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PLANAR PROJECTIVE CONFIGURATIONS

Part 1

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To Gilbert Robinson on his 80th birthday

INTRODUCTION. - The problem originated in an attempt to construct matrices A with entries 0 and 1 such that the product AA^T has each entry which is off the main diagonal either 0 or 1. Such matrices are quite common and appear as the incidence matrices of finite projective and affine planes as well as the incidence matrices of configurations such as the Desargues and Pappus configurations. The configurations which we study are all self-dual. The question of whether configurations which admit a preassigned group of collineations can be constructed is also addressed.

DEFINITIONS. We define a *planar projective configuration* of order n and deficiency k to be a system consisting of two sets P and L called points and lines respectively and an incidence relation between the sets such that (using conventional language):

1). Two points are on at most one line.

2). Two lines intersect in at most one point.

3). Every point is on exactly n+1 lines.

4). Every line passes through exactly n+1 points.

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5). For each point there are exactly k other points which are not joined to it.

6). For each line there are exactly k other lines which do not intersect it.

If we have such a system S we say that S belongs to the class [n,k], or more loosely, S is an [n,k].

If S is an [n,k] then the number of points in S is $N=n^2+n+1+k$. To see this note that each point p is on exactly n+1 lines. The total number of points on these is 1+n(n+1). Add to this the k points which are not joined to p we obtain $n^2+n+1+k$ points. The number of lines is also N. In classical terms an [n,k] configuration has been referred to in the literature as an N_{n+1} configuration. We will use both notations interchangeably.

In connection with an [n,k] configuration, we introduce the notion of the *deficiency graph* of the configuration. This graph is formed by taking the points of the configuration to be the vertices of the graph, and any two points which are not on a line of the configuration are joined by an edge in the graph. Such a graph is on N vertices and is regular of degree (valency) k. It need not be connected even when the configuration is. If U and V are configurations in the class [n,k] then the disjoint union of U and V is a configuration in the class [n,N+k]. We will refer to a configuration which is disconnected as imprimitive. A connected configuration is then called primitive.

Another concept which is useful is that of an *incidence matrix* of a configuration. If we name the points and lines of the configuration as p_i, l_j (i,j=1,2,...,N) then the matrix A with entry 1 in the i,j position if p_i is on the line l_j and 0 otherwise is called an incidence matrix of the configuration. For any configuration in the class [n,k] the corresponding incidence matrix is a matrix of order N in which each row and column has (n+1) 1's and the remaining entries all 0. By a well known theorem of Garrett Birkhoff |1|, such a matrix can be expressed as a sum of n+1 permutation matrices, usually in many ways.

Finally, we introduce the notion of a *deficient difference set* with respect to a finite group G. A group G has a left difference set in the class {n,k} if there exists a set of (n+1) elements $\{a_1,a_2,\dots,a_{n+1}\}$ of G such that all the products $a_i^{-1}a_j$ (i,j =1,2,..,n+1) with $i\neq j$ are distinct and include all but k+1 elements of G. Of course, the identity element of G is always excluded in the set of $a_i^{-1}a_j$. In what follows the adjective deficient will usually be omitted. We could also define a right difference set by using products $a_ja_i^{-1}$ instead of $a_i^{-1}a_j$. However, if $a_i^{-1}a_j=a_r^{-1}a_s$ then $a_ia_r^{-1}=a_ja_s^{-1}$. Hence, $\{a_1,a_2,\dots,a_{n+1}\}$ is a left difference set if and only if it is a right difference set. It is to be pointed out that the excluded elements may not be the same for the left differences as for the right differences. Counting included and excluded differences it follows that $|G| = n^2+n+1+k=N$. If the elements of a difference set generate G we say that the difference set is primitive, otherwise imprimitive. If $\{a_1,a_2,\dots,a_{n+1}\}$ is a difference set then if x is any element of G the set $\{a_1x,a_2x,\dots\dots,a_{n+1}x\}$ is called a right translate of the set and is clearly a difference set. If $\{a_1,a_2,\dots,a_{n+1}\}$ is a primitive difference set then some translate of the set contains the identity element G (which we denote by 1). In what follows we will assume that a primitive difference set always contains the element 1. In this case no element of G of order 2 can appear in the difference set. Also all elements of order 2 belong to the excluded set. If the group G is cyclic the group is written additively with 0 being the identity element. If n is a prime power q then the class |q,0| consists of projective planes of order q and these exist for every q. In fact among such projective planes there is for every q a *Desarguesian* projective plane of order q and by a famous theorem of Singer [6] such planes contain collineations which are cyclic on its points. This does not exclude the possibility of other planes having cyclic collineations, but at present none are known. The existence of a cyclic collineation in a Desarguesian plane of order q implies that the cyclic group C_q has a difference set in the class {q,0}. (Of course, we do not distinguish between left and right for Abelian groups).

For most finite groups it is possible to construct primitive deficient difference sets for relatively large values of n. We note the following exclusions all based on the fact that a deficient difference cannot contain elements of order 2 assuming that the identity element of the group is a member of the difference set. There is no difference set associated with an elementary Abelian group of exponent 2. There are also no primitive difference sets associated with a dihedral group or with the direct product of the symmetric group S_3 and an elementary Abelian group of exponent 2.

SOME CONSTRUCTIONS OF CONFIGURATIONS IN THE CLASS [n,k].

1.) Deletion of incidences.

Suppose the class [n,k] is non empty. Let C be a configuration in this class, and let A be an incidence matrix for C. Then A is a matrix of order N which contains (n+1) 1's in each row and each column and 0's elsewhere. Then A can be written as a sum of permutation matrices (in many ways). Let P be one of the permutation matrices which appears in the sum. Let φ be the permutation associated with P (φ is the mapping $i \to i\varphi$ for $i=1,2,\ldots,N$). Then each point p_i is incident with the line $l_{i\varphi}$. On removing these incidences, we obtain a configuration in the class [n-1,2n+k]. The different choices of the matrix P may give rise to non-isomorphic configurations. In any case we have proved the following theorem.

THEOREM 1. *If the class* [n,k] *is non empty then so is the class* [n-1,2n+k].

COROLLARY 1. Iterating this result we obtain the following: If [n,k] is non-empty, then so are the classes [n-1,2n+k], [n-2,4n-2+k], [n-3,6n-6+k], ..., $[n-i,2in-i^2+i-k]$, ..., $[1,n^2+n-2+k]$.

COROLLARY 2. If n is the order of a projective plane, then the results of corollary 1 are true with k=0. In particular, the results are true with n equal to a prime power.

Note that the results of corollary 1 and 2 may be true even when the premise is false. For example although there is no projective plane of order 6 the class [4,22] is non empty as will be shown in what follows.

2). **Stripping a projective plane.**

If there is a projective plane of order n then by removing a line and the points incident with it we obtain an affine plane. The lines of this plane separate into parallel classes and every point is on one line of each parallel class. Take one of the points in the plane and remove it and all the lines which pass through it. The resulting set of points and lines form a configuration in the class [n-1,n-2]. We now have proved the following theorem:

THEOREM 2. *If the class* [n,0] *is non empty then so is the class* [n-1,n-2].

Note that the class [n-1,n-2] may be non empty even though the class [n,0] is empty. As will be shown in what follows the class [5,4] is non empty but there is no projective plane of order 6. New classes of configurations may now be constructed by applying corollary 1 of theorem 1 to this result.

3). **The sextet substitution.**

It is a well known result on Steiner triple systems that if the system contains six points A,B,C,D,E,F lying on the four lines A B C, A F D, E B F, E C D, we may interchange the points A and E to obtain four new lines with which to replace the original four lines. The resulting system is a new Steiner triple system. The same substitution works for any configurations in the class [2,k]. Of interest is that the two configurations have the same deficiency graph. If we apply this trasformation to the Desargues configuration which is in the class [2,3] the resulting configuration serves to discriminate between those Desarguesian planes which are non-Pappian from those which are Pappian. This is discussed in detail in part 2.

4). **Construction using deficient difference sets.**

The existence of a difference set in the class {n,k} is now used to construct a configuration in the class [n,k]. If G is the group of the difference set then the collineation group of the configuration contains a subgroup isomorphic to G. If often happens that difference sets on different group yield isomorphic configurations. The construction is described in the following theorem.

THEOREM 3. *Let* $\{a_0,a_1,\ldots,a_n\}$ *be a difference set in the class* {n,k} *with* G *being the underlying group. Then there exists a configuration in the class* n, k *which admits* G *as part of its collineation group.*

Proof. We take as the lines of the configuration all the right translates of $\{a_0,a_1,\ldots,a_n\}$. Since every group element appears in some right translate the number of lines is equal to the number of points and equals $|G| = N$, and k is given by $N-(n^2+n+1)$. We next show that given any two lines, they either have 1 or 0 points in common. Indeed if the translates by x and y had two elements in common there would exist m,n,p,q such that $a_m x=a_n y$ and $a_p x=a_q y$. These imply that $a_n^{-1}a_m = a_q^{-1}a_p$. This contradicts the definition of a difference set. Hence the points and lines of the configuration form a configuration in the class [n,k].

THEOREM 4. *Let* $\{a_0,a_1,\ldots,a_n\}$ *be a difference set in the class*

$\{n,k\}$ *with* G *being the underlying group. Let* H *be the subgroup of* G *which is generated by* $a_0,a_1,\ldots,a_n$, *and suppose that* $[G:H]=m$. *Then the configuration defined by the difference set is imprimitive and consists of* m *disjoint isomorphic copies of a configuration in the class* $[n,k']$ *where* $k'=k-N(m-1)/m$, *and* $N=n^2+n+1+k$.

Proof. Let $G = H+Hg_2+\ldots+Hg_n$. The elements of a translate of $\{a_0,a_1,\ldots,a_n\}$ by an element in the coset Hg_i all lie in the set Hg_i. This allows us to separate the lines of the configuration defined by the difference set into m subsets, where the lines of the ith subset contain elements of the coset Hg_i. Call these subsets of lines $S_1,S_2,\ldots,S_n$. Each of these sets form an $[n,k']$ configuration where k' is given by the value in the statement of the theorem. Since right translations by elements of G permute the sets S_i they are all isomorphic as configurations.

THEOREM 5. *Let* $\{1,a_2,\ldots,a_r\}$ *be a difference set over a group* G. *If* $a_2,\ldots,a_r$ *generate* G *then the corresponding configuration* H *is primitive.*

Proof. Corresponding to the configuration H we introduce a graph H* where the vertices of H* are the points of H and two vertices of H* are connected by an edge iff the corresponding points of H are both incident on the same line. We show that there is a path from the vertex 1 to any other vertex of the graph. This will imply that the graph H* is connected and hence that the configuration H is primitive. From the difference set itself it follows that the vertex 1 is joined to each of the vertices $a_2,\ldots,a_r$ by an edge. Represent each element of G as a word in

$a_2,\dots,a_r$ without using inverses of generators. Inductively, we assume that there is a path from 1 to each vertex which is represented by a word of length n. Let W be a word of length n+1. Then W is of the form $W = a_m T$ for some m and T is of length n. Translate the difference set by T, we obtain that the points $T, a_2T, \dots, a_rT$ all are on a line of the configuration. Hence, in particular, there is an edge from T to a_mT. Combining this with the path from 1 to T, yields a path from 1 to W. Hence, H* is connected and H is primitive.

THEOREM 6. *If a configuration* S *is constructed using a difference set on a group* G *, then the collineation group of the configuration contains* G *as a subgroup and this subgroup acts transitively on the points and lines of* S .

Proof. The points of S are the elements of G. The set of all right translates of a point by the elements of G is simply the set of all elements of G. Hence G acts transitively on the points of S. Also, since the set of lines of S are simply the set of right translates of a single line, it follows that G acts transitively on the lines. Each right translation by an element of G induces a collineation of S, and the set of right translations is isomorphic to the right regular representation of G.

Calculation by computer yields an enormous number of deficient difference sets. At present there is no criterion which will tell us whether the configurations obtained from one group difference set are isomorphic to the configurations obtained from difference sets on other groups of the same order. Hence publication of tables of such results appears quite useless at the present time. Instead we prove two theorems which indicate the ubiquity of such configurations.

THEOREM 7. *Let* n *be a fixed integer. There exists an integer* k' , *depending on* n , *such that for every integer* $k>k'$ *the class* [n,k] *is non empty. Furthermore, each such class contains a primitive configuration which admits a collineation which acts cyclically on its points.*

Proof. Let q be the smallest prime power such that $q>(n+2)$. There is a Desarguesian plane of order q. By Singer's Theorem, this plane contains a collineation which is cyclic on its points. This implies the existence of a difference set $\{0,1,a_2,\ldots,a_q\}$ in the integers mod N, where $N=q^2+q+1$. From this difference set delete q-n elements from amongst the subset $a_2,a_3,\ldots,a_q$. There remains a set of n+1 integers which includes 0 and 1 and which form a deficient difference set S on the integers mod N. Let T be the set of differences of members of S, which are not reduced mod N. The members of T are all distinct. Let r be the largest member of T. Take M to be max(2r+1,N). Then S is a difference set in the integers mod m for every $m>M$. Also, since S contains 0 and 1, the elements generate all the integers mod m. This implies that the configuration generated by S is primitive and admits the integers mod m as collineations. The theorem now follows with $k' = M-n^2-n-1$.

THEOREM 8. *Let* G *be a finite simple group of order* N . *Then there is a primitive configuration in the class* [2,N-7] *which* *admits* G *as a group of collineations of the configuration. (In classical notation there is a configuration of type* N_3 *whose collineation group contains* G.)

Proof. We are required to find two elements a and b in G which generate G and such that {1,a,b} are a difference set over G. This requires that $a, a^{-1}, b, b^{-1}, ab^{-1}, ba^{-1}$, be all distinct, that is neither a nor b are of order 2 and neither is the square of the other. Every finite simple group contains such a pair of generators.

We remark that this is a minimal result whose purpose is to illustrate the construction. A reasonable conjecture to make is the following. If G is a group of order N and if u is the largest integer such that $u^2+u+1\leq N$, then deficient difference sets on G exist for all orders $n<n'$ where $n'<u$ but roughly of the same order of magnitude as u. How much smaller n' is than u will depend on how many involutions there are in the group G.

SOME CONCRETE EXAMPLES.

In these examples we use both notations to describe the configur ation class, the classical notation given in parentheses after our own.

EXAMPLE 1. Manipulation of the class [3,0]--(13_4). This class contains only one configuration, namely the projective plane of order 3. One can obtain a number of configurations in the class |2,6|--(13_3) by using the method of deletion of incidences, and then follw this with a sextet substitution. For example, take the lines of the projective plane to be given by L_i = i,i+1,i+3,i+9 (i=1,...,13 mod 13), we delete successively from these lines their incidences with the points 10,11,3,7,6,2,8,4, 5,13,1,12,9, obtaining the configuration whose lines are 1 2 4, 2 3 5, 4 6 12, 4 5 13, 5 8 1, 6 7 9, 7 10 3, 8 9 11, 9 10 12, 10 11 6, 11 12 7, 13 2 8, 13 1 3. Now apply the sextet substitution to the four lines 1 2 4, 1 5 8, 8 2 13, 4 5 13, by interchanging the points 4 and 8 and we obtain a second configuration whose lines are 1 2 8, 2 3 5, 4 6 12, 8 5 13, 5 4 1, 6 7 9, 7 10 3, 8 9 11,9 10 12, 10 11 6, 11 12 7, 13 2 4,13 1 3. We have not investigated how many more configurations in the class [2,6] can be obtained in this way. In fact we do not know whether or not all configurations in this class [2,6] may be obtained in this way.

EXAMPLE 2. The class [4,22]---(43_5). If there were a projective plane of order 6 configurations in this class could be constructed using the method of deletion of incidences. Since this method is unavailable we use instead the difference set { 0,1,3,7,12 } in the integers mod 43.

EXAMPLE 3. The class [5,4]---(35_6). The non existence of the projective plane of order 6 makes it impossible to use the method of stripping a projective plane. Nevertheless the class is non empty since {0,1,3,7,12,20} is a difference set in the integers mod 35.

EXAMPLE 4. The class [4,0]---(21_5). This is the projective plane of order 4. There is only one configuration in this class (apart from isomorphism). This example illustrates that difference sets on two different groups may yield the same configuration. In the group C_{21} the difference set {0,1,6,8,18} generates the configuration. The non-Abelian group of order 21 may also be used to obtain the same configuration. The group is generated by two elements a and b, which satisfy $a^7=b^3=1$, $ba=a^2b$. An appropriate difference set is $\{1,a,a^3,b,a^2b^2\}$. The differences may easily be seen to be distinct using the permutation representation a=(1234567) and b=(253)(467). The conditions under which this phenomenon occurs have not been determined, but necessary conditions under which a configuration C may be generated by two distinct groups G and H are: (1) $|G| = |C| = N$ (say); (2) both G and H have faithful transitive permutations on N symbols and; (3) if C is in the class $|n,k|$ then G and H have difference sets in the class {n,k}. When two different groups generate the same configuration, the collineation group of the configuration contains subgroups isomorphic to each of the generating groups.

EXAMPLE 5. The class [2,5]---(12_3). There are many configurations in this class. We draw attention to two of these. The difference set {0,1,3} mod 12 yields a configuration with a cyclic collineation. In part 2 we will show that this configuration can be drawn in the complex projective plane with its points on a cubic curve. The configuration can also be drawn in the real projective plane. A configuration which is non-isomorphic to this is obtained either from the alternating group A_4 or from the group $C_6 x C_2$. The difference sets are obtained as follows. Generate A_4 from the permutations (123) and (124) and take as the difference set {1,(123),(124)}. For the group $C_6 x C_2$ take generators a and b such that $a^6=b^2=1$, ab=ba, and use as the difference set $\{1,a,a^2b\}$. These difference sets generate isomorphic designs. In a subsequent paper we show the following remarkable properties of this design. The configuration can be drawn with the points lying on a real cubic curve. Furthermore, the design can be extended in a unique way to a configuration of 12 points and 16 lines such that each point is on 4 lines and each line contains 3 points. Feld [3] obtained this configuration in an entirely different way.

EXAMPLE 6. The class $[2,4]$---11_3. We draw attention to the example generated by the difference set $\{1,2,4\}$ mod 11. The configuration has a collineation cyclic on its points. It can be drawn in the real projective plane. Furthermore it can be drawn in the complex projective plane with its points all on a cubic curve, as will be shown in part 2.

EXAMPLE 7. The class $[2,3]$--10_3. There are several configurations in this class. We confine our discussion to three of these configurations. The deficiency graph is on ten vertices and is of degree three. Most of the configurations have a graph with a Hamiltonian circuit. An example of one of these is the graph generated by the difference set $\{1,2,4\}$ mod 10. This configuration is embeddable in the complex plane with its vertices on a cubic curve. It is also embeddable in a real projective plane with its points on a cubic curve. There are two configurations which correspond to a deficiency graph which is connected and is without a Hamiltonian circuit. In this case the deficiency graph is the Peterson graph. Neither can be generated by a difference set. The first is the Desargues configuration. The second, which has sometimes been referred to as a "anti Fano configuration" (a most unfortunate and inappropriate name) is obtained by applying the sextet substitution to a Desargues configuration. In terms of embeddability into a projective plane the Desargues configuration is a universal configuration in any plane which can be co-ordinated by elements of a field. On the other hand the "anti Fano configuration" has the property that it can be embedded in any non-Pappian Desarguesian plane, but cannot be embedded in a Pappian plane. It therefore serves as a closed configuration which discriminates between Desarguesian planes which are non-Pappian from those which are Pappian. Hence, a more appropriate name for this configuration would be "anti Pappian configuration" Figures 1 and 2 illustrate some of the situations. In what follows we use the term "anti Pappian configuration".

EXAMPLE 8. The class $[2,2]$---9_3. This class contains exactly 3 isomorphism classes, namely: 1) The Pappus configuration; 2) The cyclic configuration generated by the difference set {1,2,4} mod 9, and; 3) A configuration embeddable in a real cubic curve but without a collineation which is cyclic on its points. The deficiency graphs of these three configuration are; three disjoint triangles, a nonagon, and a hexagon with a triangle disjoint from it, respectively. Figure 3 illustrates the three configurations and their deficiency graphs.

EXAMPLE 9. The class $[2,1]$---8_3. There is exactly 1 configuration in this class. It can be generated in three ways: 1) using a difference set in the cyclic group of order 8, namely, {1,2,4} mod 8;

2) using the difference set {1,i,j} on the quaternion group; 3) using the method of stripping of the projective plane in the class [3,0]. The configuration cannot be embedded in the real projective plane, but lies on any complex cubic curve with nine points of inflexion. Its points are any eight of the points of inflexion. The configuration is illustrated in Figure 4, but since it cannot be draun in the real plane, the drawing distorts one of the lines into a curve.

EMBEDDABILITY OF THE CONFIGURATIONS INTO PROJECTIVE PLANES.

All of the configurations in each class [n,k] lie in some projective plane. This follows from the fact that any collection of points and lines such that any two points lie on at most one line and any two lines pass through at most one point can be embedded in projective planes in infinitely many ways by a construction given by Marshall Hall [4]. In fact if we were to take all the configurations in all the classes [n,k] for all valid values of n and k, there would be a single projective plane containing them all. This sort of situation is too general to be of interest. Our interest is mainly in the question of embedding of configurations in the planes which have been widely studied viz.; finite planes, real planes, complex planes etc. In the case of finite planes, any configuration obtained from a finite plane using either the method of stripping or the method of removal of incidences already lies in a finite plane.

In part 2 we address the question of embeddability of configurations into complex projective planes. In particular, we look at the question as to whether every class $[2,k]$---N_3, contains a configuration, all of whose points lie on a complex cubic curve, and obtain some geometric consequences of such embeddings. It is known that the Pappus configuration $(9_3,9_3)$ is embeddable in a complex non-singular cubic with two degrees of freedom. In spite of its similarities with Desargues configuration, the latter $(10_3,10_3)$ configuration is not so embeddable. These results are proved in Part 2 of this sequence where we give a systematic method of describing the actual embeddings into cubics whenever possible.

Three Configurations in the class [2,3] and their deficiency graphs

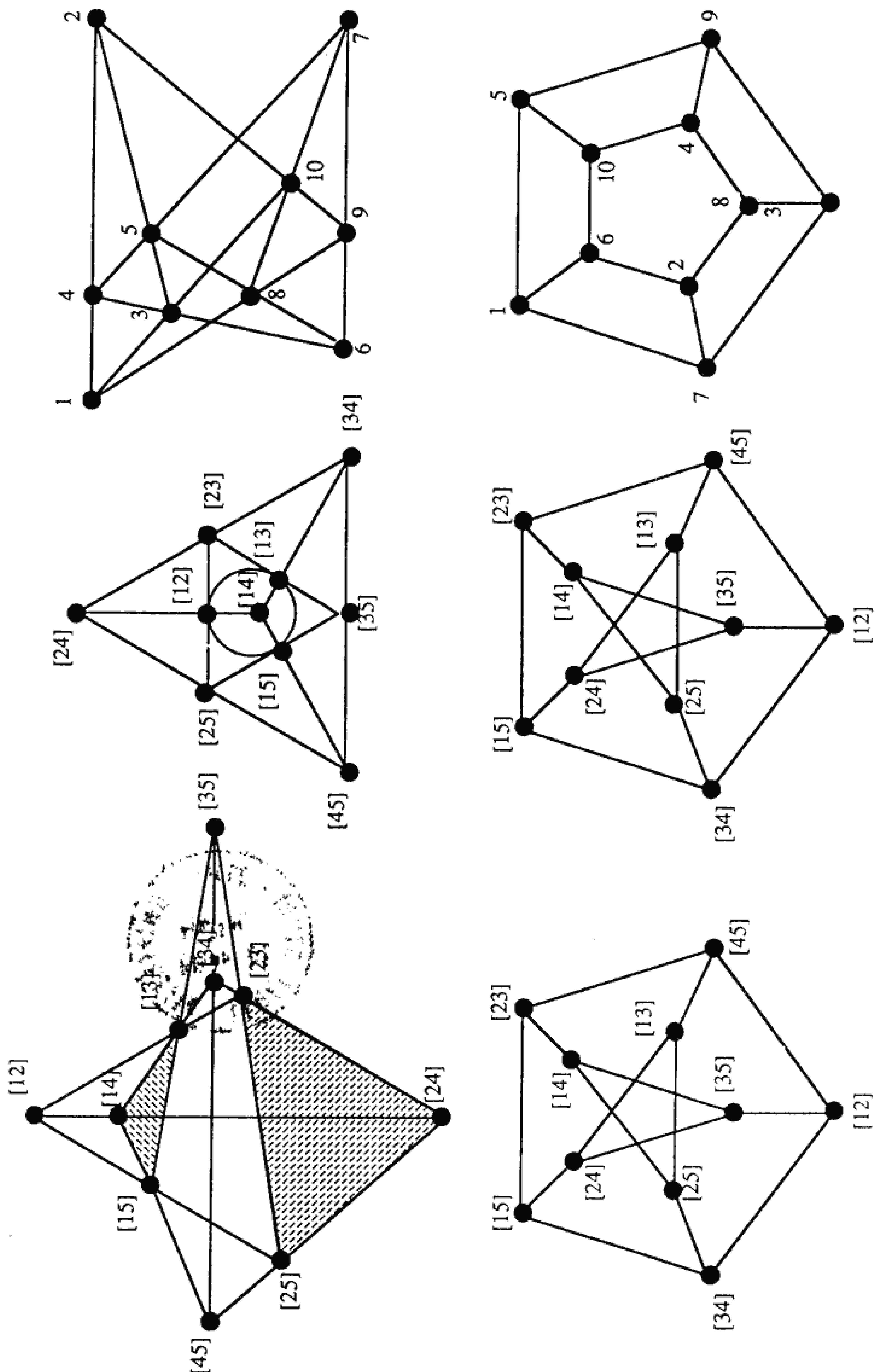


Figure 1

The Class [2,3]

How the sextet substitution transforms the Desargues Configuration into the anti Pappian Configuration

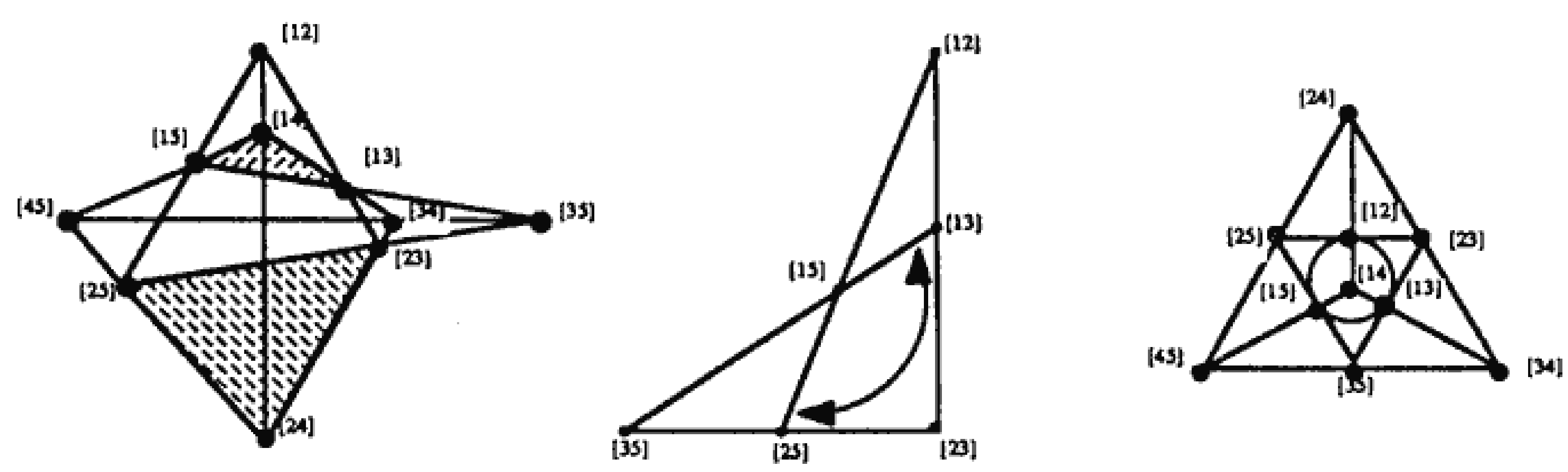
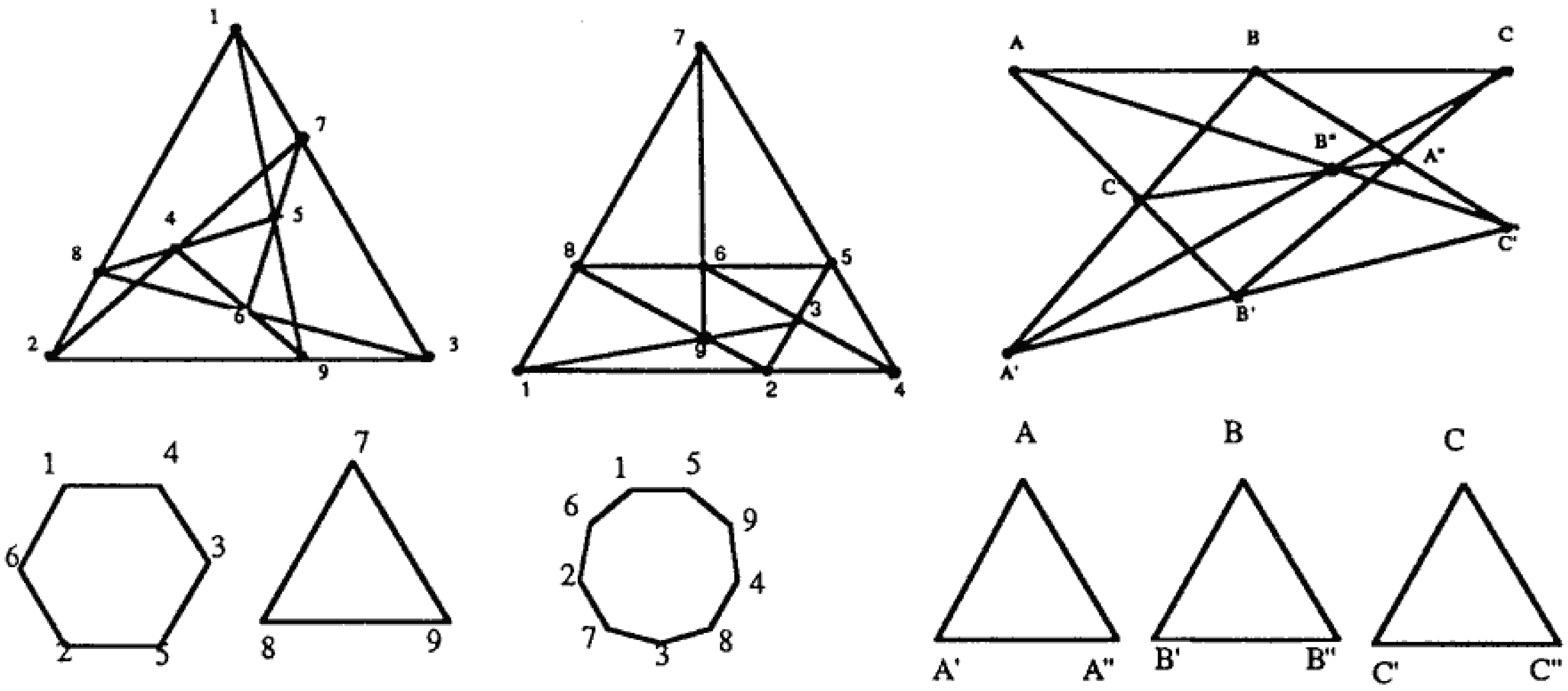


Figure 2

The Three Configurations in the Class [2,2] and their Deficiency Graphs



Non Cyclic

{1 2 4} mod 9

Pappus' Configuration

Figure 3

The Unique Configuration in the Class [2,1]

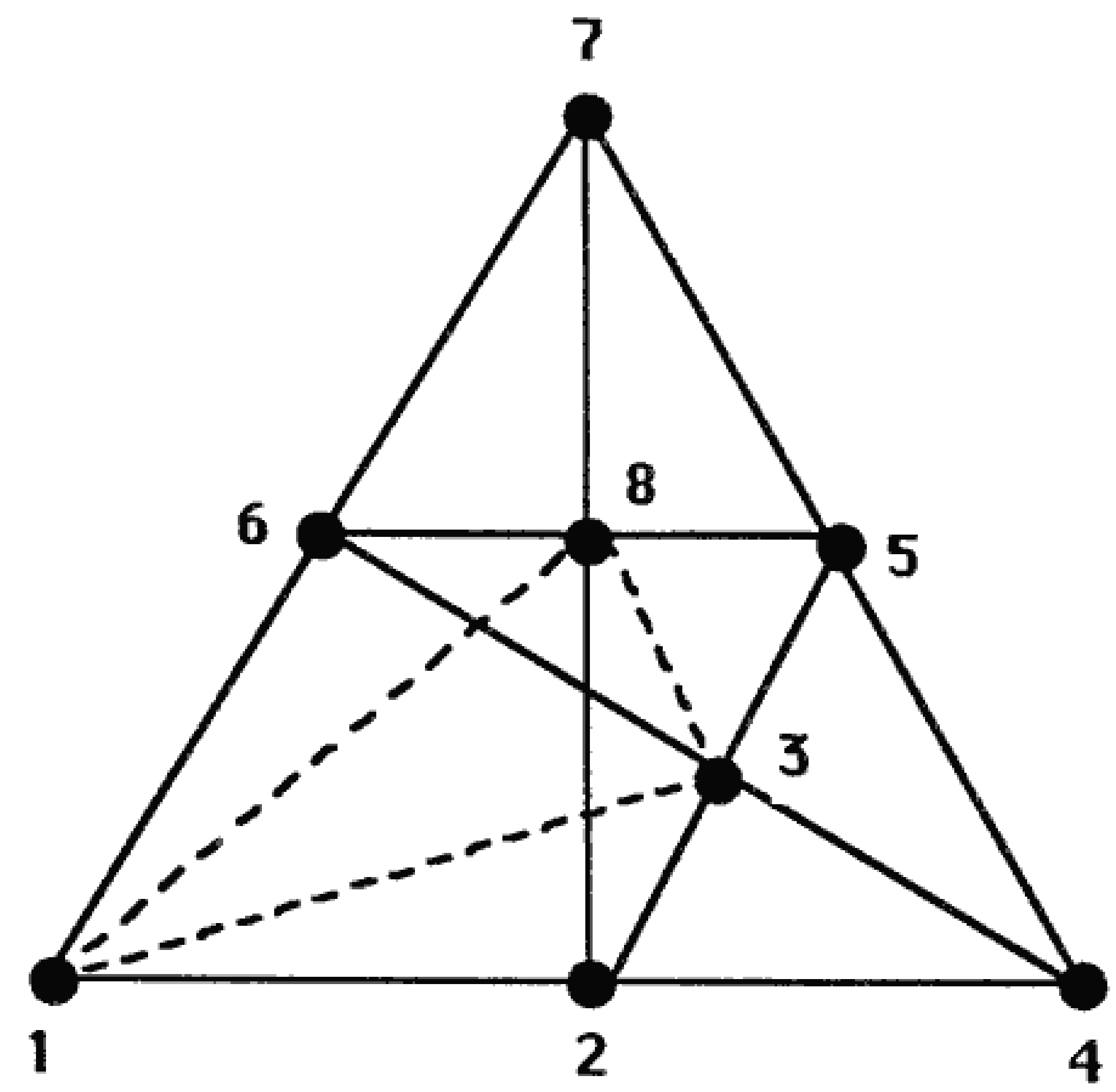


Figure 4

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