\zeta}^{-1}(T)$. If $\phi \varepsilon \tilde{\zeta}^{-1}(T)$, $s_\alpha(\phi)$ will be also a representation of T; we call $s_\alpha(\phi)$ the α-representation of T.

We note that if $\phi, \psi \varepsilon L(\mathcal{D})$ are representations of T and S respectively, the distribution $\tilde{\zeta}[s_\alpha(\phi).s_\alpha(\psi)] = \tilde{\zeta}[\phi.s_\alpha(\psi)]$ is independent of the representations ϕ and ψ of T and S. Also $\mathrm{supp}\,\phi = \mathrm{supp}\,T$ if ϕ is an α-representation of T.

3. THE FAMILY OF PRODUCTS;

Let G be a group of unimodular transformations of $\mathbb{R}^N$, such that there exists a function $\alpha \varepsilon \mathcal{D}$ with $\int \alpha = 1$ satisfying $\alpha o h = \alpha$ for all $h \varepsilon G^{(I)}$. Let N be the set of operators $\phi \varepsilon L(\mathcal{D})$ with nowhere dense support and consider the direct sum $H = \rho(C^\infty) \oplus s_\alpha(N) \subset L(\mathcal{D})$. $H \cap \mathrm{Ker}\,\tilde{\zeta} = \{0\}$ so that $\zeta = \tilde{\zeta}\,|\,H$ is an a) b) c) d) e)-isomorphism.

Now we can define the product TS of a distribution $T \varepsilon \mathcal{D}'$ by a distribution $S \varepsilon \zeta(H) = C^\infty \oplus \mathcal{D}'_n$ ($\mathcal{D}'_n$ denotes the space of distributions with nowhere dense support) relative to the pair (G, α) setting $TS = \tilde{\zeta}[\tilde{\zeta}^{-1}(T).\zeta^{-1}(S)]$.

Each (G, α)-product is coherent with the usual product of a distribution by a C^∞-function, is distributive relatively to the sum, verifies the usual law of the derivative of a product, is invariant by translations and all transformations $h \varepsilon G$. It is not commutative neither associative but it can be proved that:

1. If $T,S \in \mathcal{D}'_n$ then $\int TS = \int ST$ if T or S has compact support and the map $t \to -t$ of $\mathbb{R}^N$ onto $\mathbb{R}^N$ belongs to $G^{(II)}$.
2. If $T \in \mathcal{D}'$ and $S,U \in C^{\infty} \oplus \mathcal{D}'_n$ with $\zeta^{-1}(S), \zeta^{-1}(U) \in H$ then $T(SU) =$ $= (TS)U$.

4. EXAMPLES AND SIMPLE PHYSICAL INTREPRETATIONS.

In all examples bellow we take as G the group of orthogonal transformations of $\mathbb{R}^N$ whereas the α-function may depend on the physical examples. H and δ are the Heaviside and Dirac distributions.

a) $H\delta = \frac{1}{2}\delta$ for all α (in dimension 1). The same result is obtained by Fisher [5,6] with another approach or J.J.Lodder [7] in a context of "generalized functions" which are not distributions. Let us consider a network consisting only of a self to which is applied a current $i(t)=i_o-i_oH(t)$, where $i_o > 0$ and t is the time variable. This situation corresponds to the switching-off process at the instant t=0. Now, the energy w dissipated, (generally through an arc) can be computed as usual by $w=\int_{\mathbb{R}} e(t)\ i(t)dt$ where e(t) is the difference of potential between the extremities of the self. Clearly $e(t) = L\frac{di}{dt} = -L\ i_o\,\delta(t)$ in the sense of distributions, where L is the inductance of the self. Thus, also in the sense of distributions

$$w = \int_{\mathbb{R}}[L\ i_o^2H(t)\delta(t)-L\ i_o^2\delta(t)]dt = -\frac{1}{2}L\ i_o^2\int_{\mathbb{R}}\delta(t)dt = -\frac{1}{2}L\ i_o^2$$

where we recognise the value of the magnetic energy stored in the self before the instant t=0.

b) $\delta\delta = \alpha(o)\delta$. Let us interpret this result in dimension 1, considering an electrical network consisting only of a resistor to which is applied a current $i(t)=q_o\,\delta(t)$ where $q_o>0$. By Ohm's law we know that $e(t)=R.i(t)$ where R is the resistence of the resistor. Then, the energy dissipated in the resistor is

$$w = \int_{\mathbb{R}} e(t)i(t)dt = \int_{\mathbb{R}} R\,q_o^2\,\delta^2(t)dt = R\,q_o^2\,\alpha(o)$$

and we must choose α such that $R\,q_o^2\,\alpha(o)$ be in agreement with experiment. Some notes about this problem can be seen in Bremermann [1,2]. Different results for δ^2 with different approaches can be seen in Colombeau [3,4] and J.Silva Oliveira [8].

c) Consider the differential equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0 \qquad (1)$$

and suppose we ask for "pure shock" wave solutions, i.e., solutions of the form

$$u(x,t) = u_1+(u_2-u_1)H(x-vt) \qquad (2)$$

where t is the time variable, $x \in \mathbb{R}$ is the position variable, $v \in \mathbb{R}$ is a constant which represents the velocity of the shock wave and u_1, u_2 are complex constants with $u_2 \neq u_1$. In the sense of distributions we have $\frac{\partial u}{\partial x} = (u_2-u_1)\,\delta(x-vt)$ and $\frac{\partial u}{\partial t} =$ $= -v(u_2-u_1)\,\delta(x-vt)$. Here $\delta(x-vt)$ stands for the distribution defined by $<\delta(x-vt), \gamma(x,t)> = \int_{\mathbb{R}} \gamma(vt,t)dt$ for all $\gamma \in \mathcal{D}(\mathbb{R}^2)$.

Since we can compute the product $H(x-vt)\delta(x-vt) = \frac{1}{2}\delta(x-vt)$ for any α, it follows that (2) is a solution of (1) iff $v = \frac{1}{2}(u_1+u_2)$. This agrees with the physical reality. See Richtmyer [9]. The same result is obtained by Lodder [7] in a context of "generalised functions" which are not distributions. Classical methods leads us to an infinite number of values for the velocity of the shock wave through weak solutions of conservation laws.

For details of this paper and other material in the subject we refer the reader to [10].

(I) If G in The Lorentz group it is impossible to choose α obeying to such conditions. A distributional product Lorentz-invariant will be soon presented by the author based in the ideas of this paper

(II) In this paper the integral of a distribution is to be interpreted in the sense $\int T = <T,1>$.

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