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Jacobi-Sohncke Type Mixed Modular Equations And Their Applications

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Abstract. In this paper, we establish Jacobi-Sohncke type several new mixed modular equations for composite degrees 1, 3, n and $3n$. As an application, we establish the modular relations between the Ramanujan-Selberg continued fractions $H(q)$, $H(q^3)$, $H(q^n)$ and $H(q^{3n})$ for $n = 2, 3, 4, 5, 7, 9, 11$ and also we obtain congruence relations for color overpartitions of n with odd parts.

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1 Introduction

We start the introduction by defining a modular equation in brief. The complete elliptic integral of first kind is defined as

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \quad (1.1)$$

where $0 < k < 1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}z^n, \quad 0 \le |z| < 1,$$

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$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \quad \text{for} \quad n \text{ a positive integer}$$

and a, b, c are complex numbers such that $c \neq 0, -1, -2, \dots$. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus.

Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l' respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L}, \tag{1.2}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.2). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{1.3}$$

However, if we set

$$q = \exp\left(-\pi K'/K\right), \quad q' = \exp\left(-\pi L'/L\right), \tag{1.4}$$

we see that (1.2) is equivalent to the relation $q^n = q'$. Thus a modular equation can be viewed as an identity involving theta-functions at the arguments q and q^n. The theory of modular equations dates back to 1771 and 1775, when J. Landen records Landen's transformation in his papers [7], [8]. But actually the theory commenced when A. M. Legendre, in his paper derived a modular equation of degree 3 in 1825. C. G. J. Jacobi established modular equations of degree 3 and 5 in his famous book [6]. L. A. Sohncke established modular equations of degrees 7, 11, 13, 17 and 19 in his papers [18], [19]. Subsequently many mathematicians have contributed to the theory of modular equations. Ramanujan's contributions in the area of modular equations are immense. For more details one can see the following papers [9], [10], [11], [12].

Remark 1. The equation involving $\alpha^{1/8}$ and $\beta^{1/8}$ is referred to as Jacobi-Sohncke type modular equation.

Following are the special cases of Ramanujan's general theta function,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.5}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.6}$$

$$f(-q) := f(-q, -q^2) = \sum_{-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q)_\infty, \tag{1.7}$$

and

$$\chi(q) := (-q; q^2)_\infty, \tag{1.8}$$

where

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1,$$

and

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), |\ q\ | < 1.$$

Let K, K', L_1, L'_1, L_2, L'_2, L_3 and L'_3 denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L'_1}{L_1}, \quad n_2 \frac{K'}{K} = \frac{L'_2}{L_2} \quad \text{and} \quad n_3 \frac{K'}{K} = \frac{L'_3}{L_3} \tag{1.9}$$

hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (1.9). We say that β, γ and δ are of degrees n_1, n_2 and n_3, respectively over α. The multipliers $m = K/L_1$ and $m' = L_2/L_3$ are associated with α, β and γ, δ.

In this paper, we establish several new Jacobi-Sohncke type mixed modular equations in Section 3 by using the results enlisted in Section 2. As an application, in Section 4 we obtain several modular relations for Ramanujan-Selberg continued fractions and in Section 5, we discuss the congruence properties of overpartition $p-$tuples of n with odd parts.

2 Preliminary Results

In this section, we collect the results which are useful in proving our main results.

Lemma 2.1. (1) [3, Ch.18, p.215] If β is of degree 2 over α, then

$$\beta = \left(\frac{1 - \sqrt{1-\alpha}}{1 + \sqrt{1-\alpha}} \right)^2. \tag{2.1}$$

(2) [3, Ch.19, Entry 5 (ii), (xiii), pp. 230-231] If β has degree 3 over α, then

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{2.2}$$

$$N - \frac{1}{N} = 2\left(M - \frac{1}{M}\right), \tag{2.3}$$

where $M = (\alpha\beta)^{1/8}$ and $N = (\beta/\alpha)^{1/4}$.

(3) [3, Ch.18, p.215] If β is of degree 4 over α, then

$$\beta = \left(\frac{1-\sqrt[4]{1-\alpha}}{1+\sqrt[4]{1-\alpha}}\right)^4. \tag{2.4}$$

(4) [3, Ch.19, Entry 13 (i), p. 280] If β has degree 5 over α, then

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{2.5}$$

(5) [3, Ch.19, Entry 19 (i), p. 314] If β has degree 7 over α, then

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{2.6}$$

(6) [4, Ch.36, Entry 62, p.387] Let

$$L = 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}, \tag{2.7}$$

$$M = 64\left(\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right), \tag{2.8}$$

and

$$N = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}. \tag{2.9}$$

If β is of degree 9 over α, then

$$L^6 - N(14L^3 + LM) - 3N^2 = 0. \tag{2.10}$$

(7) [3, Ch.20, Entry 7(i), p. 363] If β has degree 11 over α, then

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \tag{2.11}$$

Lemma 2.2. Let q be defined as in (1.4), then

$$\chi(q) = 2^{1/6}(\alpha(1-\alpha)/q)^{-1/24}, \tag{2.12}$$

$$\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}, \tag{2.13}$$

where $\alpha = k^2$, k is called the modulus of K.

Proof. For proofs of (2.12) and (2.13), see [3, Entry 12 (v),(vi), Ch.17, p.124]. QED

Lemma 2.3. [3, Ch.18, Entry 24(v), p.216] If we replace α by $1-\beta$, β by $1-\alpha$, and m by n/m, where n is the degree of the modular equation, we obtain a modular equation of the same degree.

3 Mixed Modular Equations

In this section, we derive several Jacobi-Sohncke type mixed modular equations.

We set

$$P := \{\alpha\beta\gamma\delta\}^{1/16}, \tag{3.1}$$

$$Q := \left\{\frac{\alpha\beta}{\gamma\delta}\right\}^{1/16}, \tag{3.2}$$

$$R := \left\{\frac{\alpha\gamma}{\beta\delta}\right\}^{1/16} \tag{3.3}$$

and

$$T := \left\{\frac{\alpha\delta}{\beta\gamma}\right\}^{1/16}. \tag{3.4}$$

Throughout this section, we use the following notations

$$\mathbb{P}_\kappa := \left(P^\kappa + \frac{1}{P^\kappa}\right), \tag{3.5}$$

$$\mathbb{Q}_\kappa := \left(Q^\kappa + \frac{1}{Q^\kappa}\right), \tag{3.6}$$

$$\mathbb{R}_\kappa := \left(R^\kappa + \frac{1}{R^\kappa}\right) \tag{3.7}$$

and

$$\mathbb{T}_\kappa := \left(T^\kappa + \frac{1}{T^\kappa}\right). \tag{3.8}$$

By rewriting the equation (2.3), we obtain the following lemmas.

Lemma 3.1. If α, β, γ and δ are of degrees 1, 3, n and $3n$ respectively, then

$$\alpha^{1/2} = u^2\left(\frac{-a+s}{2}\right), \tag{3.9}$$

$$\gamma^{1/2} = v^2\left(\frac{-b+t}{2}\right), \tag{3.10}$$

where $a = 2\left(u - \frac{1}{u}\right)$, $b = 2\left(v - \frac{1}{v}\right)$, $u = (\alpha\beta)^{1/8}$, $v = (\gamma\delta)^{1/8}$, $s^2 = a^2 + 4$ and $t^2 = b^2 + 4$.

Lemma 3.2. If α, β, γ and δ are of degrees 1, 3, n and $3n$ respectively, then

$$\alpha^{1/4} = \left(\frac{a+s}{2u}\right), \tag{3.11}$$

$$\gamma^{1/4} = \left(\frac{b+t}{2v}\right), \tag{3.12}$$

where $a = \frac{1}{2}\left(u^2 - \frac{1}{u^2}\right)$, $b = \frac{1}{2}\left(v^2 - \frac{1}{v^2}\right)$, $u = (\beta/\alpha)^{1/8}$, $v = (\delta/\gamma)^{1/8}$, $s^2 = a^2 + 4$ and $t^2 = b^2 + 4$.

Theorem 3.3. If α, β, γ and δ are of degrees 1, 3, 3 and 9 respectively, then

$$\mathbb{Q}_6 + 2\mathbb{Q}_4 - 5\mathbb{Q}_2 + 4\mathbb{P}_2 + 12 = 4\mathbb{P}_2\mathbb{Q}_2, \tag{3.13}$$

$$\mathbb{R}_4 + 2\mathbb{R}_2 + \mathbb{T}_4 = \mathbb{R}_2\mathbb{T}_4 + 4, \tag{3.14}$$

$$T^6 - \frac{1}{T^6} - 3\left(T^2 - \frac{1}{T^2}\right) = 4\left(\frac{T^2}{P^2} - \frac{P^2}{T^2}\right). \tag{3.15}$$

Proof of (3.13). The equation (2.2) is rewritten as

$$(\alpha\gamma)^{1/4} + \{(1-\alpha)(1-\gamma)\}^{1/4} = 1, \tag{3.16}$$

where γ is of degree 3 over α.
Employing the equations (3.9) and (3.10), in the equation (3.16), we obtain,

$$\begin{aligned}
&16uv - 6u^5sv^5t + 6u^5sv^3t + 6u^3sv^5t - 6u^3sv^3t + 24u^6 + 24v^6 - 24u^2v^2 \\
&- 4u^5s - 4v^5t - 24u^6v^6 + 36u^6v^4 - 24u^6v^2 + 36u^4v^6 - 54u^4v^4 + 36u^4v^2 \\
&- 24u^2v^6 + 36u^2v^4 + 12u^6v^5t - 12u^6v^3t - 18u^4v^5t + 18u^4v^3t + 12u^5sv^6 \\
&- 18u^5sv^4 + 12u^5sv^2 + 12u^2v^5t - 12u^2v^3t - 12u^3sv^6 + 18u^3sv^4 - 8u^4 \\
&- 12u^3sv^2 - 8v^4 + 24v^{10} - 33v^8 + 368u^3v^3 - 64u^3v - 64uv^3 + 1296u^5v^5 \\
&- 684u^3v^5 - 32u^3v^2t + 8uv^2t - 32u^2v^3s + 8u^2vs + 4u^2v^2st + 64u^4v^4st \\
&- 12u^4v^2ts - 12u^2v^4st + 64u^8v^8st - 112u^8v^6st - 112u^6v^8st + 196u^6v^6st \\
&- 112u^6v^4st - 112u^4v^6st + 64u^8v^4st + 64u^4v^8st + 21v^6u^2st - 12u^8v^2st \\
&- 12v^8u^2st + 21u^6sv^2t + 108u^5v + 108uv^5 + 1296u^7v^7 - 1296u^7v^5 \\
&- 1296u^5v^7 + 684u^3v^7 - 304u^9v^3 - 304u^3v^9 + 256u^9v^9 - 576u^9v^7 - 24v^8ut \\
&- 576u^7v^9 + 576u^5v^9 + 10v^7t - 10v^9t - 108v^7u - 108u^7v - 10u^9s + 4u^{11}s \\
&+ 48u^9v - 288u^8v^5s - 288u^5v^8t + 224u^6v^9s - 288u^5v^4t + 152u^4v^3s \\
&- 288u^4v^5s - 24u^4vs + 54u^5v^2t - 24uv^4t + 54u^2v^5s - 504u^7v^6t + 10u^7s \\
&+ 504u^5v^6t - 504u^6v^7s + 504u^6v^5s - 266u^6v^3s - 266u^3v^6t + 288u^4v^7s
\end{aligned}$$

$$+152u^8v^3s+152u^3v^8t-128u^9v^8t+224u^9v^6t+288u^7v^8t-8u^{12}-8v^{12}$$
$$-128u^8v^9s+288u^8v^7s-33u^8+24u^{10}-128u^9v^4t-128u^4v^9s+4v^{11}$$
$$+42v^6ut+42u^6vs-24u^8vs-54u^7v^2t+24u^9v^2t+24v^9u^2s+576u^9v^5$$
$$+152u^3v^4t+684u^7v^3-684u^5v^3-54v^7u^2st+288u^7v^4t+48v^9u=0. \tag{3.17}$$

Eliminating s and t in the equation (3.17), we find that

$$(u+v)(u^6-4u^5v^3+2u^5v-5u^4v^2+4u^4v^4+12u^3v^3-4u^3v-4u^3v^5$$
$$-5u^2v^4+4u^2v^2+2v^5u-4v^3u+v^6)(-12288u^4v^4-135936u^6v^6$$
$$+1024u^6v^2+4096u^3v^3+904764u^9v^9+24576u^6v^4+24576u^4v^6$$
$$+1024u^2v^6-602040u^9v^7-602040u^7v^9+538816u^7v^7+173856u^9v^5$$
$$-205440u^7v^5-24384u^9v^3+23808u^7v^3+173856u^5v^9-205440u^5v^7$$
$$+105472u^5v^5-15360u^5v^3-24384u^3v^9+23808u^3v^7-15360u^3v^5$$
$$+256u^9v+256v^9u-2048u^8v^4-2048u^4v^8-54792u^{10}v^6+3456u^{10}v^2$$
$$+173856u^{13}v^9-3072u^8v^2-21024u^{10}v^4+183696u^8v^6-87752u^{13}v^7$$
$$-602040u^{11}v^9+311760u^{11}v^7+40056u^{13}v^5-87752u^{11}v^5-11352u^{13}v^3$$
$$+21312u^{11}v^3+576u^{13}v-576u^{11}v+11760u^4v^{12}-21024u^4v^{10}$$
$$-54792v^{10}u^6+173856v^{13}u^9-3072v^8u^2+183696v^8u^6+3456v^{10}u^2$$
$$-602040v^{11}u^9-87752v^{13}u^7+311760v^{11}u^7+40056v^{13}u^5-87752v^{11}u^5$$
$$-11352v^{13}u^3+21312v^{11}u^3+576v^{13}u-576v^{11}u+11760v^4u^{12}$$
$$-439008u^8v^8+538816u^{11}v^{11}+1416u^{14}v^2-2672u^{12}v^2+1416u^2v^{14}$$
$$-2672u^2v^{12}-439008u^{10}v^{10}+105472u^{13}v^{13}+332806u^{10}v^8+332806u^8v^{10}$$
$$-205440u^{13}v^{11}-205440u^{11}v^{13}+1416u^{16}v^4-2372u^{14}v^4+1416u^4v^{16}$$
$$-2372u^4v^{14}-27684u^{12}v^6-24384u^{15}v^9+21312u^{15}v^7-11352u^{15}v^5$$
$$+2160u^{15}v^3+u^{18}+v^{18}-296u^{15}v-27684u^6v^{12}-24384v^{15}u^9-159u^2v^{16}$$
$$+21312v^{15}u^7-11352v^{15}u^5+2160v^{15}u^3-296v^{15}u+11760u^{14}v^6$$
$$+256u^{17}v^9-576u^{17}v^7+576u^{17}v^5-296u^{17}v^3+42u^{17}v-54792u^8v^{12}$$
$$+11760v^{14}u^6+256v^{17}u^9-576v^{17}u^7+576v^{17}u^5-296v^{17}u^3+42v^{17}u$$
$$-54792v^8u^{12}-135936u^{12}v^{12}+4096u^{15}v^{15}+183696u^{12}v^{10}+183696u^{10}v^{12}$$
$$-15360u^{15}v^{13}-15360u^{13}v^{15}+23808u^{15}v^{11}+23808u^{11}v^{15}-2672u^{16}v^6$$
$$-2672u^6v^{16}-159u^{16}v^2+1024u^{16}v^{12}-12288u^{14}v^{14}+24576u^{14}v^{12}$$
$$+1024u^{12}v^{16}+24576u^{12}v^{14}-3072u^{16}v^{10}-2048u^{14}v^{10}-3072u^{10}v^{16}$$

$$- 2048u^{10}v^{14} + 3456u^{16}v^8 - 21024u^{14}v^8 + 3456u^8v^{16} - 21024u^8v^{14}) = 0. \tag{3.18}$$

By examining the behavior of the factors of the equation (3.18) near $q = 0$, we can find a neighbourhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighbourhood. By the Identity Theorem second factor vanishes identically.

Setting $u = PQ$ and $v = P/Q$, in the second factor, we arrive at the required equation (3.13). QED

Proof of (3.14). Employing the equations (3.11) and (3.12), in the equation (3.16), we obtain,

$$\begin{aligned}
&- 4su^2v^{12} + 20tu^{12}v^{10} - 20tu^{12}v^6 - 4tu^{12}v^2 + 6u^6v^6 + 36u^6v^{10} + 36u^{10}v^6 \\
&+ 216u^{10}v^{10} + 20su^{10}v^{12} - 20su^6v^{12} - u^3v^3 + 2v^{12} + 12u^8v^{12} + 24u^4v^{12} \\
&+ 2u^{12} + 24u^{12}v^4 + 12u^{12}v^8 + 48u^{12}v^{12} + 2u^{16}v^{12} + 2v^{16}u^{12} + 4u^{14}sv^{12} \\
&+ 4v^{14}tu^{12} - 4stu^{13}v^{13} - 8stu^{13}v^9 - 4stu^{13}v^5 + 24stu^{12}v^{12} - 24stu^{12}v^8 \\
&- 64stu^{11}v^{11} - 8stu^9v^{13} - 16stu^9v^9 - 8stu^9v^5 - 24stu^8v^{12} + 24stu^8v^8 \\
&- 4stu^5v^{13} - 8stu^5v^9 - 4stu^5v^5 - 2su^{13}v^{15} - 18su^{13}v^{11} + 18su^{13}v^7 \\
&+ 12su^{12}v^{14} + 72su^{12}v^{10} + 12su^{12}v^6 - 32su^{11}v^{13} + 32su^{11}v^9 - 4su^9v^{15} \\
&- 36su^9v^{11} + 36su^9v^7 + 4su^9v^3 - 12su^8v^{14} - 72su^8v^{10} - 12su^8v^6 - 2su^5v^{15} \\
&- 18su^5v^{11} + 18su^5v^7 + 2su^5v^3 - 2tu^{15}v^{13} - 4tu^{15}v^9 - 2tu^{15}v^5 + 12tu^{14}v^{12} \\
&- 12tu^{14}v^8 - 32tu^{13}v^{11} - 18tu^{11}v^{13} - 36tu^{11}v^9 - 18tu^{11}v^5 + 72tu^{10}v^{12} \\
&- 72tu^{10}v^8 + 32tu^9v^{11} + 18tu^7v^{13} + 36tu^7v^9 + 18tu^7v^5 + 12tu^6v^{12} - 12tu^6v^8 \\
&+ 2tu^3v^{13} + 4tu^3v^9 + 2tu^3v^5 - u^{15}v^{15} - 9u^{15}v^{11} + 9u^{15}v^7 + u^{15}v^3 \\
&+ 36u^{14}v^{10} + 6u^{14}v^6 - 16u^{13}v^{13} + 16u^{13}v^9 - 9u^{11}v^{15} - 81u^{11}v^{11} + 81u^{11}v^7 \\
&+ 9u^{11}v^3 + 36u^{10}v^{14} + 16u^9v^{13} - 16u^9v^9 + 9u^7v^{15} + 81u^7v^{11} - 81u^7v^7 \\
&- 9u^7v^3 + 6u^6v^{14} + u^3v^{15} + 2su^{13}v^3 + 6u^{14}v^{14} + 9u^3v^{11} - 9u^3v^7 = 0.
\end{aligned} \tag{3.19}$$

Eliminating s and t in the equation (3.19), we find that

$$\begin{aligned}
&(uv + 1)(u^6v^2 - v^5u^5 - u^5v - 2u^4v^4 + u^4 + 4u^3v^3 + u^2v^6 - 2u^2v^2 + v^4 \\
&- uv - uv^5)(u^2v^2 - 670u^9v^9 + u^8 + v^8 + u^5v + uv^5 + 54u^2v^{10} - 6u^6v^2 \\
&+ 913u^6v^6 - 1040u^6v^{10} - 114u^8v^4 + 1113u^8v^8 - 114u^4v^8 - 6u^2v^6 + 85u^4v^4 \\
&- 12u^3v^3 - 1040u^8v^{12} + 105u^4v^{12} - 56v^{16}u^4 + 54v^{16}u^8 + 414u^9v^{13} \\
&+ 1350u^7v^{11} - 1356u^7v^7 + 34u^7v^3 + 105u^6v^{14} + 102u^3v^{15} - 140u^3v^{11} \\
&+ 34u^3v^7 - 9u^9v^{17} + 9u^5v^{17} - 26u^5v^{13} + 414u^5v^9 - 342v^5u^5 + uv^{17} - 56u^2v^{14}
\end{aligned}$$

$$+ 9uv^{13} - 9uv^9 + u^{17}v^{13} - 9u^{17}v^9 + u^{16}v^{16} - 6u^{16}v^{12} + 54u^{16}v^8 - 56u^{16}v^4$$
$$- 12u^{15}v^{15} + 34u^{15}v^{11} - 140u^{15}v^7 + 102u^{15}v^3 + 85u^{14}v^{14} - 114u^{14}v^{10}$$
$$+ 105u^{14}v^6 - 56u^{14}v^2 + u^{13}v^{17} - 342u^{13}v^{13} + 414u^{13}v^9 - 26u^{13}v^5 + 9u^{13}v$$
$$- 6u^{12}v^{16} + 913u^{12}v^{12} - 1040u^{12}v^8 + 105u^{12}v^4 + 34u^{11}v^{15} - 1356u^{11}v^{11}$$
$$+ 1350u^{11}v^7 + u^{10}v^{18} - 114u^{10}v^{14} + 1113u^{10}v^{10} - 1040u^{10}v^6 + 54u^{10}v^2$$
$$+ 414u^9v^5 - 9u^9v + u^{18}v^{10} + 9u^{17}v^5 + u^{17}v - 140u^7v^{15} - 140u^{11}v^3) = 0. \tag{3.20}$$

By examining the behavior of the factors of the equation (3.20) near $q = 0$, we can find a neighbourhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighbourhood. By the Identity Theorem second factor vanishes identically. Setting $u = \frac{1}{RT}$ and $v = \frac{T}{R}$, in the second factor, we arrive at the required equation (3.14). $\boxed{QED}$

Proof of (3.15). Interchanging β and γ in the equation (3.13), we obtain

$$\mathbb{R}_6 + 2\mathbb{R}_4 - 5\mathbb{R}_2 + 4\mathbb{P}_2 + 12 = 4\mathbb{P}_2\mathbb{R}_2. \tag{3.21}$$

Solving the equation (3.14) for $\mathbb{R}$ and using in the above equation (3.21), we arrive at (3.15). $\boxed{QED}$

Theorem 3.4. If α, β, γ and δ are of degrees 1, 3, 2 and 6 respectively, then

$$P^4 + Q^4 - 2\left[P^2Q^2 + \frac{2}{P^2Q^2}\right] + 4 = 0, \tag{3.22}$$

$$\left[\frac{T^2}{R^2} + \frac{R^2}{T^2}\right] = 2 + \left[\frac{T}{R} + \frac{R}{T}\right]\left[T^2R^2 - \frac{1}{T^2R^2}\right]. \tag{3.23}$$

Proofs of the equations (3.22) and (3.23) are similar to the proof of equations (3.13) and (3.14) except that in place of the equation (2.2), we use the equation (2.1).

Theorem 3.5. If α, β, γ and δ are of degrees 1, 3, 4 and 12 respectively, then

$$P^8 + Q^8 + 16\left[\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right]\left[3P^2Q^2 + \frac{8}{P^2Q^2}\right] - 4\left[P^4Q^4 + \frac{16}{P^4Q^4}\right]\left[\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right]$$
$$+ 6P^4Q^4 + \frac{128}{P^4Q^4} - 32\left[P^2Q^2 + \frac{8}{P^2Q^2}\right] - 112\left[\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right] + 160 = 0, \tag{3.24}$$

$$\frac{R^4}{T^4}+\frac{T^4}{R^4}+4\left[\frac{R^2}{T^2}+\frac{T^2}{R^2}\right]-8\left[R^4T^4+\frac{1}{R^4T^4}\right]-7\left[\frac{R}{T}+\frac{T}{R}\right]\left[R^4T^4-\frac{1}{R^4T^4}\right]$$
$$-4\left[R^4T^4+\frac{1}{R^4T^4}\right]\left[\frac{T^2}{R^2}+\frac{R^2}{T^2}\right]+\left[\frac{T^3}{R^3}+\frac{R^3}{T^3}\right]\left[\frac{1}{R^4T^4}-R^4T^4\right]+22=0. \tag{3.25}$$

Proofs of the equations (3.24) and (3.25) are similar to the proof of equation (3.13) except that in place of the equation (2.2), we use the equation (2.4).

Theorem 3.6. If α, β, γ and δ are of degrees 1, 3, 5 and 15 respectively, then

$$\mathbb{Q}_6-10\mathbb{Q}_4-5\mathbb{Q}_2-60=16\mathbb{P}_4-20\mathbb{P}_2-20\mathbb{P}_2\mathbb{Q}_2, \tag{3.26}$$

$$\mathbb{R}_4+20=\mathbb{T}_6-5\mathbb{T}_4+15\mathbb{T}_2, \tag{3.27}$$

$$\mathbb{T}_{18}-5\left[3\mathbb{T}_{16}-24\mathbb{T}_{14}+149\mathbb{T}_{12}-709\mathbb{T}_{10}+2638\mathbb{T}_8-7343\mathbb{T}_6+15403\mathbb{T}_4\right.$$
$$\left.-23719\mathbb{T}_2+28094\right]=256\mathbb{P}_8-640\mathbb{P}_6+80\mathbb{P}_4\left(39-15\mathbb{T}_2+50\mathbb{T}_4-\mathbb{T}_6\right)$$
$$-40\mathbb{P}_2\left(846-670\mathbb{T}_2+425\mathbb{T}_4-184\mathbb{T}_6+55\mathbb{T}_8-10\mathbb{T}_{10}+\mathbb{T}_{12}\right). \tag{3.28}$$

Proofs of the equations (3.26), (3.27) and (3.28) are similar to the proof of equation (3.13) except that in place of the equation (2.2), we use the equation (2.5).

Theorem 3.7. If α, β, γ and δ are of degrees 1, 3, 7 and 21 respectively, then

$$\mathbb{Q}_8+28\mathbb{Q}_6+112\mathbb{Q}_4+364\mathbb{Q}_2-64\mathbb{P}_6+224\mathbb{P}_4-448\mathbb{P}_2$$
$$-308\mathbb{P}_2\mathbb{Q}_2+112\mathbb{P}_4\mathbb{Q}_2-84\mathbb{P}_2\mathbb{Q}_4+686=0, \tag{3.29}$$

$$\mathbb{R}_6-14\mathbb{R}_4+49\mathbb{R}_2-\mathbb{T}_8-14\mathbb{T}_4+7\mathbb{R}_2\mathbb{T}_4-70=0. \tag{3.30}$$

Proofs of the equations (3.29) and (3.30) are similar to the proof of equation (3.13) except that in place of the equation (2.2), we use the equation (2.6).

Theorem 3.8. If α, β, γ and δ are of degrees 1, 3, 9 and 27 respectively, then

$$\mathbb{Q}_{18}-42\mathbb{Q}_{16}-159\mathbb{Q}_{14}-2160\mathbb{Q}_{12}-2372\mathbb{Q}_{10}-40056\mathbb{Q}_8-27684\mathbb{Q}_6$$
$$-311760\mathbb{Q}_4-332806\mathbb{Q}_2-904764=2^{12}\mathbb{P}_{12}+2^{10}\mathbb{P}_{10}\left(12-15\mathbb{Q}_2\right.$$
$$\left.-\mathbb{Q}_4\right)+2^8\mathbb{P}_8\left(412-96\mathbb{Q}_2+93\mathbb{Q}_4+12\mathbb{Q}_6+\mathbb{Q}_8\right)+2^6\mathbb{P}_6\left(2124-9\mathbb{Q}_{10}\right.$$
$$\left.-54\mathbb{Q}_8-381\mathbb{Q}_6+2^5\mathbb{Q}_4-3210\mathbb{Q}_2\right)+2^4\mathbb{P}_4\left(33676+36\mathbb{Q}_{12}+167\mathbb{Q}_{10}\right.$$
$$\left.+1332\mathbb{Q}_8+1314\mathbb{Q}_6+10866\mathbb{Q}_4-11481\mathbb{Q}_2\right)+2^3\mathbb{P}_2\left(54876-37\mathbb{Q}_{14}\right.$$
$$\left.-177\mathbb{Q}_{12}-1419\mathbb{Q}_{10}-1470\mathbb{Q}_8-10969\mathbb{Q}_6+6849\mathbb{Q}_4-75255\mathbb{Q}_2\right), \tag{3.31}$$

$$\begin{aligned}&\mathbb{T}_{14}+12\mathbb{T}_{12}+85\mathbb{T}_{10}+342\mathbb{T}_8+913\mathbb{T}_6+1356\mathbb{T}_4+1113\mathbb{T}_2+670=\mathbb{R}_{16}\\&+\mathbb{R}_{12}\left(102+56\mathbb{T}_2+9\mathbb{T}_4\right)-\mathbb{R}_8\left(\mathbb{T}_{10}+9\mathbb{T}_8+54\mathbb{T}_6+140\mathbb{T}_4+105\mathbb{T}_2+26\right)\\&+\mathbb{R}_4\left(\mathbb{T}_{12}+6\mathbb{T}_{10}+34\mathbb{T}_8+114\mathbb{T}_6+414\mathbb{T}_4+1040\mathbb{T}_2+1350\right).\end{aligned} \tag{3.32}$$

Proofs of the equations (3.31) and (3.32) are similar to the proof of equation (3.13) except that in place of the equation (2.2), we use the equation (2.10).

Theorem 3.9. If α, β, γ and δ are of degrees 1, 3, 11 and 33 respectively, then

$$\begin{aligned}&\mathbb{Q}_{12}+88\mathbb{Q}_{10}+374\mathbb{Q}_8+3696\mathbb{Q}_6+4015\mathbb{Q}_4+17336\mathbb{Q}_2+12980=2^{10}\mathbb{P}_{10}\\&+5632\mathbb{P}_8+9856\mathbb{P}_6+23936\mathbb{P}_4+21032\mathbb{P}_2-4532\mathbb{P}_2\mathbb{Q}_2+10560\mathbb{P}_4\mathbb{Q}_4\\&-2816\mathbb{P}_8\mathbb{Q}_2-14432\mathbb{P}_4\mathbb{Q}_2-14080\mathbb{P}_6\mathbb{Q}_2+3520\mathbb{P}_6\mathbb{Q}_4+836\mathbb{P}_2\mathbb{Q}_8\\&-2464\mathbb{P}_4\mathbb{Q}_6+10472\mathbb{P}_2\mathbb{Q}_4-1804\mathbb{P}_2\mathbb{Q}_6,\end{aligned} \tag{3.33}$$

$$\begin{aligned}&\mathbb{R}_{10}+11\left(2\mathbb{R}_8+5\mathbb{R}_6-20\mathbb{R}_4+56\mathbb{R}_2-64\right)=\mathbb{T}_{12}-11\left(2\mathbb{T}_8+25\mathbb{T}_4\right.\\&\left.+10\mathbb{T}_4\mathbb{R}_4-\mathbb{T}_4\mathbb{R}_6-15\mathbb{T}_4\mathbb{R}_2\right).\end{aligned} \tag{3.34}$$

Proofs of the equations (3.33) and (3.34) are similar to the proof of equation (3.13) except that in place of the equation (2.2), we use the equation (2.11).

4 Modular identities for Ramanujan-Selberg continued fraction

The continued fraction identity

$$\begin{aligned}H(q) &:= \frac{q^{\frac{1}{8}}}{1+}\,\frac{q}{1+}\,\frac{q^2+q}{1+}\,\frac{q^3}{1+}\,\frac{q^4+q^2}{1+\cdots}\\&= \frac{q^{\frac{1}{8}}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}, \qquad |q|<1,\end{aligned} \tag{4.1}$$

appears as Formula 5 [16, p.290] and was first proved by Selberg [1, eq.(54)]. Other proofs have been given by Ramanathan [15] and Andrews, Berndt, Jacobsen and Lamphere [2]. C. Adiga, M. S. Mahadeva Naika and Ramya Rao [1] have obtained two integral representations for $H(q)$, also derived a relation between $H(q)$ and $H(q^n)$ and some explicit evaluations of $H(q)$. Mahadeva Naika et al. [13],[14] have obtained several integral representations and also Rogers-Ramanujan type functions for $H(q)$.

In this section, we establish modular relations between Ramanujan-Selberg continued fractions $H(q)$, $H(q^3)$, $H(q^n)$ and $H(q^{3n})$ for $n = 2, 3, 4, 5, 7, 9, 11$. We define

$$w := 4H(q)H(q^3)H(q^n)H(q^{3n}), \tag{4.2}$$

$$x := \frac{H(q)H(q^3)}{H(q^n)H(q^{3n})}, \tag{4.3}$$

$$y := \frac{H(q)H(q^n)}{H(q^3)H(q^{3n})}, \tag{4.4}$$

$$z := \frac{H(q)H(q^{3n})}{H(q^n)H(q^3)}. \tag{4.5}$$

Throughout this section, we use the following notations

$$\mathbb{W}_\kappa := \left(w^\kappa + \frac{1}{w^\kappa}\right), \tag{4.6}$$

$$\mathbb{X}_\kappa := \left(x^\kappa + \frac{1}{x^\kappa}\right), \tag{4.7}$$

$$\mathbb{Y}_\kappa := \left(y^\kappa + \frac{1}{y^\kappa}\right), \tag{4.8}$$

$$\mathbb{Z}_\kappa := \left(z^\kappa + \frac{1}{z^\kappa}\right). \tag{4.9}$$

Lemma 4.1 ([13], Th.3.2)**.** We have

$$H(q) = \frac{\alpha^{1/8}}{\sqrt{2}},$$

where q is as defined in (1.4) and $\alpha = k^2$, k is called the modulus of K.

Using the above lemma in the corresponding Jacobi-Sohncke type mixed modular equations obtained in the previous section, we deduce the following theorems.

Theorem 4.2. If α, β, γ and δ are of degrees 1, 3, 2 and 6 respectively, then

$$w^2 + x^2 - 2\left(wx + \frac{2}{wx}\right) + 4 = 0, \tag{4.10}$$

$$\left(\frac{y}{z} + \frac{z}{y}\right) = 2 + \left(\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}}\right)\left(yz - \frac{1}{yz}\right). \tag{4.11}$$

Theorem 4.3. If α, β, γ and δ are of degrees 1, 3, 3 and 9 respectively, then

$$\mathbb{X}_3 + 2\mathbb{X}_2 - 5\mathbb{X}_1 + 4\mathbb{W}_1 + 12 = 4\mathbb{W}_1\mathbb{X}_1, \tag{4.12}$$

$$\mathbb{Y}_2 + 2\mathbb{Y}_1 + \mathbb{Z}_2 = \mathbb{Y}_1\mathbb{Z}_2 + 4, \tag{4.13}$$

$$z^3 - \frac{1}{z^3} - 3\left(z - \frac{1}{z}\right) = 4\left(\frac{z}{w} - \frac{w}{z}\right). \tag{4.14}$$

Theorem 4.4. If α, β, γ and δ are of degrees 1, 3, 4 and 12 respectively, then

$$\begin{aligned} &w^4 + x^4 + 16\left(\frac{w}{x} + \frac{x}{w}\right)\left(3wx + \frac{8}{wx}\right) - 4\left(w^2x^2 + \frac{16}{w^2x^2}\right)\left(\frac{w}{x} + \frac{x}{w}\right) \\ &+ 6w^2x^2 + \frac{128}{w^2x^2} - 32\left(wx + \frac{8}{wx}\right) - 112\left(\frac{w}{x} + \frac{x}{w}\right) + 160 = 0, \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\frac{y^2}{z^2} + \frac{z^2}{y^2} + 4\left(\frac{y}{z} + \frac{z}{y}\right) - 8\left(y^2z^2 + \frac{1}{y^2z^2}\right) - 7\left(\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}}\right)\left(y^2z^2 - \frac{1}{y^2z^2}\right) \\ &- 4\left(y^2z^2 + \frac{1}{y^2z^2}\right)\left(\frac{y}{z} + \frac{z}{y}\right) + \left(\sqrt{\frac{y^3}{z^3}} + \sqrt{\frac{z^3}{y^3}}\right)\left(\frac{1}{y^2z^2} - y^2z^2\right) + 22 = 0. \end{aligned} \tag{4.16}$$

Theorem 4.5. If α, β, γ and δ are of degrees 1, 3, 5 and 15 respectively, then

$$\mathbb{X}_3 - 10\mathbb{X}_2 - 5\mathbb{X}_1 - 60 = 16\mathbb{W}_2 - 20\mathbb{W}_1 - 20\mathbb{W}_1\mathbb{X}_1, \tag{4.17}$$

$$\mathbb{Y}_2 + 20 = \mathbb{Z}_3 - 5\mathbb{Z}_2 + 15\mathbb{Z}_1, \tag{4.18}$$

$$\begin{aligned} &\mathbb{Z}_9 - 5\left[3\mathbb{Z}_8 - 24\mathbb{Z}_7 + 149\mathbb{Z}_6 - 709\mathbb{Z}_5 + 2638\mathbb{Z}_4 - 7343\mathbb{Z}_3 + 15403\mathbb{Z}_2\right. \\ &\left.-23719\mathbb{Z}_1 + 28094\right] = 256\mathbb{W}_4 - 640\mathbb{W}_3 + 80\mathbb{W}_2\left(39 - 15\mathbb{Z}_1 + 50\mathbb{Z}_2 - \mathbb{Z}_3\right) \\ &- 40\mathbb{W}_1\left(846 - 670\mathbb{Z}_1 + 425\mathbb{Z}_2 - 184\mathbb{Z}_3 + 55\mathbb{Z}_4 - 10\mathbb{Z}_5 + \mathbb{Z}_6\right). \end{aligned} \tag{4.19}$$

Theorem 4.6. If α, β, γ and δ are of degrees 1, 3, 7 and 21 respectively, then

$$\begin{aligned} &\mathbb{X}_4 + 28\mathbb{X}_3 + 112\mathbb{X}_2 + 364\mathbb{X}_1 - 64\mathbb{W}_3 + 224\mathbb{W}_2 - 448\mathbb{W}_1 \\ &- 308\mathbb{W}_1\mathbb{X}_1 + 112\mathbb{W}_2\mathbb{X}_1 - 84\mathbb{W}_1\mathbb{X}_2 + 686 = 0, \end{aligned} \tag{4.20}$$

$$\mathbb{Y}_3 - 14\mathbb{Y}_2 + 49\mathbb{Y}_1 - \mathbb{Z}_4 - 14\mathbb{Z}_2 + 7\mathbb{Y}_1\mathbb{Z}_2 - 70 = 0. \tag{4.21}$$

Theorem 4.7. If α, β, γ and δ are of degrees 1, 3, 9 and 27 respectively, then

$$\begin{aligned}
&\mathbb{X}_9 - 42\mathbb{X}_8 - 159\mathbb{X}_7 - 2160\mathbb{X}_6 - 2372\mathbb{X}_5 - 40056\mathbb{X}_4 - 27684\mathbb{X}_3 \\
&- 311760\mathbb{X}_2 - 332806\mathbb{X}_1 - 904764 = 2^{12}\mathbb{W}_6 + 2^{10}\mathbb{W}_5\left(12 - 15\mathbb{X}_1 - \mathbb{X}_2\right) \\
&+ 2^8\mathbb{W}_4\left(412 - 96\mathbb{X}_1 + 93\mathbb{X}_2 + 12\mathbb{X}_3 + \mathbb{X}_4\right) + 2^6\mathbb{W}_3\left(2124 - 9\mathbb{X}_5 - 54\mathbb{X}_4\right. \\
&\left.-381\mathbb{X}_3 + 2^5\mathbb{X}_2 - 3210\mathbb{X}_1\right) + 2^4\mathbb{W}_2\left(33676 + 36\mathbb{X}_6 + 167\mathbb{X}_5 + 1332\mathbb{X}_4\right. \\
&\left.+1314\mathbb{X}_3 + 10866\mathbb{X}_2 - 11481\mathbb{X}_1\right) + 2^3\mathbb{W}_1\left(54876 - 37\mathbb{X}_7 - 177\mathbb{X}_6\right. \\
&\left.-1419\mathbb{X}_5 - 1470\mathbb{X}_4 - 10969\mathbb{X}_3 + 6849\mathbb{X}_2 - 75255\mathbb{X}_1\right),
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&\mathbb{Z}_7 + 12\mathbb{Z}_6 + 85\mathbb{Z}_5 + 342\mathbb{Z}_4 + 913\mathbb{Z}_3 + 1356\mathbb{Z}_2 + 1113\mathbb{Z}_1 + 670 = \mathbb{Y}_8 \\
&+ \mathbb{Y}_6\left(102 + 56\mathbb{Z}_1 + 9\mathbb{Z}_2\right) - \mathbb{Y}_4\left(\mathbb{Z}_5 + 9\mathbb{Z}_4 + 54\mathbb{Z}_3 + 140\mathbb{Z}_2 + 105\mathbb{Z}_1 + 26\right) \\
&+ \mathbb{Y}_2\left(\mathbb{Z}_6 + 6\mathbb{Z}_5 + 34\mathbb{Z}_4 + 114\mathbb{Z}_3 + 414\mathbb{Z}_2 + 1040\mathbb{Z}_1 + 1350\right).
\end{aligned} \tag{4.23}$$

Theorem 4.8. If α, β, γ and δ are of degrees 1, 3, 11 and 33 respectively, then

$$\begin{aligned}
&\mathbb{X}_6 + 88\mathbb{X}_5 + 374\mathbb{X}_4 + 3696\mathbb{X}_3 + 4015\mathbb{X}_2 + 17336\mathbb{X}_1 + 12980 = 2^{10}\mathbb{W}_5 \\
&+ 5632\mathbb{W}_4 + 9856\mathbb{W}_3 + 23936\mathbb{W}_2 + 21032\mathbb{W}_1 - 4532\mathbb{W}_1\mathbb{X}_1 + 10560\mathbb{W}_2\mathbb{X}_2 \\
&- 2816\mathbb{W}_4\mathbb{X}_1 - 14432\mathbb{W}_2\mathbb{X}_1 - 14080\mathbb{W}_3\mathbb{X}_1 + 3520\mathbb{W}_3\mathbb{X}_2 + 836\mathbb{W}_1\mathbb{X}_4 \\
&- 2464\mathbb{W}_2\mathbb{X}_3 + 10472\mathbb{W}_1\mathbb{X}_2 - 1804\mathbb{W}_1\mathbb{X}_3,
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&\mathbb{Y}_5 + 11\left(2\mathbb{Y}_4 + 5\mathbb{Y}_3 - 20\mathbb{Y}_2 + 56\mathbb{Y}_1 - 64\right) = \mathbb{Z}_6 - 11\left(2\mathbb{Z}_4 + 25\mathbb{Z}_2\right. \\
&\left.+10\mathbb{Z}_2\mathbb{Y}_2 - \mathbb{Z}_2\mathbb{Y}_3 - 15\mathbb{Z}_2\mathbb{Y}_1\right).
\end{aligned} \tag{4.25}$$

5 Congruence relations for overpartition with odd parts

An overpartition of n is a non increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of an integer n. For convenience, we set $\overline{p}(0) = 1$. For example, there are four overpartitions of 2 $\quad 2, \quad \overline{2}, \quad 1+1,$ $\overline{1}+1$.

S. Corteel and J. Lovejoy [5] provided the generating function $\overline{p}(n)$ as

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{5.1}$$

Similarly, let $\mathbb{P}(n)$ be the number of overpartitions of n in which only odd parts are considered.

$$\sum_{n=0}^{\infty} \mathbb{P}(n)q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}. \tag{5.2}$$

Now from (2.12) and (2.13), we can write

$$(1-\alpha)^{-1/8} = \frac{\chi(q)}{\chi(-q)} = \sum_{n=0}^{\infty} \mathbb{P}(n)q^n. \tag{5.3}$$

By Lemma 2.3, equations (3.1), (3.2), (3.3) and (3.4) reduces to

$$A := \{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/16}, \tag{5.4}$$

$$B := \left\{\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right\}^{1/16}, \tag{5.5}$$

$$C := \left\{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right\}^{1/16}, \tag{5.6}$$

$$D := \left\{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right\}^{1/16}. \tag{5.7}$$

Let l, m be positive integers with $l \neq m$, we define the following :

$$\sum_{n=0}^{\infty} \mathbb{P}^p_{l,m}(n)q^n = \frac{\chi^p(q)\chi^p(q^l)\chi^p(q^m)\chi^p(q^{lm})}{\chi^p(-q)\chi^p(-q^l)\chi^p(-q^m)\chi^p(-q^{lm})}, \tag{5.8}$$

where $\mathbb{P}^p_{l,m}(n)$ denotes the number of overpartitions of n with odd parts in $4p$ colors in which, p colors appears in only multiples of l, another p colors appears in multiples of m and remaining p colors appears only in multiples of lm.

Let

$$\sum_{n=0}^{\infty} \mathbb{Q}^p_{l,m}(n)q^n = \frac{\chi^p(q)\chi^p(q^l)\chi^p(-q^m)\chi^p(-q^{lm})}{\chi^p(-q)\chi^p(-q^l)\chi^p(q^m)\chi^p(q^{lm})}, \tag{5.9}$$

where $\mathbb{Q}^p_{l,m}(n)$ denotes the number of overpartitions of n in $2p$ colors in which p colors appears only in odd parts that are not multiples of m, and another p colors appears only in odd parts that are multiples of l but are not multiples of lm.

Let

$$\sum_{n=0}^{\infty} \mathbb{R}^p_{l,m}(n)q^n = \frac{\chi^p(q)\chi^p(-q^l)\chi^p(q^m)\chi^p(-q^{lm})}{\chi^p(-q)\chi^p(q^l)\chi^p(-q^m)\chi^p(q^{lm})}, \tag{5.10}$$

where $\mathbb{R}^p_{l,m}(n)$ denotes the number of overpartitions of n in $2p$ colors in which p colors appears only in odd parts that are not multiples of l, and another p colors appears only in odd parts that are multiples of m but are not multiples of lm.

For l, m relatively prime, let $\mathbb{T}^p_{l,m}(n)$ denotes the number of overpartitions of n into odd parts with p colors that are not multiples of l or m. The generating function for $\mathbb{T}^p_{l,m}(n)$ satisfies,

$$\sum_{n=0}^{\infty} \mathbb{T}^p_{l,m}(n)q^n = \frac{\chi^p(q)\chi^p(-q^l)\chi^p(-q^m)\chi^p(q^{lm})}{\chi^p(-q)\chi^p(q^l)\chi^p(q^m)\chi^p(-q^{lm})}. \tag{5.11}$$

Theorem 5.1. We have

$$\mathbb{Q}^3_{3,5}(2n) \equiv \mathbb{P}^2_{3,5}(2n) \pmod 5, \tag{5.12}$$

$$\mathbb{R}^2_{3,5}(2n) \equiv \mathbb{T}^3_{3,5}(2n) \pmod 5, \tag{5.13}$$

$$\mathbb{T}^9_{3,5}(2n) \equiv \mathbb{P}^4_{3,5}(2n) \pmod 5. \tag{5.14}$$

Proof. Using Lemma (2.3) in equation (3.26), we get

$$\left(B^6 + \frac{1}{B^6}\right) \equiv \left(A^4 + \frac{1}{A^4}\right) \pmod 5. \tag{5.15}$$

Employing (5.3) in the equations (5.4) and (5.5), the above equation becomes

$$\frac{\chi^3(q)\chi^3(q^3)\chi^3(-q^5)\chi^3(-q^{15})}{\chi^3(-q)\chi^3(-q^3)\chi^3(q^5)\chi^3(q^{15})} + \frac{\chi^3(-q)\chi^3(-q^3)\chi^3(q^5)\chi^3(q^{15})}{\chi^3(q)\chi^3(q^3)\chi^3(-q^5)\chi^3(-q^{15})}$$
$$\equiv \frac{\chi^2(q)\chi^2(q^3)\chi^2(q^5)\chi^2(q^{15})}{\chi^2(-q)\chi^2(-q^3)\chi^2(-q^5)\chi^2(-q^{15})} + \frac{\chi^2(-q)\chi^2(-q^3)\chi^2(-q^5)\chi^2(-q^{15})}{\chi^2(q)\chi^2(q^3)\chi^2(q^5)\chi^2(q^{15})} \pmod 5. \tag{5.16}$$

With the aid of (5.8) and (5.9), we arrive at (5.12). QED

Proofs of the equations (5.13) and (5.14) are similar to the proof of equation (5.12) and except that in place of the equation (3.26), we use the equations (3.27) and (3.28).

Theorem 5.2. We have

$$\mathbb{Q}^4_{3,7}(2n) \equiv \mathbb{P}^3_{3,7}(2n) \pmod 7, \tag{5.17}$$

$$\mathbb{R}^3_{3,7}(2n) \equiv \mathbb{T}^4_{3,7}(2n) \pmod 7. \tag{5.18}$$

Proofs of the equations (5.17) and (5.18) are similar to the proof of equation (5.12) and except that in place of the equation (3.26), we use the equations (3.29) and (3.30).

Theorem 5.3. We have

$$\mathbb{Q}^9_{3,9}(2n) \equiv \mathbb{Q}^7_{3,9}(2n) \pmod 2. \tag{5.19}$$

Proof of the equation (5.19) is similar to the proof of equation (5.12) and except that in place of the equation (3.26), we use the equation (3.31).

Theorem 5.4. We have

$$\mathbb{Q}^6_{3,11}(2n) \equiv \mathbb{Q}^2_{3,11}(2n) \pmod 2, \tag{5.20}$$

$$\mathbb{Q}^6_{3,11}(2n) \equiv \mathbb{P}^5_{3,11}(2n) \pmod{11}, \tag{5.21}$$

$$\mathbb{R}^5_{3,11}(2n) \equiv \mathbb{T}^6_{3,11}(2n) \pmod{11}. \tag{5.22}$$

Proofs of the equations (5.20), (5.21) and (5.22) are similar to the proof of equation (5.12) and except that in place of the equation (3.26), we use the equations (3.33) and (3.34).

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References

[1] C. Adiga, M. S. Mahadeva Naika, Ramya Rao: *Integral representations and some explicit evaluations of a continued fraction of Ramanujan*, JP Jour. Algebra, Number Theory & Appl., **2**, n. 1, (2002), 5–20.

[2] G. E. Andrews, B. C. Berndt, L. Jacobsen, R. L. Lamphere: *The continued fractions found in the unorganized portions of Ramanujan's notebooks*, Memoir No. 477, Amer. Math. Soc., **99**, Providence, Rhode Island, 1992.

[3] B. C. Berndt: Ramanujan's Notebooks Part III, Springer-Verlag, New York, 1991.

[4] B. C. Berndt: Ramanujan's Notebooks Part V, Springer-Verlag, New York, 1998.

[5] S. Corteel, J. Lovejoy: *Overpartitions*, Trans. Amer. math. Soc., **356**, (2004), 1623–1635.

[6] C. G. J. Jacobi: Fundamenta Nova Theoriae Functionum Ellipticarum, Sumptibus Fratrum Bornträger, Regiomonti, 1829.

[7] J. Landen: *A disquisition concerning certain fluents, which are assignable by the arcs of the conic sections; where in are investigated some new and useful theorems for computing such fluents*, Philos. Trans. R. Soc., London, **61**, (1771), 298–309.

[8] J. Landen: *An investigation of a general theorem for finding the length of any arc of any conic hyperbola, by means of two elliptic arcs, with some other new and useful theorems deduced therefrom*, Philos. Trans. R. Soc., London, **65**, (1775), 283–289.

[9] M. S. Mahadeva Naika, S. Chandankumar: *Some new modular equations and their applications*, Tamsui Oxf. J. Math. Sci., **28**, n. 4, (2012), 437–469.

[10] M. S. Mahadeva Naika, S. Chandankumar, M.Manjunatha: *On some new modular equations and their applications to continued fractions*, Int. Math. Forum., **6**, n. 58, (2011), 2881–2905.

[11] M. S. Mahadeva Naika, S. Chandankumar, M. Harish: *On some new P-Q mixed modular equations*, Annali dell'Università di Ferrara., **59**, n. 2, (2013), 353–374.

[12] M. S. Mahadeva Naika, S. Chandankumar, K. Sushan Bairy: *Modular equations for the ratios of Ramanujan's theta function ψ and evaluations*, New Zealand J. Math., **40**, (2010), 33–48.

[13] M. S. Mahadeva Naika, Remy Y Denis, K. Sushan Bairy: *On some Ramanujan–Selberg continued fraction*, Indian J. Math., **51**, n. 3, (2009), 585–596.

[14] M. S. Mahadeva Naika, K. Sushan Bairy , M. Manjunatha: *A continued fraction of order 4 found in Ramanujan's 'lost' notebook*, South East Asian J. Math. & Math. Sc., **9**, n. 3, (2011), 43–63.

[15] K. G. Ramanathan: *Ramanujan's continued fraction*, Indian J. Pure Appl. Math., **16**, (1985), 695–724.

[16] S. Ramanujan: Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[17] A. Selberg: *Uber einige arithmetische identitaten*, Avh. Norske Vid.-Akad. Oslo I. Mat. -Naturv. Kl., (1936), 2–23.

[18] L. A. Sohncke: *Aequationes modulares pro transformatione functionum ellipticarum et undecimi et decimi terti et decimi septimi ordinis*, J. Reine Angew. Math., **12**, (1834), 178.

[19] L. A. Sohncke: *Aequationes modulares pro transformatione functionum ellipticarum*, J. Reine Angew. Math., **16**, (1837), 97–130.