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SYNTESIZING JUDGEMENTS GIVEN BY PROBABILITY DISTRIBUTION FUNCTIONS

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ABSTRACT. -*In this paper we study some methods of synthesizing* n *probability distribution functions assigned by* n *individuals to a given situation in order to obtain a consensual probability distribution function.*

1. - Let us assume that a group of n individuals assigns to a given situation n probability distribution functions $F_1, F_2, \dots F_n$ each of which expresses, in the opinion of its producer, the best description of the analyzed problem. Then the problem of consensus arises. How to build another distribution function $F = s(F_1, F_2, \dots F_n)$ in order to arrive to a consensual assignement? This question, in the case of discrete distributions, has already a large literature (see [5]) and some interesting methods of averaging α-percent quantiles of the experts distributions have been investigated in [7] and [9]. In our approach we follow the axiomatic way by requiring conditions on the synthesizing functions s in a similar manner to the procedure that, for numerical assignements, have been developed in [2],[3],[4].

We will denote by Δ^+ the space of distribution functions (d.f.) vanishing at zero, i.e.

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$\Delta^+ = \{F \mid F: [-\infty,+\infty] \to [0,1] \mid F$ is non-decreasing, left-continuous and $F(0) = 0\}$. Among the elements of Δ^+ we find for any $a \geq 0$ and for any b in $[0,1]$ the functions:

$$\varepsilon_a(x) = \begin{cases} 0, & x \leq a \\ 1, & x > a \end{cases} \quad \text{and} \quad C_b(x) = \begin{cases} 0, & x \leq 0 \\ b, & x > 0 \end{cases}$$

Of course, $\varepsilon_0 = C_1$ and for any F in Δ^+, $F \circ C_b = C_{F(b)}$. Obviously, for any non-negative random variable (r.v.) X its associated distribution function F_X belongs to Δ^+.

Any binary operation τ in Δ^+ such that τ is associative, commutative, nondecreasing and ε_0 is a unit will be called a triangle function ([8]) and τ will be Archimedean if τ is continuous (with respect to the topology of weak convergence in Δ^+) and $\tau(F,F)<F$ for all F in $\Delta^+ \setminus \{\varepsilon_0, \varepsilon_\infty\}$. Note that using the associativity of a triangle function τ, for all n we can define $G_n = \tau(F_1,\dots,F_n)$ recursively by $G_1 = F_1$, $G_2 = \tau(F_1,F_2),\dots,G_n = \tau(G_{n-1},F_n)$.

Finally we introduce the spaces

$$L(\Delta^+) = \{f \mid f:[0,1] \to [0,1],\ f \text{ bijective},\ f(0) = 0,\ f(1) = 1 \text{ and } f \circ F \in \Delta^+, \text{ for all } F \text{ in } \Delta^+\},$$

$$D(\Delta^+) = \{g \mid g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0,\ F \circ g \in \Delta^+, \text{ for all } F \text{ in } \Delta^+\}.$$

2. - Our first model is based on the following:

DEFINITION 1. Let τ be an Archimedean triangle function in Δ^+ and let Φ be a given function in $L(\Delta^+)$. A function s from $(\Delta^+)^n$ into Δ^+ will be called an (n,τ,Φ)-*synthesizing function of distribution functions* if the following conditions hold:

(I) $s(\overbrace{F,F,\dots,F}^{n}) = F$;

(II) $s(F_1,F_2,\dots,F_n) = \tau(\Phi\circ F_1,\Phi\circ F_2,\dots,\Phi\circ F_n)$;

(III) $s(F_1\circ h,\ F_2\circ h,\dots,F_n\circ h)=s(F_1,F_2,\dots,F_n)\circ h$ for all h in $D(\Delta^+)$.

Condition (I) in the *unanimity principle,* i.e. if all individuals propose the same distribution F then the consensual distribution must be F. Condition (II) is the *separation principle* and (III) is the *invariance property* which states the good behaviour of s with respect to all possible uniform changes of scale.

THEOREM 1. *Let* s *be an* (n,τ,Φ)*-synthesizing function of d.f. Then there exists a continuous strictly decreasing function* t *from* $[0,1]$ *into* $\mathbb{R}^+$ *such that* $\Phi(x) = t^{-1}(t(x)/n)$ *and*

$$s(F_1,F_2,\dots,F_n)(x) = t^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} t(F_i(x))\Big),$$

i.e., s is the quasi-arithmetic mean generated by t .

Proof. Let s be an (n,τ,Φ)-synthesizing function. Given $a_1,a_2,\dots,a_n$ in $[0,1]$ we have for any $x,y > 0$:

$$\begin{aligned} s(C_{a_1},C_{a_2},\dots,C_{a_n})(x) &= s(C_{a_1},C_{a_2},\dots,C_{a_n})(C_x(y)) \\ &= s(C_{a_1}\circ C_x, C_{a_2}\circ C_x,\dots,C_{a_n}\circ C_x)(y) \\ &= s(C_{a_1},C_{a_2},\dots,C_{a_n})(y), \end{aligned}$$

i.e., $s(C_{a_1},C_{a_2},\dots,C_{a_n})$ is constant on $(0,+\infty)$. Consequently if we define $M : [0,1]^n \to [0,1]$ by

$$M(a_1,a_2,\dots,a_n) = s(C_{a_1},C_{a_2},\dots,C_{a_n})(1),$$

we will have

$$s(C_{a_1},C_{a_2},\dots,C_{a_n}) = C_{M(a_1,a_2,\dots,a_n)}. \tag{1}$$

Since s satisfies (I), if we substitute $a_1=a_2=\dots=a_n= a$ in (1), we obtain

$$M(a,a,\dots,a) = a,$$

i.e., M is an averaging function. Moreover, for all $F_1,F_2,\dots,F_n$ in Δ^+:

$$\begin{aligned} s(F_1,F_2,\dots,F_n)(x) &= s(F_1,F_2,\dots,F_n)(C_x(1)) = s(F_1\circ C_x,F_2\circ C_x,\dots,F_n\circ C_x)(1) \\ &= s(C_{F_1(x)},C_{F_2(x)},\dots,C_{F_n(x)})(1) \\ &= M(F_1(x),F_2(x),\dots,F_n(x)), \end{aligned}$$

i.e., s is a pointwise averaging function. Next we define $T : [0,1] \times [0,1] \to [0,1]$ by

$$T(a,b) = M(\Phi^{-1}(a), \Phi^{-1}(b), 1,\dots,1).$$

Then we have for all $x > 0$:

$$\begin{aligned} \tau(C_a,C_b)(x) &= \tau(C_a,C_b,\varepsilon_0,\dots,\varepsilon_0)(x) \\ &= \tau(\Phi\circ C_{\Phi^{-1}(a)},\Phi\circ C_{\Phi^{-1}(b)},\Phi\circ C_1,\dots \Phi\circ C_1)(x) \\ &= s(C_{\Phi^{-1}(a)}, C_{\Phi^{-1}(b)},C_1,\dots C_1)(x) \end{aligned}$$

$$= M(C_{\Phi^{-1}(a)}(x),\ C_{\Phi^{-1}(b)}(x), C_1(x), \dots C_1(x))$$

$$= M(\Phi^{-1}(a),\ \Phi^{-1}(b), 1, \dots, 1) = T(a,b),$$

whence

$$\tau(C_a, C_b) = C_{T(a,b)}. \tag{2}$$

Since τ is an Archimedean triangle function it follows from (2) that T is an Archimedean ordered abelian semigroup and according to the theorem of Ling (see [8]) there exists a continuous strictly decreasing function t from $[0,1]$ into $\mathbb{R}^+$ such that T is representable in the form $T(x,y) = t^{(-1)}(t(x)+t(y))$ where $t^{(-1)}$ denotes the pseudo-inverse of t, i.e., $t^{(-1)}(x)=t^{-1}(x)$ for x in $[0,t(0)]$ and $t^{(-1)}(x) = 0$ whenever $x \geq t(0)$. Then for all x in $(0,1)$ we have:

$$0 < x = M(x,x,\dots,x) = s(C_x, C_x, \dots, C_x)(1) =$$

$$= \tau(C_{\Phi(x)}, C_{\Phi(x)}, \dots, C_{\Phi(x)})(1)$$

$$= T(\Phi(x), \Phi(x), \dots, \Phi(x)) = t^{(-1)}\Big(\sum_{i=1}^{n} t(\Phi(x))\Big)$$

$$= t^{(-1)}(n\ t(\Phi(x))),$$

hence $\Phi(x) = t^{-1}(t(x)/n)$ and $t^{(-1)} = t^{-1}$, i.e., $t(0) = +\infty$.

Finally we have for all $F_1, F_2, \dots, F_n$ in Δ^+ :

$$\begin{aligned}
s(F_1,F_2,\dots,F_n)(x) &= M(F_1(x),F_2(x),\dots,F_n(x))\\
&= C_{M(F_1(x),F_2(x),\dots,F_n(x))}(1)\\
&= s(C_{F_1(x)},C_{F_2(x)},\dots,C_{F_n(x)})(1)\\
&= \tau(\Phi\circ C_{F_1(x)},\Phi\circ C_{F_2(x)},\dots,\Phi\circ C_{F_n(x)})(1)\\
&= \tau(C_{\Phi(F_1(x))},C_{\Phi(F_2(x))},\dots C_{\Phi(F_n(x))})(1)\\
&= C_{T(\Phi(F_1(x)),\Phi(F_2(x),\dots,\Phi(F_n(x)))}(1)\\
&= T(\Phi(F_1(x)),\Phi(F_2(x),\dots,\Phi(F_n(x)))\\
&= t^{-1}(\sum_{i=1}^{n} t(\Phi(F_i(x))))\\
&= t^{-1}(\frac{1}{n}\sum_{i=1}^{n} t(F_i(x)))
\end{aligned}$$

The theorem is proved.

3. - Motivated by this last result we turn our attention here to the case where we really want to synthesize distribution functions through operations on random variables. To this end we introduce the following

DEFINITION 2. Given a Borel measurable function g from $(\mathbb{R}^+)^n$ into $\mathbb{R}^+$ and a non-empty subset H of $D(\Delta^+)$, a function f from $(\Delta^+)^n$, into Δ^+ is called an (n,g,H)-*probabilistic synthesizing* *function* if the following conditions are satisfied for all positive random variables $X, X_1, X_2, \ldots, X_n$:

(i) $f(F_X, F_X, \ldots, F_X) = F_X$;

(ii) $f(F_{X_1}, F_{X_2}, \ldots, F_{X_n}) = F_{g(X_1, X_2, \ldots, X_n)}$;

(iii) $f(F_{X_1} \circ h, F_{X_2} \circ h, \ldots, F_{X_n} \circ h) = f(F_{X_1}, F_{X_2}, \ldots, F_{X_n}) \circ h$, for all h in H.

Conditions (i) and (iii) expresses as in the previous section the unanimity principle and the invariance property respect to an admissible space H of changes of scale. Obviously condition (ii) states that f is necessarily derivable from an operation on random variables.

In the next theorems we will characterize completely the (n,g,H)-probabilistic synthesizing functions when either H is the space of affine transformations or g is a weighted quasiarithmetic mean.

THEOREM 2. *Let* f *be an* (n,g,H)*-probabilistic synthesizing function where* $H = \{h_{ab} \mid h_{ab} : \mathbb{R}^+ \to \mathbb{R}^+, h_{ab}^{-1}(x) = ax+b,\ a,b \geq 0\}$.

Then $f(F_X, F_X, \ldots, F_X) = F_X$ *and for* $\{F_{X_1}, F_{X_2}, \ldots, F_{X_n}\}$ *which contains two different function we have* $f(F_{X_1}, \ldots, F_{X_n}) = F_{g(X_1, \ldots, X_n)}$ *where*

$$g(x_1, \ldots, x_n) = \mu + \sigma \cdot G\left(\frac{x_1 - \mu}{\sigma}, \ldots, \frac{x_n - \mu}{\sigma}\right),$$

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i, \qquad \sigma = \left[\frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2\right]^{\frac{1}{2}}$$

and $G : (\mathbb{R}^+)^n \to \mathbb{R}^+$ *is an arbitrary function.*

Proof. Let f be an (n,g,H)-probabilistic synthesizing function. Consider $X_1 = X_2 = \ldots = X_n \equiv x$ to be constant random variables of value $x \geq 0$. Then $F_{X_1} = F_{X_2} = \ldots = F_{X_n} = \varepsilon_x$ and by (i) and (ii) of Definition 2:

$$\varepsilon_x = f(\varepsilon_x, \varepsilon_x, \ldots, \varepsilon_x) = F_{g(X_1, X_2, \ldots, X_n)} = \varepsilon_{g(x,x,\ldots,x)} ,$$

i.e.,

$$g(x,x,\ldots,x) = x. \tag{3}$$

Thus $g : (\mathbb{R}^+)^n \to \mathbb{R}^+$ is an averaging function. Given $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^+$ and $a,b \geq 0$, let $X_1 \equiv x_1, \ldots, X_n \equiv x_n$ be a collection of constant r.v.

By (iii) and (ii) of Definition 2 we will have

$$\begin{aligned}\varepsilon_{ag(x_1,\ldots,x_n)+b} = \varepsilon_{h_{ab}^{-1}(g(x_1,\ldots,x_n))} &= \varepsilon_{g(x_1,\ldots,x_n)} \circ h_{ab} \\ &= f(\varepsilon_{x_1}, \ldots, \varepsilon_{x_n}) \circ h_{ab} = f(\varepsilon_{x_1} \circ h_{ab}, \ldots, \varepsilon_{x_n} \circ h_{ab}) \\ &= f(\varepsilon_{ax_1+b}, \ldots, \varepsilon_{ax_n+b}) \\ &= \varepsilon_{g(ax_1+b,\ldots,ax_n+b)},\end{aligned}$$

whence

$$ag(x_1,\ldots,x_n)+b = g(ax_1+b,\ldots,ax_n+b). \tag{4}$$

Writing (4) first for a=1 and then for b=0, and in view of (3), we can apply theorem 2 p 236 in [1], thus g must be of the form

$$g(x,x,\dots,x) = x,$$

and for $x_1,x_2,\dots,x_n$ with two different values $x_i \neq x_j$,

$$g(x_1,\dots,x_n) = \mu + \sigma G\left(\frac{x_1-\mu}{\sigma},\dots,\frac{x_n-\mu}{\sigma}\right),$$

where

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right]^{\frac{1}{2}}$$

and G arbitrary. The theorem follows

THEOREM 3. *Let f be an (n,g,H)-probabilistic synthesizing function where g is a weighted quasiarithmetic mean of the form:*

$$g(x_1,\dots,x_n) = \Phi^{-1}\left(\sum_{i=1}^{n} w_i\, \Phi(x_i)\right),$$

where $\Phi:\mathbb{R}^+ \to \mathbb{R}^+$ *is a bijection and* $w_1,\dots,w_n \geq 0$ *with* $\sum_{i=1}^{n} w_i = 1$. *Then* $f(F_{X_1},\dots,F_{X_n}) = F_{\Phi^{-1}(\sum_{i=1}^{n} w_i \Phi(X_i))}$ *and* $H=\{h:\mathbb{R}^+ \to \mathbb{R}^+ \mid h^{-1}(x) = \Phi^{-1}(a\Phi(x)+b), a,b > 0\}$.

Proof. For any h in H and $x_1,x_2,\dots,x_n \geq 0$ we have:

$$\varepsilon_{h^{-1}(\Phi^{-1}(\sum_{i=1}^{n} w_i\Phi(x_i)))} = \varepsilon_{g(x_1,\dots,x_n)}\circ h = f(\varepsilon_{x_1},\dots,\varepsilon_{x_n})\circ h$$

$$= f(\varepsilon_{x_1}\circ h,\dots,\varepsilon_{x_n}\circ h)$$

$$= f(\varepsilon_{h^{-1}(x_1)},\dots,\varepsilon_{h^{-1}(x_n)})$$

$$= \varepsilon_{g(h^{-1}(x_1),\dots,h^{-1}(x_n))}$$

$$= \varepsilon_{\Phi^{-1}(\sum_{i=1}^{n} w_i \Phi(h^{-1}(x_i)))} \quad ,$$

whence

$$h^{-1}(\Phi^{-1}(\sum_{i=1}^{n} w_i \Phi(x_i))) = \Phi^{-1}(\sum_{i=1}^{n} w_i \Phi(h^{-1}(x_i)))$$

i.e. the function $J = \Phi \circ h^{-1} \circ \Phi^{-1}$ satisfies the generalized Jensen equation:

$$J(\sum_{i=1}^{n} w_i x_i) = \sum_{i=1}^{n} w_i J(x_i).$$

Since J is bounded (see [1], Theorem 2, p.67) J has the form:

$$J(x) = ax + b,$$

where a and b are arbitrary positive constans, $a \neq 0$. Then necessarily $h^{-1}(x) = \Phi^{-1}(a\Phi(x)+b)$ and

$$f(F_{X_1},\dots,F_{X_n}) = F_{\Phi^{-1}(\sum_{i=1}^{n} w_i \Phi(X_i))}.$$

COROLLARY 1. Let f be an (n,g,H)-probabilistic synthesizing function where $g : (\mathbb{R}^+) \to \mathbb{R}^+$ is given by

$$g(x_1,\dots,x_n) = G(x_1)\circ\dots\circ G(x_n),$$

G is a measurable function from $\mathbb{R}^+$ into $\mathbb{R}^+$ into $\mathbb{R}^+$ and $(\mathbb{R}^+, \circ)$ is a topological semigroup representable in the form $x \circ y = \Phi^{-1}(\Phi(x)+\Phi(y))$ for some bijection Φ from $\mathbb{R}^+$ onto itself. Then $H = \{h : \mathbb{R}^+ \to \mathbb{R}^+ | h^{-1}(x) = \Phi^{-1}(a\Phi(x)+b),\ a,b>0\}$ and

$$f(F_{X_1},\ldots,F_{X_n}) = F_{\Phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\Phi(X_i)\right)},$$

Proof. Since f satisfies (i) and (iii) of Definition 2 if we take $F_{X_i} = \varepsilon_x$, $x \geq 0$, for $i=1,\ldots,n$, then

$$\varepsilon_x = f(\varepsilon_x,\ldots,\varepsilon_x) = \varepsilon_{g(x,\ldots,x)} = \varepsilon_{G(x)\circ\ldots\circ G(x)}$$

$$= \varepsilon_{\Phi^{-1}\left(\sum_{i=1}^{n}\Phi(G(x))\right)} = \varepsilon_{\Phi^{-1}(n\,\Phi(G(x)))},$$

whence $G(x) = \Phi^{-1}(\Phi(x)/n)$ and g is the quasiarithmetic mean generated by 0. Then we apply the previous theorem.

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