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Affine distributions on a four-dimensional extension of the semi-Euclidean group

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Abstract. The invariant affine distributions on a four-dimensional central extension of the semi-Euclidean group are classified (up to group automorphism). This classification is briefly discussed in the context of invariant control theory and sub-Riemannian geometry.

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1 Introduction

In the last few decades affine distributions (and their equivalence) have been considered by several authors. These developments have primarily been inspired and motivated by geometric control theory. Elkin [9, 10] studied equivalence of affine distributions on low-dimensional manifolds and obtained normal forms for the associated control systems. More recently, Clelland et al. [8] investigated the geometry of so-called point-affine distributions and computed (local) invariants for a class of such distributions (using Cartan's method of equivalence). Invariant affine distributions on low-dimensional Lie groups (or rather their associated control systems) have attracted particular attention (see, e.g., [1, 3, 11, 12, 13, 15, 16]).

Among the four-dimensional Lie algebras, only four can be described as nontrivial central extensions of three-dimensional Lie algebras [6]. These are (in Mubarakzyanov's notation [14]):

(1) the Engel algebra $\mathfrak{g}_{4.1}$, a central extension of the Heisenberg algebra $\mathfrak{h}_3$;

(2) the algebra $\mathfrak{g}_{4.3}$, a central extension of $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$;

(3) the algebra $\mathfrak{g}_{4.8}^{-1}$, a central extension of the semi-Euclidean algebra $\mathfrak{se}(1,1)$;

(4) the oscillator algebra $\mathfrak{g}_{4.9}^{0}$, a central extension of the Euclidean algebra $\mathfrak{se}(2)$.

Moreover, the only indecomposable four-dimensional Lie algebras admitting an invariant scalar product are $\mathfrak{g}_{4.9}^{0}$ and $\mathfrak{g}_{4.8}^{-1}$. The oscillator algebra $\mathfrak{g}_{4.9}^{0}$ and its associated groups were studied in [4].

In this paper we consider the algebra $\mathfrak{g}_{4.8}^{-1}$ (which we denote $\mathfrak{e}_{1,1}^{\bowtie}$) and its associated simply connected Lie group $\mathsf{E}_{1,1}^{\bowtie}$. More specifically, we are interested in the equivalence of left-invariant affine distributions on $\mathsf{E}_{1,1}^{\bowtie}$. We regard two distributions as being equivalent if they are related by a group automorphism. In section 2, a characterization of this equivalence relation in terms of Lie algebra automorphisms is provided. In section 3, the group $\mathsf{E}_{1,1}^{\bowtie}$ and its Lie algebra $\mathfrak{e}_{1,1}^{\bowtie}$ are introduced and the vector subspaces of $\mathfrak{e}_{1,1}^{\bowtie}$ are classified. (As corollaries, we obtain an exhaustive list of the subalgebras as well as the ideals.) In section 4, the invariant affine distributions on $\mathsf{E}_{1,1}^{\bowtie}$ are classified. Finally, in section 5, two extensive examples interpreting this classification in the context of control theory and sub-Riemannian geometry are presented.

2 Invariant affine distributions

An *affine distribution* on a (real, finite-dimensional) connected Lie group G is a (smooth) map $\mathcal{D}$ that assigns to every point $g \in \mathsf{G}$ an affine subspace $\mathcal{D}_g$ of $T_g\mathsf{G}$. $\mathcal{D}$ is said to be *left-invariant* if $(L_g)_*\mathcal{D} = \mathcal{D}$, i.e., $T_hL_g \cdot \mathcal{D}_h = \mathcal{D}_{gh}$. (Here $T_hL_g : T_h\mathsf{G} \to T_{gh}\mathsf{G}$ is the tangent map of the left translation $L_g : h \mapsto gh$.) A left-invariant affine distribution $\mathcal{D}$ is determined by its associated affine subspace $\mathcal{D}_\mathbf{1} \subseteq \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. (If $\mathcal{D}_\mathbf{1}$ is a vector subspace, then $\mathcal{D}$ is a left-invariant vector distribution on G.) We say that $\mathcal{D}$ is *bracket generating* if $\mathcal{D}_\mathbf{1}$ is bracket generating, i.e., the subalgebra $\mathrm{Lie}(\mathcal{D}_\mathbf{1})$ generated by $\mathcal{D}_\mathbf{1}$ is $\mathfrak{g}$.

Two left-invariant affine distributions $\mathcal{D}$ and $\mathcal{D}'$ on G are called $\mathfrak{L}$-*equivalent* if there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $\phi_*\mathcal{D} = \mathcal{D}'$.

Proposition 1. *$\mathcal{D}$ is $\mathfrak{L}$-equivalent to $\mathcal{D}'$ if and only if there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_\mathbf{1}\phi \cdot \mathcal{D}_\mathbf{1} = \mathcal{D}'_\mathbf{1}$.*

Proof. If $\phi : \mathsf{G} \to \mathsf{G}$ is an automorphism such that $\phi_* \mathcal{D} = \mathcal{D}'$, then clearly $T_{\mathbf{1}}\phi \cdot \mathcal{D}_{\mathbf{1}} = \mathcal{D}'_{\mathbf{1}}$. Conversely, suppose there exists an automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_{\mathbf{1}}\phi \cdot \mathcal{D}_{\mathbf{1}} = \mathcal{D}'_{\mathbf{1}}$. We have $\phi = L_{\phi(g)} \circ \phi \circ L_{g^{-1}}$ for $g \in \mathsf{G}$. By left invariance, we get $T_g\phi \cdot \mathcal{D}_g = \mathcal{D}'_{\phi(g)}$. QED

Corollary 1. *When G is simply connected, $\mathcal{D}$ is $\mathfrak{L}$-equivalent to $\mathcal{D}'$ if and only if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \mathcal{D}_{\mathbf{1}} = \mathcal{D}'_{\mathbf{1}}$.*

Accordingly, the classification of left-invariant affine distributions on a simply connected Lie group reduces to a classification of affine subspaces of its Lie algebra. By a slight abuse of terminology, we say that two affine subspaces $\Gamma = \mathcal{D}_{\mathbf{1}}$ and $\Gamma' = \mathcal{D}'_{\mathbf{1}}$ are *$\mathfrak{L}$-equivalent* if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$. We shall write $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$, where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$ and $B_1, \ldots, B_\ell$ are linearly independent.

3 $\mathsf{E}^{\rtimes}_{1,1}$ and its Lie algebra

The connected, simply connected four-dimensional matrix Lie group

$$\mathsf{E}^{\rtimes}_{1,1} = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & e^\theta & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z, \theta \in \mathbb{R} \right\}$$

is a (nontrivial) central extension of the semi-Euclidean group $\mathsf{SE}(1,1)$. Indeed, the mapping $\phi : \mathsf{E}^{\rtimes}_{1,1} \to \mathsf{SE}(1,1)$,

$$\begin{bmatrix} 1 & y & x \\ 0 & e^\theta & z \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2}\left(ye^{-\theta} + z\right) & \cosh\theta & -\sinh\theta \\ \frac{1}{2}\left(ye^{-\theta} - z\right) & -\sinh\theta & \cosh\theta \end{bmatrix}$$

is a Lie group epimorphism with $\ker\phi = Z(\mathsf{E}^{\rtimes}_{1,1})$. Moreover, $\mathsf{E}^{\rtimes}_{1,1}$ decomposes as the semi-direct product $\mathsf{H}_3 \rtimes \mathsf{SO}(1,1)_0$ of the Heisenberg subgroup

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

and the pseudo-orthogonal subgroup

$$\mathsf{SO}(1,1)_0 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta \in \mathbb{R} \right\}.$$

(That is, $\mathsf{E}_{1,1}^{\bowtie} = \mathsf{H}_3\,\mathsf{SO}(1,1)_0$, $\mathsf{H}_3 \cap \mathsf{SO}(1,1)_0 = \{\mathbf{1}\}$ and H_3 is normal in $\mathsf{E}_{1,1}^{\bowtie}$.) The Lie algebra of $\mathsf{E}_{1,1}^{\bowtie}$

$$\mathfrak{e}_{1,1}^{\bowtie} = \left\{ \begin{bmatrix} 0 & y & x \\ 0 & \theta & z \\ 0 & 0 & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3 + \theta E_4 : x, y, z, \theta \in \mathbb{R} \right\}$$

is unimodular and completely solvable. Its (nonzero) commutator relations are

$$[E_2, E_3] = E_1, \qquad [E_2, E_4] = E_2, \qquad [E_3, E_4] = -E_3.$$

Proposition 2 (cf. [7])**.** *The automorphism group* $\mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})$ *is given by*

$$\left\{ \begin{bmatrix} xy & wx & vy & u \\ 0 & x & 0 & v \\ 0 & 0 & y & w \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -xy & -vx & -wy & u \\ 0 & 0 & y & v \\ 0 & x & 0 & w \\ 0 & 0 & 0 & -1 \end{bmatrix} : u, v, w, x, y \in \mathbb{R},\ xy \neq 0 \right\}.$$

The group of inner automorphisms $\mathsf{Int}(\mathfrak{e}_{1,1}^{\bowtie}) = \{\mathrm{Ad}_g : g \in \mathsf{E}_{1,1}^{\bowtie}\}$ *takes the form*

$$\mathsf{Int}(\mathfrak{e}_{1,1}^{\bowtie}) = \left\{ \begin{bmatrix} 1 & -ze^{-\theta} & y & -yze^{-\theta} \\ 0 & e^{-\theta} & 0 & ye^{-\theta} \\ 0 & 0 & e^{\theta} & -z \\ 0 & 0 & 0 & 1 \end{bmatrix} : y, z, \theta \in \mathbb{R} \right\}.$$

(*In each case, the automorphisms are identified with their matrices with respect to* (E_1, E_2, E_3, E_4).)

Remark 1. The group of automorphisms $\mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})$ decomposes as the semidirect product of the normal subgroup $\mathsf{Int}(\mathfrak{e}_{1,1}^{\bowtie})$ and

$$\left\{ \begin{bmatrix} \sigma r^2 & 0 & 0 & u \\ 0 & \sigma r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\sigma r^2 & 0 & 0 & u \\ 0 & 0 & \sigma r & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} : u \in \mathbb{R},\ r \neq 0,\ \sigma \in \{-1, 1\} \right\}.$$

There exists an invariant scalar product on $\mathfrak{e}_{1,1}^{\bowtie}$, i.e., a nondegenerate bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ such that $\langle\!\langle [A, B], C \rangle\!\rangle = \langle\!\langle A, [B, C] \rangle\!\rangle$ for every $A, B, C \in \mathfrak{e}_{1,1}^{\bowtie}$.

Proposition 3. *The Lie algebra* $\mathfrak{e}_{1,1}^{\bowtie}$ *admits exactly one family* $(\omega_\alpha)_{\alpha \in \mathbb{R}}$ *of invariant scalar products. In coordinates* (*with respect to* (E_1, E_2, E_3, E_4)),

$$\omega_\alpha = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & \alpha \end{bmatrix}.$$

The orthogonal complement of a subspace Γ (with respect to the symmetric invariant scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle = \omega_0$) is the subspace

$$\Gamma^\perp = \{A \in \mathfrak{e}_{1,1}^{\bowtie} : \langle\!\langle A, B\rangle\!\rangle = 0 \text{ for every } B \in \Gamma\}.$$

Lemma 1. *Let Γ_1 and Γ_2 be vector subspaces of $\mathfrak{e}_{1,1}^{\bowtie}$ and $\varphi \in \mathsf{Int}(\mathfrak{e}_{1,1}^{\bowtie})$. If $\varphi \cdot \Gamma_1^\perp = \Gamma_2^\perp$, then $\varphi \cdot \Gamma_1 = \Gamma_2$.*

Vector subspaces of $\mathfrak{e}_{1,1}^{\bowtie}$

We classify the vector subspaces of $\mathfrak{e}_{1,1}^{\bowtie}$ under $\mathfrak{L}$-equivalence. As corollaries, we obtain enumerations of the subalgebras and the ideals. The following simple lemmas prove useful in distinguishing between equivalence classes. Let $\psi : \mathfrak{e}_{1,1}^{\bowtie} \to \mathfrak{e}_{1,1}^{\bowtie}$ be an automorphism and let Γ be a subspace of $\mathfrak{e}_{1,1}^{\bowtie}$. (Below E^4 denotes the corresponding element of the dual basis.)

Lemma 2. $E^4(\Gamma) = \{0\}$ *if and only if* $E^4(\psi \cdot \Gamma) = \{0\}$.

Lemma 3. $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \Gamma$ *if and only if* $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \psi \cdot \Gamma$.

Proposition 4. *Any proper vector subspace of $\mathfrak{e}_{1,1}^{\bowtie}$ is $\mathfrak{L}$-equivalent to exactly one of the following subspaces*:

$$\begin{gathered}\langle E_1\rangle, \quad \langle E_2\rangle, \quad \langle E_2 + E_3\rangle, \quad \langle E_4\rangle,\\ \langle E_1, E_2\rangle, \quad \langle E_2, E_3\rangle, \quad \langle E_1, E_4\rangle, \quad \langle E_2, E_4\rangle,\\ \langle E_1, E_2 + E_3\rangle, \quad \langle E_1 + E_2, E_4\rangle, \quad \langle E_2 + E_3, E_4\rangle,\\ \langle E_1, E_2, E_3\rangle, \quad \langle E_1, E_2, E_4\rangle, \quad \langle E_2, E_3, E_4\rangle, \quad \langle E_1, E_2 + E_3, E_4\rangle.\end{gathered}$$

Proof. Throughout the proof, ς denotes the automorphism

$$\begin{bmatrix} -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}.$$

Let $\Gamma = \langle \sum a_i E_i\rangle$ be a one-dimensional subspace. If $a_4 \neq 0$ (i.e., $E^4(\Gamma) \neq \{0\}$), then

$$\psi = \begin{bmatrix} 1 & -\frac{a_3}{a_4} & -\frac{a_2}{a_4} & \frac{2a_2a_3 - a_1a_4}{a_4^2}\\ 0 & 1 & 0 & -\frac{a_2}{a_4}\\ 0 & 0 & 1 & -\frac{a_3}{a_4}\\ 0 & 0 & 0 & 1\end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_4 \rangle$. If $a_4 = 0$ and $a_2, a_3 \neq 0$, then

$$\psi = \begin{bmatrix} a_2a_3 & 0 & -a_1a_2 & 0 \\ 0 & a_3 & 0 & -a_1 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_2 + E_3 \rangle$. If $a_2, a_4 = 0$ and $a_3 \neq 0$, then

$$\psi = \begin{bmatrix} -1 & 0 & \frac{a_1}{a_3} & 0 \\ 0 & 0 & \frac{1}{a_3} & 0 \\ 0 & a_3 & 0 & -a_1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_2 \rangle$. Likewise, if $a_2 \neq 0$ and $a_3, a_4 = 0$, then Γ is $\mathfrak{L}$-equivalent to $\langle E_2 \rangle$. Lastly, if $a_2, a_3, a_4 = 0$, then $\Gamma = \langle E_1 \rangle$.

Let $\Gamma = \langle \sum a_i E_i, \sum b_i E_i \rangle$ be a two-dimensional subspace. Suppose $E^4(\Gamma) \neq \{0\}$. We may assume $a_4 = 0$ and $b_4 = 1$. Then

$$\psi_1 = \begin{bmatrix} 1 & -b_3 & -b_2 & 2b_2b_3 - b_1 \\ 0 & 1 & 0 & -b_2 \\ 0 & 0 & 1 & -b_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_1 \cdot \Gamma = \langle a_1' E_1 + a_2 E_2 + a_3 E_3, E_4 \rangle$. If $a_2, a_3 \neq 0$, then

$$\psi_2 = \begin{bmatrix} a_2a_3 & -\frac{a_1'a_3}{2} & -\frac{a_1'a_2}{2} & 0 \\ 0 & a_3 & 0 & -\frac{a_1'}{2} \\ 0 & 0 & a_2 & -\frac{a_1'}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma = \langle E_2 + E_3, E_4 \rangle$. If $a_2 = 0$ and $a_1', a_3 \neq 0$, then

$$\psi_2 = \begin{bmatrix} \frac{1}{a_1'} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a_3} & 0 \\ 0 & -\frac{a_3}{a_1'} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is an automorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma = \langle E_1 + E_2, E_4 \rangle$. On the other hand, if $a_1', a_2 = 0$ and $a_3 \neq 0$, then $\varsigma \cdot \psi_1 \cdot \Gamma = \langle E_2, E_4 \rangle$. Similarly Γ is $\mathfrak{L}$-equivalent to $\langle E_1 + E_2, E_4 \rangle$ (if $a_1', a_2 \neq 0$, $a_3 = 0$) or $\langle E_2, E_4 \rangle$ (if $a_2 \neq 0$, $a_1', a_3 = 0$). If $a_2, a_3 = 0$, then $\psi_1 \cdot \Gamma = \langle E_1, E_4 \rangle$.

Suppose $E^4(\Gamma) = \{0\}$ and $b_2, b_3 \neq 0$. Then

$$\psi_1 = \begin{bmatrix} b_2 b_3 & -\frac{b_1 b_3}{2} & -\frac{b_1 b_2}{2} & 0 \\ 0 & b_3 & 0 & -\frac{b_1}{2} \\ 0 & 0 & b_2 & -\frac{b_1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_1 \cdot \Gamma = \langle a_1' E_1 + a_2' E_2, E_2 + E_3 \rangle$. If $a_2' \neq 0$, then we have an automorphism

$$\psi_2 = \begin{bmatrix} 1 & -\frac{a_1'}{a_2'} & \frac{a_1'}{a_2'} & 0 \\ 0 & 1 & 0 & \frac{a_1'}{a_2'} \\ 0 & 0 & 1 & -\frac{a_1'}{a_2'} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

such that $\psi_2 \cdot \psi_1 \cdot \Gamma = \langle E_2, E_3 \rangle$. If $a_2' = 0$, then $\psi_1 \cdot \Gamma = \langle E_1, E_2 + E_3 \rangle$.

Suppose $E^4(\Gamma) = \{0\}$, $b_2 = 0$ and $b_3 \neq 0$. We may assume $a_3 = 0$ and $b_3 = 1$. If $a_2 \neq 0$, then

$$\psi = \begin{bmatrix} \frac{1}{a_2} & -\frac{a_1}{a_2^2} & -\frac{b_1}{a_2} & 0 \\ 0 & \frac{1}{a_2} & 0 & -\frac{b_1}{a_2} \\ 0 & 0 & 1 & -\frac{a_1}{a_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle E_2, E_3 \rangle$. If $a_2 = 0$, then $\varsigma \cdot \Gamma = \langle E_1, E_2 \rangle$. Likewise, if $E^4(\Gamma) = \{0\}$, $b_2 \neq 0$ and $b_3 = 0$, then Γ is $\mathfrak{L}$-equivalent to $\langle E_2, E_3 \rangle$ or $\langle E_1, E_2 \rangle$.

Suppose $E^4(\Gamma) = \{0\}$ and $b_2, b_3 = 0$. We may assume $a_1 = 0$ and $b_1 = 1$. If $a_2, a_3 \neq 0$, then $\psi = \operatorname{diag}(a_2 a_3, a_3, a_2, 1)$ is an automorphism such that $\psi \cdot \Gamma = \langle E_2 + E_3, E_4 \rangle$. If $a_2 = 0$ and $a_3 \neq 0$, then $\varsigma \cdot \Gamma = \langle E_1, E_2 \rangle$. If $a_2 \neq 0$ and $a_3 = 0$, then clearly $\Gamma = \langle E_1, E_2 \rangle$.

Let Γ be a three-dimensional subspace with orthogonal complement $\Gamma^\perp = \langle \sum a_i E_i \rangle$. If $E^4(\Gamma^\perp) = \{0\}$ and $a_2, a_3 = 0$, then $\Gamma = \langle E_2, E_3, E_4 \rangle$. Suppose $E^4(\Gamma^\perp) = \{0\}$ and $a_2, a_3 \neq 0$. We may assume $a_3 = 1$. Then

$$\varphi = \begin{bmatrix} 1 & 0 & -a_1 & 0 \\ 0 & 1 & 0 & -a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an inner automorphism such that $\varphi \cdot \Gamma^\perp = \langle a_2 E_2 + E_3 \rangle$. Consequently, $\varphi \cdot \Gamma = \langle E_1, -a_2 E_2 + E_3, E_4 \rangle$; hence $\psi = \operatorname{diag}\left(-\frac{1}{a_2}, -\frac{1}{a_2}, 1, 1\right)$ is an automorphism such that $\psi \cdot \varphi \cdot \Gamma = \langle E_1, E_2 + E_3, E_4 \rangle$.

Suppose $E^4(\Gamma^\perp) = \{0\}$, $a_2 = 0$ and $a_3 \neq 0$. We may assume $a_3 = 1$. Then

$$\varphi = \begin{bmatrix} 1 & 0 & -a_1 & 0 \\ 0 & 1 & 0 & -a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an inner automorphism such that $\varphi \cdot \Gamma^\perp = \langle E_3 \rangle$. Hence $\varsigma \cdot \varphi \cdot \Gamma = \langle E_1, E_2, E_4 \rangle$. Likewise, if $E^4(\Gamma^\perp) = \{0\}$, $a_2 \neq 0$ and $a_3 = 0$, then Γ is $\mathfrak{L}$-equivalent to $\langle E_1, E_2, E_4 \rangle$.

Suppose $E^4(\Gamma^\perp) \neq \{0\}$. We may assume $a_4 = 1$. Then

$$\varphi = \begin{bmatrix} 1 & -a_3 & -a_2 & a_2 a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an inner automorphism such that $\varphi \cdot \Gamma^\perp = \langle a_1' E_1 + E_4 \rangle$. Hence $\varphi \cdot \Gamma = \langle E_2, E_3, -a_1' E_1 + E_4 \rangle$ and so

$$\psi = \begin{bmatrix} 1 & 0 & 0 & a_1' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \varphi \cdot \Gamma = \langle E_2, E_3, E_4 \rangle$.

Finally, using a straightforward argument (together with the foregoing lemmas), one verifies that none of the representatives obtained are $\mathfrak{L}$-equivalent to each other. *QED*

Corollary 2. *Any proper subalgebra of* $\mathfrak{e}_{1,1}^{\bowtie}$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following subalgebras*:

$$\langle E_1 \rangle, \quad \langle E_2 \rangle, \quad \langle E_2 + E_3 \rangle, \quad \langle E_4 \rangle,$$
$$\langle E_1, E_2 \rangle, \quad \langle E_1, E_4 \rangle, \quad \langle E_2, E_4 \rangle, \quad \langle E_1, E_2 + E_3 \rangle,$$
$$\langle E_1, E_2, E_3 \rangle, \quad \langle E_1, E_2, E_4 \rangle.$$

Among these subalgebras, only $\langle E_2, E_4 \rangle \cong \mathfrak{aff}(\mathbb{R})$, $\langle E_1, E_2, E_4 \rangle \cong \mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$ *and* $\langle E_1, E_2, E_3 \rangle \cong \mathfrak{h}_3$ *are not Abelian.*

Corollary 3. *Any proper ideal of* $\mathfrak{e}_{1,1}^{\bowtie}$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following ideals*:

$$\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) = \langle E_1 \rangle, \quad \langle E_1, E_2 \rangle, \quad \mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie})^\perp = \langle E_1, E_2, E_3 \rangle.$$

The ideals $\langle E_1 \rangle$ *and* $\langle E_1, E_2, E_3 \rangle$ *are fully characteristic* (*i.e.*, $\psi \cdot \mathfrak{i} = \mathfrak{i}$ *for every* $\psi \in \mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})$).

4 Classification

We classify the bracket-generating left-invariant affine (and vector) distributions on $\mathsf{E}_{1,1}^{\bowtie}$. This is accomplished by classifying the bracket-generating affine (and vector) subspaces of $\mathfrak{e}_{1,1}^{\bowtie}$. Henceforth, all subspaces under consideration are assumed to be bracket generating.

The following classification of vector subspaces follows from proposition 4.

Theorem 1. *Any (bracket-generating) proper vector subspace* Γ *is* $\mathfrak{L}$*-equivalent to exactly one of the following subspaces:*

$$\Gamma^{(2,0)} = \langle E_2 + E_3, E_4\rangle$$

$$\Gamma_1^{(3,0)} = \langle E_2, E_3, E_4\rangle \qquad \mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \not\subseteq \Gamma$$

$$\Gamma_2^{(3,0)} = \langle E_1, E_2 + E_3, E_4\rangle \qquad \mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \Gamma.$$

We now proceed to classify the affine subspaces of $\mathfrak{e}_{1,1}^{\bowtie}$. We provide details for both the one- and two-dimensional case; the three-dimensional case is similar and so the proof will be omitted. As before, we denote by ς the automorphism

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Theorem 2. *Any (bracket-generating) one-dimensional strictly affine subspace* $\Gamma = A + \Gamma^0$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following affine subspaces:*

$$\Gamma_{1,\beta}^{(1,1)} = \beta E_1 + E_2 + E_3 + \langle E_4\rangle \qquad E^4(\Gamma^0) \neq \{0\}$$

$$\Gamma_{2,\alpha}^{(1,1)} = \alpha E_4 + \langle E_2 + E_3\rangle \qquad E^4(\Gamma^0) = \{0\}.$$

Here $\alpha > 0$ *and* $\beta \geq 0$ *parametrize families of class representatives, each different value corresponding to a distinct (non-equivalent) representative.*

Proof. Since Γ^0 is a one-dimensional vector subspace, it is $\mathfrak{L}$-equivalent to exactly one of $\langle E_1\rangle$, $\langle E_2\rangle$, $\langle E_2 + E_3\rangle$ or $\langle E_4\rangle$ (see proposition 4). However, no subspace $A + \langle E_1\rangle$ or $A + \langle E_2\rangle$ is bracket generating (for any $A \in \mathfrak{e}_{1,1}^{\bowtie}$).

Suppose $E^4(\Gamma^0) \neq \{0\}$. Then there exists $\psi_1 \in \mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})$ such that $\psi_1 \cdot \Gamma = a_1E_1 + a_2E_2 + a_3E_3 + \langle E_4\rangle$, where $a_2, a_3 \neq 0$ (by the bracket-generating condition). Hence $\psi_2 = \operatorname{diag}\left(\frac{1}{a_2a_3}, \frac{1}{a_2}, \frac{1}{a_3}, 1\right)$ is an automorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma = a_1'E_1 + E_2 + E_3 + \langle E_4\rangle$. If $a_1' < 0$, then $\varsigma \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = -a_1'E_1 + E_2 + E_3 + \langle E_4\rangle$. Therefore Γ is $\mathfrak{L}$-equivalent to $\Gamma_{1,\beta}^{(1,1)}$, where $\beta = |a_1'| \geq 0$.

Suppose $E^4(\Gamma^0) = \{0\}$. Then there exists $\psi_1 \in \mathsf{Aut}(\mathfrak{e}^{\bowtie}_{1,1})$ such that $\psi_1 \cdot \Gamma = \sum a_i E_i + \langle E_2 + E_3 \rangle$. Since $\psi_1 \cdot \Gamma$ is bracket generating, $a_4 \neq 0$ and so

$$\psi_2 = \begin{bmatrix} 1 & \frac{a_2-a_3}{2a_4} & -\frac{a_2-a_3}{2a_4} & -\frac{(a_2-a_3)^2+2a_1a_4}{2a_4^2} \\ 0 & 1 & 0 & -\frac{a_2-a_3}{2a_4} \\ 0 & 0 & 1 & \frac{a_2-a_3}{2a_4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma = a_4 E_4 + \langle E_2 + E_3 \rangle$. If $a_4 < 0$, then $\varsigma \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = -a_4 E_4 + \langle E_2 + E_3 \rangle$. Therefore Γ is $\mathfrak{L}$-equivalent to $\Gamma^{(1,1)}_{2,\alpha}$, where $\alpha = |a_4| > 0$.

As $\langle E_4 \rangle$ and $\langle E_2 + E_3 \rangle$ are not $\mathfrak{L}$-equivalent (proposition 4), it follows that $\Gamma^{(1,1)}_{1,\beta}$ is not $\mathfrak{L}$-equivalent to $\Gamma^{(1,1)}_{2,\alpha}$. We claim that $\Gamma^{(1,1)}_{1,\beta}$ is $\mathfrak{L}$-equivalent to $\Gamma^{(1,1)}_{1,\beta'}$ only if $\beta = \beta'$. Indeed, suppose

$$\psi = \begin{bmatrix} xy & wx & vy & u \\ 0 & x & 0 & v \\ 0 & 0 & y & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma^{(1,1)}_{1,\beta} = \Gamma^{(1,1)}_{1,\beta'}$. Then $(wx + vy + \beta xy - \beta')E_1 + (x-1)E_2 + (y-1)E_3 \in \beta' E_1 + E_2 + E_3 + \langle E_4 \rangle$ and $uE_1 + vE_2 + wE_3 \in \langle E_4 \rangle$. Thus $u = v = w = 0$, $x = y = 1$ and so $\beta = \beta'$. On the other hand, if

$$\psi = \begin{bmatrix} -xy & -vx & -wy & u \\ 0 & 0 & y & v \\ 0 & x & 0 & w \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma^{(1,1)}_{1,\beta} = \Gamma^{(1,1)}_{1,\beta'}$, then $u = v = w = 0$ and $x = y = 1$; whence $\beta = -\beta'$. As $\beta, \beta' \geq 0$, this implies that $\beta = \beta' = 0$. Likewise, $\Gamma^{(1,1)}_{2,\alpha}$ is $\mathfrak{L}$-equivalent to $\Gamma^{(1,1)}_{2,\alpha'}$ only if $\alpha = \alpha'$. QED

Theorem 3. *Let $\Gamma = A + \Gamma^0$ be a (bracket-generating) two-dimensional strictly affine subspace.*

(*i*) *If $E^4(\Gamma^0) \neq \{0\}$, then Γ is $\mathfrak{L}$-equivalent to exactly one of the following affine subspaces:*

$\Gamma^{(2,1)}_{1} = E_3 + \langle E_2, E_4 \rangle$ $\quad\quad$ $\mathrm{Lie}(\Gamma^0) \neq \mathfrak{e}^{\bowtie}_{1,1}, \ \mathsf{Z}(\mathfrak{e}^{\bowtie}_{1,1}) \not\subseteq \langle \Gamma \rangle$

$\Gamma^{(2,1)}_{2,\gamma} = \gamma E_3 + \langle E_1 + E_2, E_4 \rangle$ $\quad\quad$ $\mathrm{Lie}(\Gamma^0) \neq \mathfrak{e}^{\bowtie}_{1,1}, \ \mathsf{Z}(\mathfrak{e}^{\bowtie}_{1,1}) \not\subseteq \langle \Gamma \rangle$

$$\Gamma_3^{(2,1)} = E_2 + E_3 + \langle E_1, E_4\rangle \qquad \mathrm{Lie}(\Gamma^0) \neq \mathfrak{e}_{1,1}^{\bowtie},\ \mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \langle\Gamma\rangle$$
$$\Gamma_{4,\beta}^{(2,1)} = \beta E_1 + E_3 + \langle E_2 + E_3, E_4\rangle \qquad \mathrm{Lie}(\Gamma^0) = \mathfrak{e}_{1,1}^{\bowtie},\ \mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \not\subseteq \langle\Gamma\rangle$$
$$\Gamma_5^{(2,1)} = E_1 + \langle E_2 + E_3, E_4\rangle \qquad \mathrm{Lie}(\Gamma^0) = \mathfrak{e}_{1,1}^{\bowtie},\ \mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \langle\Gamma\rangle.$$

(*ii*) *If $E^4(\Gamma^0) = \{0\}$, then Γ is $\mathfrak{L}$-equivalent to exactly one of the following affine subspaces*:

$$\Gamma_{6,\alpha}^{(2,1)} = \alpha E_4 + \langle E_2, E_3\rangle \qquad \mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \not\subseteq \langle\Gamma\rangle$$
$$\Gamma_{7,\alpha}^{(2,1)} = \alpha E_4 + \langle E_1, E_2 + E_3\rangle \qquad \mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \langle\Gamma\rangle.$$

Here $\alpha > 0$, $\beta \geq 0$ and $\gamma \neq 0$ parametrize families of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

Proof. Since Γ^0 is a two-dimensional vector subspace, it is $\mathfrak{L}$-equivalent to exactly one of $\langle E_1, E_2\rangle$, $\langle E_1, E_4\rangle$, $\langle E_2, E_3\rangle$, $\langle E_2, E_4\rangle$, $\langle E_1, E_2{+}E_3\rangle$, $\langle E_1{+}E_2, E_4\rangle$ or $\langle E_2 + E_3, E_4\rangle$ (see proposition 4). However, no subspace $A + \langle E_1, E_2\rangle$ is bracket generating (for any $A \in \mathfrak{e}_{1,1}^{\bowtie}$).

(*i*) Assume $E^4(\Gamma^0) \neq \{0\}$. First, suppose $\mathrm{Lie}(\Gamma^0) \neq \mathfrak{e}_{1,1}^{\bowtie}$ and $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \not\subseteq \langle\Gamma\rangle$. Then there exists $\psi_1 \in \mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})$ such that $\psi_1 \cdot \Gamma = a_1E_1 + a_3E_3 + \langle E_2, E_4\rangle$ or $\psi_1 \cdot \Gamma = a_2E_2 + a_3E_3 + \langle E_1 + E_2, E_4\rangle$, where $a_3 \neq 0$. If $\psi_1 \cdot \Gamma = a_1E_1 + a_3E_3 + \langle E_2, E_4\rangle$, then we have an automorphism

$$\psi_2 = \begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} & 0 \\ 0 & a_3 & 0 & -a_1 \\ 0 & 0 & \frac{1}{a_3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

such that $\psi_2 \cdot \psi_1 \cdot \Gamma = E_3 + \langle E_2, E_4\rangle = \Gamma_1^{(2,1)}$. If $\psi_1 \cdot \Gamma = a_2E_2 + a_3E_3 + \langle E_1 + E_2, E_4\rangle$, then

$$\psi_2 = \begin{bmatrix} a_3 & 0 & a_2 & a_2 \\ 0 & a_3 & 0 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma = a_3E_3 + \langle E_1 + E_2, E_4\rangle = \Gamma_{2,\gamma}^{(2,1)}$, where $\gamma = a_3 \neq 0$.

Suppose $\mathrm{Lie}(\Gamma^0) \neq \mathfrak{e}_{1,1}^{\bowtie}$ and $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \langle\Gamma\rangle$. Then there exists $\psi_1 \in \mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})$ such that $\psi_1 \cdot \Gamma = a_2E_2 + a_3E_3 + \langle E_1, E_4\rangle$ with $a_2, a_3 \neq 0$. Hence $\psi_2 = \mathrm{diag}\left(\frac{1}{a_2a_3}, \frac{1}{a_2}, \frac{1}{a_3}, 1\right)$ is an automorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma = E_2 + E_3 + \left\langle E_1, E_4\right\rangle = \Gamma_3^{(2,1)}$.

Suppose $\text{Lie}(\Gamma^0) = \mathfrak{e}_{1,1}^{\bowtie}$ and $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \not\subseteq \langle\Gamma\rangle$. Then there exists an automorphism ψ_1 such that $\psi_1 \cdot \Gamma = a_1E_1 + a_3E_3 + \langle E_2 + E_3, E_4\rangle$, where $a_3 \neq 0$. Hence we have an automorphism $\psi_2 = \text{diag}\left(\frac{1}{a_3^2}, \frac{1}{a_3}, \frac{1}{a_3}, 1\right)$ such that $\psi_2 \cdot \psi_1 \cdot \Gamma = a_1'E_1 + E_3 + \langle E_2 + E_3, E_4\rangle$. If $a_1' < 0$, then

$$\psi_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is an automorphism such that $\psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = -a_1E_1 + E_3 + \langle E_2 + E_3, E_4\rangle$. Hence Γ is $\mathfrak{L}$-equivalent to $\Gamma^{(2,1)}_{4,\beta}$, where $\beta = |a_1| \geq 0$.

Lastly, suppose $\text{Lie}(\Gamma^0) = \mathfrak{e}_{1,1}^{\bowtie}$ and $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \langle\Gamma\rangle$. Then there exists an automorphism ψ_1 such that $\psi_1 \cdot \Gamma = a_1E_1 + \langle E_2 + E_3, E_4\rangle$. Hence we have an automorphism $\psi_2 = \text{diag}\left(\frac{1}{a_1}, \frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_1}}, 1\right)$ such that $\psi_2 \cdot \psi_1 \cdot \Gamma = E_1 + \langle E_2 + E_3, E_4\rangle = \Gamma^{(2,1)}_5$.

(*ii*) Assume $E^4(\Gamma^0) = \{0\}$. First, suppose $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \not\subseteq \langle\Gamma\rangle$. Then there exists an automorphism ψ_1 such that $\psi_1 \cdot \Gamma = a_1E_1 + a_4E_4 + \langle E_2, E_3\rangle$, where $a_4 \neq 0$. Hence

$$\psi_2 = \begin{bmatrix} 1 & 0 & 0 & -\frac{a_1}{a_4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma = a_4E_4 + \langle E_2, E_3\rangle$. If $a_4 < 0$, then $\varsigma \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = -a_4E_4 + \langle E_2, E_3\rangle$. Thus Γ is $\mathfrak{L}$-equivalent to $\Gamma^{(2,1)}_{6,\alpha}$, where $\alpha = |a_4| > 0$.

On the other hand, suppose $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \langle\Gamma\rangle$. Then there exists an automorphism ψ_1 such that $\psi_1 \cdot \Gamma = a_3E_3 + a_4E_4 + \langle E_1, E_2 + E_3\rangle$ with $a_4 \neq 0$. Hence

$$\psi_2 = \begin{bmatrix} 1 & -\frac{a_3}{a_4} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{a_3}{a_4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma = a_4E_4 + \langle E_1, E_2 + E_3\rangle$. If $a_4 < 0$, then $\varsigma \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = -a_4E_4 + \langle E_1, E_2 + E_3\rangle$. Therefore Γ is $\mathfrak{L}$-equivalent to $\Gamma^{(2,1)}_{7,\alpha}$, where $\alpha = |a_4| > 0$.

Since the conditions $E^4(\Gamma^0) = \{0\}$, $\text{Lie}(\Gamma^0) = \mathfrak{e}_{1,1}^{\bowtie}$ and $\mathsf{Z}(\mathfrak{e}_{1,1}^{\bowtie}) \subseteq \langle\Gamma\rangle$ are invariant under automorphisms, in most cases it follows that no two (families of) representatives are $\mathfrak{L}$-equivalent. The only exception is $\Gamma^{(2,1)}_1$ and $\Gamma^{(2,1)}_{2,\gamma}$, which are not $\mathfrak{L}$-equivalent as $\langle E_2, E_4\rangle$ is not $\mathfrak{L}$-equivalent to $\langle E_1 + E_2, E_4\rangle$. It remains to be shown that within each one-parameter family of affine subspaces, different values of the parameter yield distinct representatives. We shall treat only the family $\Gamma^{(2,1)}_{2,\gamma}$; the remaining cases are very similar.

We claim that $\Gamma^{(2,1)}_{2,\gamma}$ is $\mathfrak{L}$-equivalent to $\Gamma^{(2,1)}_{2,\gamma'}$ only if $\gamma = \gamma'$. Indeed, suppose

$$\psi = \begin{bmatrix} xy & wx & vy & u \\ 0 & x & 0 & v \\ 0 & 0 & y & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma^{(2,1)}_{2,\gamma} = \Gamma^{(2,1)}_{2,\gamma'}$. Then $(\gamma v y)E_1 + \gamma y E_3 \in \gamma' E_3 + \langle E_1 + E_2, E_4\rangle$ and $\langle x(w+y)E_1 + xE_2, uE_1 + vE_2 + wE_3 + E_4\rangle = \langle E_1 + E_2, E_4\rangle$. Hence $w = 0$, $y = 1$ and so $\gamma = \gamma'$. On the other hand, if

$$\psi = \begin{bmatrix} -xy & -vx & -wy & u \\ 0 & 0 & y & v \\ 0 & x & 0 & w \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma^{(2,1)}_{2,\gamma} = \Gamma^{(2,1)}_{2,\gamma'}$, then $w = 0$ and $\gamma y = 0$, a contradiction. QED

Theorem 4. *Let $\Gamma = A + \Gamma^0$ be a (bracket-generating) three-dimensional strictly affine subspace.*

(*i*) *If $E^4(\Gamma^0) \neq \{0\}$, then Γ is $\mathfrak{L}$-equivalent to exactly one of the following affine subspaces:*

$$\Gamma^{(3,1)}_1 = E_3 + \langle E_1, E_2, E_4\rangle \qquad \mathrm{Lie}(\Gamma^0) \neq \mathfrak{e}^{\bowtie}_{1,1}$$
$$\Gamma^{(3,1)}_2 = E_1 + \langle E_2, E_3, E_4\rangle \qquad \mathrm{Lie}(\Gamma^0) = \mathfrak{e}^{\bowtie}_{1,1},\ \mathsf{Z}(\mathfrak{e}^{\bowtie}_{1,1}) \not\subseteq \Gamma^0$$
$$\Gamma^{(3,1)}_3 = E_3 + \langle E_1, E_2 + E_3, E_4\rangle \qquad \mathrm{Lie}(\Gamma^0) = \mathfrak{e}^{\bowtie}_{1,1},\ \mathsf{Z}(\mathfrak{e}^{\bowtie}_{1,1}) \subseteq \Gamma^0.$$

(*ii*) *If $E^4(\Gamma^0) = \{0\}$, then Γ is $\mathfrak{L}$-equivalent to exactly one of the following affine subspaces:*

$$\Gamma^{(3,1)}_{4,\alpha} = \alpha E_4 + \langle E_1, E_2, E_3\rangle.$$

Here $\alpha > 0$ parametrizes a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

5 Two demonstrative examples

Invariant control systems

To every invariant control affine system one can canonically associate an invariant affine distribution (on the same state space). Two systems are (detached feedback) equivalent if and only if their associated affine distributions are $\mathfrak{L}$-equivalent. Accordingly, the classification of affine distributions on $\mathsf{E}^{\bowtie}_{1,1}$ may be interpreted as a classification of (invariant) control affine systems.

A *left-invariant control affine system* Σ on a (real, finite-dimensional) matrix Lie group G may be regarded as a family $(\Xi_u)_{u\in\mathbb{R}^\ell}$ of left-invariant vector fields on G affinely parametrized by controls, i.e.,

$$\Xi_u(g) = g(A + u_1B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

Here $A, B_1, \ldots, B_\ell$ are elements of the Lie algebra $\mathfrak{g}$ of G and $B_1, \ldots, B_\ell$ are linearly independent. It is assumed that $A, B_1, \ldots, B_\ell$ generate $\mathfrak{g}$. We write a control affine system Σ in the abbreviated form $\Sigma : A + u_1B_1 + \cdots + u_\ell B_\ell$. An *admissible control* is a piecewise continuous map $u(\cdot) : [0, T] \to \mathbb{R}^\ell$. A *trajectory*, corresponding to an admissible control $u(\cdot)$, is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = \Xi_{u(t)}(g(t))$ for almost every $t \in [0, T]$.

Two systems Σ and Σ' on G are *detached feedback equivalent* if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi_u(g) = \Xi'_{\varphi(u)}(\phi(g))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. The map ϕ establishes a one-to-one correspondence between trajectories of equivalent systems. We associate to each system Σ a (bracket-generating) affine distribution $\mathcal{D}$ given by $\mathcal{D}_g = \{\Xi_u(g) : u \in \mathbb{R}^\ell\}$. Σ is detached feedback equivalent to Σ' exactly when their associated distributions $\mathcal{D}$ and $\mathcal{D}'$ are $\mathfrak{L}$-equivalent (cf. [3, 5]).

Accordingly, we have the following classification of systems on $\mathsf{E}^{\bowtie}_{1,1}$.

Proposition 5. *Every left-invariant control affine system is detached feedback equivalent to exactly one of the following systems*:

$\Sigma^{(1,1)}_{1,\beta} : \beta E_1 + E_2 + E_3 + uE_4$ $\qquad$ $\Sigma^{(1,1)}_{2,\alpha} : \alpha E_4 + u(E_2 + E_3)$

$\Sigma^{(2,0)} : u_1(E_2 + E_3) + u_2E_4$ $\qquad$ $\Sigma^{(2,1)}_{1} : E_3 + u_1E_2 + u_2E_4$

$\Sigma^{(2,1)}_{2,\gamma} : \gamma E_3 + u_1(E_1 + E_2) + u_2E_4$ $\qquad$ $\Sigma^{(2,1)}_{3} : E_2 + E_3 + u_1E_1 + u_2E_4$

$\Sigma^{(2,1)}_{4,\beta} : \beta E_1 + E_3 + u_1(E_2 + E_3) + u_2E_4$ $\qquad$ $\Sigma^{(2,1)}_{5} : E_1 + u_1(E_2 + E_3) + u_2E_4$

$\Sigma^{(2,1)}_{6,\alpha} : \alpha E_4 + u_1E_2 + u_2E_3$ $\qquad$ $\Sigma^{(2,1)}_{7,\alpha} : \alpha E_4 + u_1E_1 + u_2(E_2 + E_3)$

$\Sigma^{(3,0)}_{1} : u_1E_2 + u_2E_3 + u_3E_4$ $\qquad$ $\Sigma^{(3,0)}_{2} : u_1E_1 + u_2(E_2 + E_3) + u_3E_4$

$\Sigma^{(3,1)}_{1} : E_3 + u_1E_1 + u_2E_2 + u_3E_4$ $\qquad$ $\Sigma^{(3,1)}_{2} : E_1 + u_1E_2 + u_2E_3 + u_3E_4$

$\Sigma_3^{(3,1)} : E_3 + u_1 E_1 + u_2(E_2 + E_3) + u_3 E_4 \quad \Sigma_{4,\alpha}^{(3,1)} : \alpha E_4 + u_1 E_1 + u_2 E_2 + u_3 E_3$
$\Sigma^{(4,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4.$

Here $\alpha > 0$, $\beta \geq 0$ and $\gamma \neq 0$ parametrize families of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

Invariant sub-Riemannian structures

A *left-invariant sub-Riemannian structure* on a (real, finite-dimensional) connected Lie group G consists of a nonintegrable left-invariant distribution $\mathcal{D}$ and a left-invariant Riemannian metric $\mathcal{G}$ on $\mathcal{D}$. It is assumed that $\mathcal{D}$ is bracket generating.

Two sub-Riemannian structures $(\mathcal{D}, \mathcal{G})$ and $(\mathcal{D}', \mathcal{G}')$ on G are *isometric* if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathcal{G} = \phi^* \mathcal{G}'$. If, in addition, ϕ is a group automorphism, then we shall say that they are $\mathfrak{L}$-*isometric*. If G is simply connected, then $(\mathcal{D}, \mathcal{G})$ is $\mathfrak{L}$-isometric to $(\mathcal{D}', \mathcal{G}')$ if and only if there exists a Lie algebra automorphism ψ such that $\psi \cdot \mathcal{D}_{\mathbf{1}} = \mathcal{D}'_{\mathbf{1}}$ and $\mathcal{G}_{\mathbf{1}}(X, Y) = \mathcal{G}'_{\mathbf{1}}(\psi \cdot X, \psi \cdot Y)$ for every $X, Y \in \mathcal{D}_{\mathbf{1}}$ (cf. [17, 2]).

Accordingly, one need only normalize the metric in order to obtain a classification of sub-Riemannian structures on $\mathsf{E}_{1,1}^{\bowtie}$.

Proposition 6. *Let $(\mathcal{D}, \mathcal{G})$ be a left-invariant sub-Riemannian structure on $\mathsf{E}_{1,1}^{\bowtie}$.*

(*i*) *If $\operatorname{rank} \mathcal{D} = 2$, then $(\mathcal{D}, \mathcal{G})$ is $\mathfrak{L}$-isometric to exactly one of the following structures:*

$$\mathcal{D}_{\mathbf{1}} = \langle E_2 + E_3, E_4 \rangle, \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \qquad \beta < 1.$$

($\mathcal{G}_{\mathbf{1}}$ *is identified with its matrix with respect to* $(E_2 + E_3, E_4)$.)

(*ii*) *If $\operatorname{rank} \mathcal{D} = 3$ and $E^4(\mathcal{D}_{\mathbf{1}}^{\perp}) \neq \{0\}$, then $(\mathcal{D}, \mathcal{G})$ is $\mathfrak{L}$-isometric to exactly one of the following structures:*

$$\mathcal{D}_{\mathbf{1}} = \langle E_2, E_3, E_4 \rangle, \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} \alpha_1 & \gamma & 1 \\ \gamma & \alpha_2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \begin{matrix} \alpha_1 \alpha_2 - \gamma^2 > 0, \\ \det[\mathcal{G}_{\mathbf{1}}] > 0 \end{matrix}$$

$$\mathcal{D}_{\mathbf{1}} = \langle E_2, E_3, E_4 \rangle, \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} \alpha & \beta & 1 \\ \beta & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \alpha - \beta^2 > 1$$

$$\mathcal{D}_{\mathbf{1}} = \langle E_2, E_3, E_4 \rangle, \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & \beta & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \beta < 1.$$

($\mathcal{G}_{\mathbf{1}}$ *is identified with its matrix with respect to* (E_2, E_3, E_4).)

(*iii*) *If* $\operatorname{rank} \mathcal{D} = 3$ *and* $E^4(\mathcal{D}_{\mathbf{1}}^{\perp}) = \{0\}$, *then* $(\mathcal{D}, \mathcal{G})$ *is* $\mathfrak{L}$*-isometric up to scale with exactly one of the following structures*:

$$\mathcal{D}_{\mathbf{1}} = \langle E_1, E_2 + E_3, E_4 \rangle, \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & \beta \\ 0 & \beta & 1 \end{bmatrix} \qquad \beta < 1.$$

($\mathcal{G}_{\mathbf{1}}$ *is identified with its matrix with respect to* $(E_1, E_2 + E_3, E_4)$.)

Here $\lambda, \alpha > 0$, $\alpha_1 \geq \alpha_2 > 0$, $\beta \geq 0$ *and* $\gamma \in \mathbb{R}$ (*with additional constraints given adjacent to each representative*) *parametrize families of class representatives, with different values corresponding to distinct* (*non-equivalent*) *representatives.*

Proof. We treat only the case when $\mathcal{D}$ has rank two. (The rank-three case is similar.) Let $(\mathcal{D}, \mathcal{G})$ be a left-invariant sub-Riemannian structure on $\mathsf{E}_{1,1}^{\bowtie}$, where $\mathcal{D}$ is a rank-two distribution. By theorem 1, there exists $\psi_1 \in \mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})$ such that $\psi_1 \cdot \mathcal{D}_{\mathbf{1}} = \langle E_2 + E_3, E_4 \rangle$. Hence $(\mathcal{D}, \mathcal{G})$ is $\mathfrak{L}$-isometric to a structure $(\mathcal{D}', \mathcal{G}')$, where $\mathcal{D}'_{\mathbf{1}} = \langle E_2 + E_3, E_4 \rangle$. The subgroup of automorphisms leaving $\mathcal{D}'_{\mathbf{1}}$ invariant is given by

$$\mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})\Big|_{\mathcal{D}'_{\mathbf{1}}} = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & -1 \end{bmatrix} : x \neq 0 \right\}.$$

(The elements of $\mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})\Big|_{\mathcal{D}'_{\mathbf{1}}}$ are written with respect to $(E_2 + E_3, E_4)$.) Let $\mathcal{G}'_{\mathbf{1}} = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$. We have $\psi_2 = \operatorname{diag}\left(\sqrt{\frac{a_2}{a_1}}, 1\right) \in \mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})\Big|_{\mathcal{D}'_{\mathbf{1}}}$ such that

$$\psi_2^{\top} \mathcal{G}'_{\mathbf{1}} \psi_2 = a_2 \begin{bmatrix} 1 & b' \\ b' & 1 \end{bmatrix}.$$

If $b' < 0$, then $\psi_3 = \operatorname{diag}(1, -1) \in \mathsf{Aut}(\mathfrak{e}_{1,1}^{\bowtie})\Big|_{\mathcal{D}'_{\mathbf{1}}}$ and

$$\psi_3^{\top} \psi_2^{\top} \mathcal{G}'_{\mathbf{1}} \psi_2 \psi_3 = a_2 \begin{bmatrix} 1 & -b' \\ -b' & 1 \end{bmatrix}.$$

Therefore $(\mathcal{D}, \mathcal{G})$ is $\mathfrak{L}$-isometric to $(\mathcal{D}', \mathcal{G}'')$, where $\mathcal{G}''_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix}$ with $\lambda > 0$ and $\beta \geq 0$. As $\mathcal{G}''_{\mathbf{1}}$ is positive definite, we have $\beta < 1$. It is easy to verify that two representatives are $\mathfrak{L}$-isometric only if their parameters are equal. QED

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