COMPLETING SEQUENCES AND SEMI-LB-SPACES (*)

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SUMMARY. - Given a completing sequence in a locally convex space, we associate to it a Fréchet space and we use it to obtain localization results both in webbed spaces and semi-LB-spaces. Finally the fact that every convex webbed space is absolutely convex webbed is also proved.

INTRODUCTION. - The vector spaces we shall use here are defined over the field $\mathbb{K}$ of real or complex numbers. The word "space" means "separated locally convex space". Given a space $E$, we denote by $\hat{E}$ its completion. $\mathbb{N}$ is the set of positive integers.

If $A$ is a bounded, absolutely convex set in a space $E$, we denote by $E_A$ the linear hull of $A$ endowed with the norm of the Minkowski functional of $A$. A fundamental system of neighbourhoods of the origin in $E_A$ is the family

$$\{\frac{1}{n}A : n = 1, 2, \ldots\}.$$ 

It is said that $A$ is a Banach disc when $E_A$ is a Banach space. A space $A$ is unordered Baire-like if, given any sequence $(A_n)$ of closed and absolutely convex subsets of $E$ converging $E$, there

(*) This research was undertaken while the author visited the University of Lecce during the spring of 1985, at the invitation of Prof. V.B. Moscatelli.

(**) Supported in part by CAICYT (pr. 83-2622).
is a positive integer \( p \) such that \( A_p \) is a neighbourhood of the origin \([5]\). As an immediate consequence, if \((E_n)\) is a sequence of subspaces of an unordered Baire-like space \( E \) that covers \( E \), there is a positive integer \( p \) such that \( E_p \) is unordered Baire-like and dense in \( E \).

Following De Wilde \([1]\) and \([2]\), we define a web in a space \( E \) as a family

\[ \mathcal{W} = \{ C_{m_1, m_2, \ldots, m_n} \} \]

of subsets of \( E \), where \( n, m_1, m_2, \ldots, m_n \) are positive integers, and such that the following relations are satisfied:

\[ E = \bigcup \{ C_{m_1} : m_1 = 1, 2, \ldots \}, \]

\[ C_{m_1, m_2, \ldots, m_n} = \bigcup \{ C_{m_1, m_2, \ldots, m_n, m} : m = 1, 2, \ldots \}, \quad n \geq 1. \]

A web \( \mathcal{W} \) is said to be convex (absolutely convex) if the sets defining it are convex (absolutely convex). A web \( \mathcal{W} \) is completing, or a \( \mathcal{W} \)-web, if the following condition is satisfied: for every sequence \((m_n)\) of positive integers there is a sequence \((\lambda_n)\) of positive numbers such that for

\[ x_n \in C_{m_1, m_2, \ldots, m_n}, \quad 0 \leq |\mu_n| \leq \lambda_n, \quad \mu_n \in \mathbb{K}, \quad n=1, 2, \ldots, \]

the series

\[ \sum_{n=1}^{\infty} \mu_n x_n \]
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converges in E. We shall say that a space E is a convex (absolutely convex) webbed space if it admits a convex (absolutely convex) $\mathcal{G}$-web.

We shall say that a sequence $a_n$ in $\mathbb{N}^\infty$, with

$$a_n = (a_{n,p})_{p=1}^\infty, \quad n=1,2,\ldots,$$

is semi-stationary if, given any positive integer $p$, we have another positive integer $q$ such that

$$a_{n,p} = a_{q,p}, \quad n > q.$$  

A semi-LB-representation in a space $F$ is a family of Banach discs

$$\{A_\alpha : \alpha \in \mathbb{N}^\infty\}$$

verifying the following two conditions:

1. $\bigcup \{A_\alpha : \alpha \in \mathbb{N}^\infty\} = F$.

2. If $(a_n)$ is a semi-stationary sequence in $\mathbb{N}^\infty$, we have $\alpha$ in $\mathbb{N}^\infty$ such that

$$A_{a_n} \subset A_\alpha, \quad n = 1,2,\ldots.$$  

We shall call a semi-LB-space a space admitting a semi-LB-representation.

1. ABSOLUTERY CONVEX $\mathcal{G}$-COMPLETING SEQUENCES.

In a space $F$, let $\mathcal{G}$ be a family of Banach discs that converges $F$ and such that the finite union of members of $\mathcal{G}$ is contained
in some member of \( \mathcal{B} \). We shall say that a sequence \((A_k)\) of subsets of \( F \) is absolutely convex \( \mathcal{B} \)-completing if it is a decreasing sequence, every \( A_k \) is absolutely convex, and there is a sequence \((\lambda_k)\) of positive numbers such that given

\[
0 \leq |u_k| \leq \lambda_k \cdot x_k \in A_k, \quad k = 1, 2, \ldots, \]

there is a \( B \) in \( \mathcal{B} \) with

\[
x_k \in F_B, \quad k = 1, 2, \ldots, \]

and the series

\[
\sum_{k=1}^{\infty} u_k x_k
\]

converges in \( F_B \). In what follows we shall suppose that

\[
\lambda_1 = 1, \quad \lambda_k > \lambda_{k+1}, \quad \lambda_k < \frac{1}{2k}, \quad k = 2, 3, \ldots, \]

which does not imply any loss of generality.

When \( \mathcal{B} \) is the family of all the Banach discs in \( F \), the former concept coincides with the absolutely convex completing sequences of De Wilde (see [2, Proposition IV: 1.9]). We are going to consider the family \( \mathcal{B} \) in order to obtain results that can be applied to the class of semi-LB-spaces.

We take a positive integer \( k \) and we write

\[
B_k = \bigcup \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : x_n \in A_{ktn-1}, \quad n = 1, 2, \ldots \right\}.
\]

It is immediate that \( B_k \) is absolutely convex and contains \( A_k \).
Of course \((B_n)\) is a decreasing sequence.

**Proposition 1.** If \(W\) is a neighbourhood of the origin in \(F\), there are a positive integer \(k\) together with a positive number \(\lambda\) such that

\[\lambda B_k \subset W.\]

**Proof.** It is not a restriction to assume that \(W\) is closed and absolutely convex. It is clear that the condition required for \((B_k)\) is equivalent to the corresponding one with \((A_k)\). But the latter is easy to prove. Let us suppose that the property does not hold. For every positive integer \(k\) there is a point \(x_k\) in \(A_k\) such that

\[\lambda_k x_k \notin W.\]

The series

\[\sum_{k=1}^{\infty} \lambda_k x_k\]

converges in \(F\), consequently the sequence \((\lambda_k x_k)\) converges to the origin in \(F\). So we have a positive integer \(p\) such that

\[\lambda_k x_k \in W \quad \text{if} \quad k \geq p,

which is a contradiction. \(\quad q.e.d.\)

Let \(G\) be a dense subspace of a metrizable space \(E\). Let \(T\) be a linear mapping from \(G\) into \(F\). We write

\[T^{-1}(A_k) = U_k, \quad T^{-1}(B_k) = V_k.\]
$ar{U}_k$ will be the closure of $U_k$ in $E$ and $\bar{U}_k$ the interior of $\bar{U}_k$ in the same space $E$. Let us suppose that $\bar{U}_k$ is a neighbourhood of the origin in $E$, $k=1,2,\ldots$.

**PROPOSITION 2.** If the graph of $T$ meets $\text{ExF}_B$ in a closed subspace for every $B$ in $\mathcal{B}$, we have that

$$\bar{U}_k \subset V_k, \quad k = 1,2,\ldots.$$  

**Proof.** We fix a positive integer $k$ and we take any point $x$ in $\bar{U}_k$. Let

$$\{W_n : n = 1,2,\ldots\}$$

be a fundamental system of neighbourhoods of the origin in $E$ such that

$$W_n \subset \bar{U}_{n+k}, \quad n = 1,2.$$  

We take $x_1$ in $U_k$ such that

$$y_1 = x - x_1 \in \lambda_2 W_1.$$  

Proceeding by recurrence, it is assumed that for a positive integer $m$ we have found

$$y_m \in \lambda_{m+1} W_m.$$  

We now determine

$$x_{m+1} \in U_{m+k}$$

such that
\[ y_{m+1} = y_m - \lambda_{m+1} x_{m+1} e \lambda_{m+2} W_{m+1}. \]

The sequence \((y_n)\) obviously converges to the origin in \(E\), and

\[ y_n = x - x_1 - \lambda_2 x_2 - \ldots - \lambda_n x_n \]

for every positive integer \(n\). Consequently, we have in \(E\)

\[ x = \sum_{n=1}^{\infty} \lambda_n x_n. \]

For every positive integer \(j\),

\[ T x_j \in A_{k+j-1}; \]

since \((A_n)\) is \(\mathcal{A}\)-completing, we have a \(B\) in \(\mathcal{A}\) such that

\[ T x_j \in F_B \]

and the series

\[ \sum_{n=1}^{\infty} \lambda_n T x_n \]

converges in \(F_B\) to a vector \(u\) that obviously belongs to \(B_k\). The fact that \(Tx = u\) follows from the fact that the graph of \(T\) meets \(E \times F_B\) in a closed set. Then \(x\) belongs to \(V_k\) and the proof is com-plete.

q.e.d.

**PROPOSITION 3.** The set

\[ M := \bigcap \{ A_k : k = 1, 2, \ldots \} \]

is contained in a Banach disc.
Proof. If $W$ is a neighbourhood of the origin in $F$, we apply Proposition 1 to obtain $\lambda > 0$ and a positive integer $p$ such that

$$\lambda M \subset \lambda B_p \subset W$$

and thus $M$ is a bounded subset of $F$. Let $\psi$ be the canonical injection of $F_M$ into $F$. We can extend $\psi$ to a linear mapping $\hat{\psi}$ from the completion $H$ of $F_M$ into $\hat{F}$. Let $G$ be equal to $\hat{\psi}^{-1}(F)$. If $\varphi$ is the restriction of $\hat{\psi}$ to $G$, we have that the graph of $\varphi$ is closed in $H \times F$. If we denote by $U_k$ the set $\varphi^{-1}(A_k)$ and by $V_k$ the set $\varphi^{-1}(B_k)$, we have that the closure $\hat{U}_k$ of $U_k$ in $H$ is a neighbourhood of the origin in this space. Therefore, if we apply Proposition 2 we obtain that

$$\hat{U}_k \subset V_k,$$

from which it follows that $H=G$. Consequently, the image through $\varphi$ of the closed unit ball of $H$ is a Banach disc in $F$ containing the set $M$.

q.e.d.

Let us take $v_k$ in $A_k$, $k = 1, 2, ..., n$, and let us denote by $X_k$ the absolutely convex cover of

$$\{v_1, v_2, ..., v_k\} \cup A_k.$$

PROPOSITION 4. $(X_k)$ is an absolutely convex $w$-completing sequence.

Proof. Let us take $x_k$ in $X_k$. There is $y_k$ in $A_k$ and
\[ b_k, a_{kj} \in \mathbb{K}, \quad j = 1, 2, \ldots, k, \]
such that
\[ \sum_{j=1}^{k} |a_{kj}| + |b_k| \leq 1, \quad x_k = \sum_{j=1}^{k} a_{kj} v_j + b_k y_k. \]

If
\[ 0 \leq |u_k| \leq 2^{-k} \lambda_k \]
we have
\[ \sum_{k=1}^{\infty} \mu_k x_k = \sum_{k=1}^{\infty} \mu_k \left( \sum_{j=1}^{k} a_{kj} v_j + b_k y_k \right) \]
\[ = \sum_{j=1}^{\infty} \left( \sum_{k=j}^{\infty} \mu_k a_{kj} \right) v_j + \sum_{k=1}^{\infty} \left( \mu_k b_k \right) y_k. \]

Since
\[ \left| \sum_{k=j}^{\infty} \mu_k a_{kj} \right| \leq \sum_{k=j}^{\infty} 2^{-k} \lambda_k \leq \lambda_j, \]
\[ |\mu_k b_k| \leq |\mu_k| \leq \lambda_k, \]
it follows that the series
\[ \sum_{k=1}^{\infty} \mu_k x_k \]
belongs to some $F_B$, $B \in \mathcal{A}$, and it converges in this space.

q.e.d.

**PROPOSITION 5. Iff**
\[
\nu_k \in A_k, \ b_k \in \mathbb{K}, \ k=1,2,..., \text{ and } \sum_{k=1}^{\infty} |b_k| < \infty ,
\]

then the series
\[
\sum_{k=1}^{\infty} b_k \nu_k
\]
converges in \( P \) and the set
\[
A = \{ \sum_{k=1}^{\infty} a_k \nu_k : \sum_{k=1}^{\infty} |a_k| \leq 1 \}
\]
is a Banach disc.

Proof. We write \( X_k \) to denote the absolutely convex cover of
\[
\{ \nu_1, \nu_2, \ldots, \nu_k \} \cup A_k.
\]

According to the former proposition, \( (X_k) \) is an absolutely convex \( \mathcal{S} \)-completing sequence. We know that
\[
\cap \{X_k : k = 1,2,...\}
\]
is contained in a Banach disc \( P \) by Proposition 3. Let us observe that
\[
\nu_k \in P, \ k = 1,2,...
\]
and the conclusion now is obvious.

q.e.d.

The former proposition ensures that the following sets are well defined:
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\[ C_k = \{ \lim_{j \to \infty} \sum_{j=1}^{\infty} a_j x_j : x_j \in A_{k+j-1}, a_j \in K, \quad j = 1, 2, \ldots, \quad \sum_{j=1}^{\infty} |a_j| \leq 1 \}, \quad k = 1, 2, \ldots. \]

We write \( D_k \) for the linear hull of \( C_k \). We set

\[ F(A_k) = \cap \{ D_r : r = 1, 2, \ldots \}. \]

According to Proposition 1, the family

\[ \frac{1}{r}(F(A_k) \cap C_r), \quad r = 1, 2, \ldots, \]

is a fundamental system of neighbourhoods of the origin in \( F(A_k) \) for a locally convex and metrizable topology finer than the topology induced by \( F \) on \( F(A_k) \). Let us suppose that \( F(A_k) \) is endowed with this metrizable topology.

**PROPOSITION 6.** \( F(A_k) \) is a Fréchet space.

**Proof.** Let \( (y_r) \) be a Cauchy sequence in \( F(A_k) \). We select a subsequence \( (z_r) \) of \( (y_r) \) such that

\[ 2^{2r}(z_{r+1} - z_r) \in C_r. \]

Then we have

\[ x_{jr} \in A_{r+j-1}, a_{jr} \in K, \quad j = 1, 2, \ldots, \quad \sum_{j=1}^{\infty} |a_{jr}| \leq 1, \]

such that

\[ 2^{2r}(z_{r+1} - z_r) = \sum_{j=1}^{\infty} a_{jr} x_{jr}. \]
We fix a positive integer. Then
\[
\sum_{r=s}^{\infty} (z_{r+1} - z_r) = \sum_{r=s}^{\infty} \sum_{j=1}^{\infty} a_{jr} 2^{r+j} \cdot r = \sum_{m=s}^{\infty} \sum_{r=s}^{m} \frac{a_{(m-r+1)r}}{2^{2r}} \cdot (m-r+1)r.
\]

We put
\[
\nu_m = \sum_{r=1}^{m} \frac{|a_{(m-r+1)r}|}{2^{2r}}, \quad m = s, s+1, \ldots,
\]
and \(y_m = 0\) if \(\nu_m = 0\),
\[
y_m = \sum_{r=s}^{m} \frac{a_{(m-r+1)r}}{2^{2r}} \cdot (m-r+1)r \quad \text{if} \quad \nu_m \neq 0.
\]

Clearly, \(y_m\) belongs to \(A_m\) and
\[
\sum_{r=s}^{\infty} (z_{r+1} - z_r) = \sum_{m=s}^{\infty} \nu_m y_m.
\]

On the other hand,
\[
\sum_{m=s}^{\infty} \nu_m = \sum_{m=s}^{\infty} \sum_{r=s}^{m} \frac{|a_{(m-r+1)r}|}{2^{2r}} = \sum_{r=s}^{\infty} \sum_{j=1}^{\infty} \frac{|a_{jr}|}{2^{2r}} \cdot r = \sum_{r=s}^{\infty} \frac{1}{2^{2r}} \cdot \frac{1}{2^r} \cdot \frac{1}{2^s}.
\]

Consequently, the series \((1)\) is convergent in \(F\) and its sum belongs to \(\frac{1}{2^{2s}}\). Therefore, if
\[
\sum_{r=1}^{\infty} (z_{r+1} - z_r) = u
\]
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in $F$, we have $(z_r)$ converging to $u - z_1$ in $F$. On the other hand,

$$\sum_{r=s}^{\infty} (z_{r+1} - z_r) = u - z_1 - z_s e - \frac{1}{2^s} C_s,$$

from which it follows that

$$u \in D_s, \quad s = 1, 2, \ldots,$$

and this

$$u \in F^{(A_k)}.$$

It also follows from (1) that $(z_s)$ converges to $u-z_1$ in $F^{(A_k)}$.

Finally, it is obvious that $(y_T)$ also converges to $u-z_1$ in $F^{(A_k)}$.

q.e.d.

**Theorem 1.** Let $f$ be a linear mapping from a metrizable space $E$ into $F$ such that the graph of $f$ meets $E \times F_B$ in a closed set for every $B$ of $\mathcal{B}$. If the closure of $f^{-1}(A_k)$ in $E$ is a neighbourhood of the origin, then $f(E) \subset F^{(A_k)}$ and $f : E \to F^{(A_k)}$ is continuous.

**Proof.** We fix a positive integer $k$. According to Proposition 2, $f^{-1}(B_k)$ is a neighbourhood of the origin in $E$ and, consequently, $f(E)$ is contained in the linear hull of $B_k$. From the definitions, it is clear that $2C_k$ contains $B_k$. Thus we have $f(E) \subset D_k$ and so

$$f(E) \subset F^{(A_k)}.$$

If $(x_n)$ is a sequence in $E$ converging to the origin and $r$ is a positive integer, there is another positive integer $p$ such that
\[ x_n \in \frac{1}{2^n} B_1, \quad n \geq p. \]

Then

\[ f(x_n) \in \frac{1}{2^n} (F^k \cap C_1), \quad n \geq p, \]

from which the continuity of \( f \) follows.

q.e.d.

**PROPOSITION 7.** Let \( f \) be a continuous and injective linear mapping from a space \( E \) into \( F \). If the closure \( M_k \) of \( f^{-1}(A_k) \) in \( E \) is a neighbourhood of the origin, then the family

\[ \left\{ \frac{1}{k} M_k : k = 1, 2, \ldots \right\} \]

is a fundamental system of neighbourhoods of the origin for a metrizable locally convex topology on \( E \).

**Proof.** We must show that

\[ \bigcap_{k=1}^{\infty} \frac{1}{k} M_k = \{0\}. \]

Let us take a point \( x \) in \( E \), \( x \neq 0 \). We find a neighbourhood of the origin \( U \) in \( F \), closed and absolutely convex and such that

\[ f(x) \notin U. \]

Then

\[ x \in f^{-1}(U). \]
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According to Proposition 1 there is a positive integer $k$ such that

$$\frac{1}{k} A_k \subset U$$

and, therefore,

$$\frac{1}{k} M_k \subset f^{-1}(U),$$

showing that $x$ does not belong to $\frac{1}{k} M_k$.

q.e.d.

**Theorem 2.** Let $f$ be a linear mapping with closed graph from a space $E$ into $F$. Let us suppose that for every positive integer $k$, the closure of $f^{-1}(A_k)$ in $E$ is a neighbourhood of the origin. Then we have

$$f(E) \subset F^{(A_k)}$$

and $f : E \to F^{(A_k)}$ is continuous.

**Proof.** Since the graph of $f$ is closed, there is a Hausdorff and locally convex topology $\psi$ on $F$, coarser than the original one, and such that

$$f : E \to F[\psi]$$

is continuous, (cf. [3] and [4]). The sequence $(A_k)$ is also a $\mathcal{A}$-completing sequence of absolutely convex subsets in $F[\psi]$ and $f^{-1}(0)$ is closed in $E$. Let $\phi$ be the canonical mapping from $E$ onto $G := E/f^{-1}(0)$ and $\psi$ the canonical injection from $G$ into $F$, with
\[ f = \psi \circ \varphi. \]

According to the former proposition, and denoting by \( M_k \) the closure of \( \psi^{-1}(A_k) \) in \( G \), \( k=1,2,\ldots \), we obtain the family

\[ \left\{ \frac{1}{k} M_k : k = 1,2,\ldots \right\} \]

as a fundamental system of neighbourhoods of the origin in \( G \) for a metrizable and locally convex topology \( \mathcal{U} \) on \( G \). Then the closure of \( \psi^{-1}(A_k) \) in \( G[\mathcal{U}] \) coincides with \( M_k \) and, therefore, it is a neighbourhood of the origin in this space. Now the conclusion follows applying Theorem 1.

\[ \text{q.e.d.} \]

2. ABSOLUTELY CONVEX WEBBED SPACES

In all this section

\[ W = \{ C_{m_1,m_2,\ldots,m_n} \} \tag{2} \]

will be an absolutely convex and completing web in a space \( E \).

If \( \alpha = (a_n) \) is an element of \( \mathbb{N}^\mathbb{N} \), we have an absolutely convex and completing sequence

\[ (C_{a_1,a_2,\ldots,a_k})_{k=1}^{\infty}. \]

We shall write \( E_\alpha \) to denote the Fréchet space \( E \)
\[ (C_{a_1,a_2,\ldots,a_k}) \]

and we say that

\[ \{ E_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \} \]
is the family of Fréchet spaces associated to the web (2).

**Theorem 3.** Let \( f \) be a linear mapping from a metrizable and unordered Baire-like space \( F \) into the space \( E \). If the graph of \( f \) meets \( F \times E_B \) in a closed subspace for every Banach disc \( B \) of \( E \), there is an \( a \) in \( \mathbb{N}^\infty \) such that \( f(F) \in E_a \) and \( f : F \rightarrow E_a \) is continuous.

**Proof.** Given a sequence \( (p_n) \) of positive integers, we denote by \( L_{p_1, p_2, \ldots, p_n} \) the linear hull of \( f^{-1}(C_{p_1, p_2, \ldots, p_n}) \) in \( F \), \( n = 1, 2, \ldots \). We have

\[
F = \bigcap_{n=1}^{\infty} L_n,
\]

from which it follows that for a positive integer \( m_1 \) the space \( L_{m_1} \) is unordered Baire-like and dense in \( F \). Proceeding by recurrence, let us suppose that the positive integers \( m_1, m_2, \ldots, m_p \) have been obtained in such a way that the space \( L_{m_1, m_2, \ldots, m_p} \) is unordered Baire-like and dense in \( F \). We have

\[
L_{m_1, m_2, \ldots, m_p} = \bigcup_{m=1}^{\infty} L_{m_1, m_2, \ldots, m_p, m}
\]

from which we have again a positive integer \( m_{p+1} \) such that the space \( L_{m_1, m_2, \ldots, m_p, m_{p+1}} \) is unordered Baire-like and dense in \( F \).

Obviously, the closures in \( F \) of \( f^{-1}(C_{m_1, m_2, \ldots, m_k}) \), \( k=1, 2, \ldots \), are neighbourhood of the origin in \( F \). Therefore according to Theorem...
we obtain for $\alpha = (a_k)$ that $f(F) \subseteq E_\alpha$ and $f : F \rightarrow E_\alpha$ is continuous.

q.e.d.

**Theorem 4.** If $f$ is a linear mapping with closed graph from an unordered Baire-like space $F$ into the space $E$, then there exists $\alpha$ in $\mathbb{N}^N$ such that $f(F) \subseteq E_\alpha$ and $f : F \rightarrow E_\alpha$ is continuous.

**Proof.** Proceeding as we have done in the former theorem we can obtain $\alpha = (a_k)$ in $\mathbb{N}^N$ such that $f^{-1}(C_{m_1^k, m_2, \ldots, m_k})$ is a neighbourhood of the origin in $F$, $k=1, 2, \ldots$. The conclusion now follows applying Theorem 2.

q.e.d.

**Corollary.** Every continuous linear mapping from an unordered Baire-like space $F$ into $E$ can be extended to a continuous linear mapping from $F$ into $E$.

3. **Semi-LB-Spaces.**

Let

$$
\{ A_\alpha : \alpha \in \mathbb{N}^N \}
$$

be a semi-LB-representation in a space $E$. Given positive integers $k, m_1, m_2, \ldots, m_k$, we write

$$
M_{m_1^k, m_2, \ldots, m_k} = \bigcup \{ A_\alpha : \alpha = (a_n) \in \mathbb{N}^N, a_n = m_n, n=1, 2, \ldots, k \}.
$$

Let $C_{m_1^k, m_2, \ldots, m_k}$ be the absolutely convex cover of $M_{m_1^k, m_2, m_k}$. 
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We denote by Ω the family (3) of Banach discs.

PROPOSITION 8. Given \((m_k)\) in \(\mathbb{N}^N\), the sequence

\[
C(m_1, m_2, \ldots, m_k)
\]

is absolutely convex and Ω-completing.

Proof. Let \(x_k\) be a vector in \(C_{m_1, m_2, \ldots, m_k}\), \(k = 1, 2, \ldots\). There are

\[
x_k = \sum_{j=1}^{p(k)} a_{kj} x_{kj}, \quad j=1, 2, \ldots, p(k)
\]

such that

\[
x_k = \sum_{j=1}^{p(k)} a_{kj} x_{kj}, \quad \sum_{j=1}^{p(k)} |a_{kj}| \leq 1.
\]

Let

\[
a_{kj} = (a_{n,kj}) \to \mathbb{N}^N, \quad a_{n,kj} = m_n, \quad n = 1, 2, \ldots, k,
\]

and

\[
x_{kj} \in \mathbb{A}_{a_{kj}}, \quad j = 1, 2, \ldots, p(k).
\]

The sequence

\[
a_{11}, a_{12}, \ldots, a_{1p(1)}, a_{21}, a_{22}, \ldots, a_{2p(2)}, \ldots, a_{k1}, a_{k2}, \ldots, a_{kp(k)},
\]

obviously is semi-stationary; therefore, we have \(a\) in \(\mathbb{N}^N\) such that

\[
A_{a_{kj}} \in A_{a}, \quad j = 1, 2, \ldots, p(k), \quad k = 1, 2, \ldots.
\]
Consequently,
\[ x_k \in A_q , \quad k = 1, 2, \ldots, \]
and if
\[ b_k \in k, \quad k = 1, 2, \ldots, \quad \text{and} \quad \sum_{k=1}^{\infty} |b_k| < 1, \]
the series
\[ \sum_{k=1}^{\infty} b_k x_k \]
converges in \( E_{A_q} \).

q.e.d.

If \( a = (m_k) \in \mathbb{N}^\mathbb{N} \) we denote by \( E_a \) the Fréchet space \( E_{m_1, m_2, \ldots, m_k} \) and we shall say that
\[ \{ E_a : a \in \mathbb{N}^\mathbb{N} \} \]
in the family of Fréchet spaces associated to the semi-LB-representation (3).

The following two theorems are proved using Theorem 1 and Theorem 2 respectively.

**THEOREM 5.** Let \( f \) be a linear mapping from a metrizable Baire space \( F \) into the space \( E \). If the graph of \( f \) meets \( F \times E_{A\beta} \) in a closed subspace for every \( \beta \) in \( \mathbb{N}^\mathbb{N} \) there is \( a \) in \( \mathbb{N}^\mathbb{N} \) such that \( f(F) \subset E_a \) and \( f : F \rightarrow E_a \) is continuous.

**THEOREM 6.** If \( f \) is a linear mapping with closed graph from a Baire space \( F \) into the space \( E \), there is \( a \) in \( \mathbb{N}^\mathbb{N} \) such that
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f(F) c E_q and \( f : F \rightarrow E_q \) is continuous.

In the set \( \mathbb{N}^\mathbb{N} \) we consider the following order relation "\( \preceq \)"; for \( \alpha = (a_n) \) and \( \beta = (b_n) \) in \( \mathbb{N}^\mathbb{N} \) we say that \( \alpha \preceq \beta \) if and only if \( a_n \preceq b_n \) for every positive integer \( n \).

A quasi-LB-representation in a space G is a family

\[ \{ B_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \} \]

of Banach discs satisfying the following conditions:

1. \( \bigcup \{ B_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \} = G \).

2. If \( \alpha, \beta \in \mathbb{N}^\mathbb{N} \) and \( \alpha \preceq \beta \), then \( B_\alpha \subset B_\beta \)

We say that a space admitting a quasi-LB-representation is a quasi-LB-space.

It is obvious that a quasi-LB-representation is a semi-LB-representation, and thus, a quasi-LB-space is a semi-LB-space.

Lifting theorems have been proved in [6] for quasi-LB-representations. These results can be formulated with some minor modifications for semi-LB-representations.

4. CONVEX WEBBED SPACES

Let

\[ \mathcal{V} = \{ L_{n_1, n_2, \ldots, n_k} \} \]
be a convex \( \mathcal{W} \)-web in a space \( E \). If \( M_{n_1,n_2,\ldots,n_k} \) is the convex cover of

\[
\{0\} \cup \bigcup_{n_1,n_2,\ldots,n_k}
\]

we write

\[
A_{n_1,n_2,\ldots,n_k} = M_{n_1,n_2,\ldots,n_k} - M_{n_1,n_2,\ldots,n_k}.
\]

We denote by \( T \) an injective mapping from \( \mathbb{N}^2 \) onto \( \mathbb{N} \). When \( (p_1,r_1) \) belongs to \( \mathbb{N}^2 \) and \( T(p_1,r_1) = n_1 \), we put

\[
B_{n_1} = p_1 A_{r_1}.
\]

Proceeding by recurrence, let us suppose that for a positive integer \( k > 1 \) we have constructed the subsets

\[
B_{n_1,n_2,\ldots,n_{k-1}},
\]

where \( n_1,n_2,\ldots,n_{k-1} \) are arbitrary positive integers. Given positive integers \( p_1,r_1,p_2,r_2,\ldots,p_k,r_k \) we write

\[
p_{n_1,n_2,\ldots,n_k} = p_1 p_2,\ldots,p_k A_{r_1,r_2,\ldots,r_k},
\]

\[
B_{n_1,n_2,\ldots,n_k} = p_{n_1,n_2,\ldots,n_k} \cap B_{n_1,n_2,\ldots,n_{k-1}},
\]

where

\[
T(p_j,r_j) = n_j, \quad j = 1,2,\ldots,k.
\]
Since $p_1^n A_{r_1}$ contains $L_{r_1}$, it follows that

$$U \left\{ B_{n_1} : n_1 = 1, 2, \ldots \right\} = E.$$  

Let us now take a point $x$ in $B_{n_1, n_2, \ldots, n_k}$, then $x$ belongs to $p_{n_1, n_2, \ldots, n_k}$ and therefore there exist two points $y$ and $z$ in

$$p_1 p_2 \ldots p_k L_{r_1, r_2, \ldots, r_k}$$

together with two numbers $\alpha$ and $\beta$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, such that

$$x = \alpha y - \beta x.$$  

If $y$ coincides with $z$, there is a positive integer $r_{k+1}$ such that

$$y = z \in p_1 p_2 \ldots p_k L_{r_1, r_2, \ldots, r_k, r_{k+1}}$$

and so

$$x \in p_1 p_2 \ldots p_k A_{r_1, r_2, \ldots, r_{k+1}}.$$  

If $y$ is not equal to $z$, we take

$$H = \{ \lambda z + (1 - \lambda) y : 0 \leq \lambda \leq 1 \}.$$  

Since $H$ is contained in

$$p_1 p_2 \ldots p_k L_{r_1, r_2, \ldots, r_k},$$
there exists a positive integer \( r_{k+1} \) such that
\[
 p_1 p_2 \cdots p_k \mathcal{L}_{r_1, r_2, \ldots, r_{k+1}}
\]
meets \( H \) in two points at least. A positive integer \( p_{k+1} \) can be determined such that
\[
 2H \mathcal{c} p_1 p_2 \cdots p_k p_{k+1} A_{r_1, r_2, \ldots, r_k, r_{k+1}}
\]
and therefore
\[
 x \in p_1 p_2 \cdots p_k p_{k+1} A_{r_1, r_2, \ldots, r_k, r_{k+1}} \tag{5}
\]
Consequently, (5) holds in the two cases considered. If
\[
 T(p_{k+1}, r_{k+1}) = n_{k+1},
\]
it follows that
\[
 x \in p_{n_1, n_2, \ldots, n_{k+1}} \cap B_{n_1, n_2, \ldots, n_k} = B_{n_1, n_2, \ldots, n_k, n_{k+1}}
\]
from which we have
\[
 B_{n_1, n_2, \ldots, n_k} = U \{ B_{n_1, n_2, \ldots, n_k, n_{k+1}} : n_{k+1} = 1, 2, \ldots \}, \tag{6}
\]

**PROPOSITION 9.** The family

\[
 \mathcal{U} = \{ B_{n_1, n_2, \ldots, n_k} \}
\]
is a completing web in \( E \).

**Proof.** From (4) and (6) we know that \( \mathcal{U} \) is a web in \( E \). Given a sequence of positive integers \( (r_k) \) we determine a sequence
(λ_k) of positive numbers such that the series

\[ \sum_{k=1}^{\infty} u_k x_k \]

converges in E whenever

\[ 0 \leq \mu_k \leq \lambda_k, \quad x_k \in L_{r_1, r_2, \ldots, r_k}, \quad k = 1, 2, \ldots. \]

Let us now suppose that for the sequence (n_j) in \( \mathbb{N} \) we have

\[ T^{-1}(n_j) = (p_j, r_j), \quad j = 1, 2, \ldots. \]

If we take \( z_k \) in \( B_{n_1, n_2, \ldots, n_k} \) we have

\[ z_k \in p_1 p_2 \cdots p_k A_{r_1, r_2, \ldots, r_k} \]

and we can find \( u_k \) and \( v_k \) in \( L_{r_1, r_2, \ldots, r_k} \) together with

\[ 0 \leq a_k \leq 1, \quad 0 \leq \beta_k \leq 1, \]

such that

\[ z_k = p_1 p_2 \cdots p_k (a_k u_k - \beta_k v_k). \]

Let us now take

\[ 0 \leq \mu_k \leq (p_1 p_2 \cdots p_k)^{-1} \lambda_k \]

and we have the convergent series

\[ \sum_{k=1}^{\infty} u_k p_1 p_2 \cdots p_k a_k u_k \quad \text{and} \quad \sum_{k=1}^{\infty} u_k p_1 p_2 \cdots p_k \beta_k v_k \]

from which it follows that the series
\[ \sum_{k=1}^{\infty} u_k z_k \]

also converges in \( E \).

Q.E.D.

When \( E \) is a real space we write

\[ C_{n_1, n_2, \ldots, n_k} = B_{n_1, n_2, \ldots, n_k} \]

and in case of \( E \) being a complex space we write

\[ C_{n_1, n_2, \ldots, n_k} = B_{n_1, n_2, \ldots, n_k} \cap i \ B_{n_1, n_2, \ldots, n_k} \]

whenever \( k, n_1, n_2, \ldots, n_k \) are positive integers.

**Proposition 10.** The family

\[ W = \{ C_{n_1, n_2, \ldots, n_k} \} \]

is an absolutely convex and completely web in \( E \).

**Proof.** The result is obvious when \( E \) is a real space. Let us now suppose that \( E \) is a complex space. If \( x \) is any point of \( E \) there are two positive integers \( p_1 \) and \( r_1 \) such that the strongt line with and-points in \( x \) and \( ix \) is contained in \( p_1 A_{r_2} \). It now follows that both \( x \) and \( ix \) are in \( B_{n_1} \), where \( n_1 = T(p_1, r_1) \). Thus we have

\[ U \{ C_{n_1} : n_1 = 1, 2, \ldots \} = E . \]

If \( x \) is any point in \( C_{n_1, n_2, \ldots, n_k} \), we know that \( x \) and \( ix \) belong
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We put \((p_j, r_j) = T^{-1}(n_j), j = 1, 2, \ldots\). Then we have

\[ x, ix \in p_1 p_2 \ldots p_k A_{r_1, r_2, \ldots, r_k} \]

and therefore there are two positive integers \(p_{k+1}\) and \(r_{k+1}\) such that

\[ x, ix \in p_1 p_2 \ldots p_{k+1} A_{r_1, r_2, \ldots, r_{k+1}} \]

Consequently, we have

\[ x, ix \in B_{n_1, n_2, \ldots, n_{k+1}} \]

where \(T(p_{k+1}, r_{k+1}) = n_{k+1}\) and so

\[ x \in C_{n_1, n_2, \ldots, n_{k+1}} \]

Thus

\[ U\{C_{n_1, n_2, \ldots, n_{k+1}} : n_{k+1} = 1, 2, \ldots\} = C_{n_1, n_2, \ldots, n_k} \]

and hence \(W\) is a web in \(E\). Finally it is clear that \(W\) is absolutely convex and completing.

q.e.d.

The following theorem is now clear:

**Theorem 7.** If \(F\) is a convex webbed space, then \(F\) is an absolutely convex webbed space.
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