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On Nonlocal $p(x)-$Laplacian Problems Involving No Flux Boundary Condition

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Abstract. In this work, we give some results on the existence and multiplicity of solutions concerned a class of $p(x)-$Kirchhoff problem with No-flux boundary condition. Our methods involve the variational approach and the degree of Larey-Shauder.

Keywords: boundary value problem, $p(x)-$Kirchhoff problem, topological degree, critical point.

MSC 2010 classification: 35J30 , 35J60, 35J92.

Introduction

The interest in the study of Kirchhoff type problem has increased, because of the various applications which raise many difficult mathematical problems. Indeed, this type of equations describe many natural phenomena like elastic mechanics, image restoration, electrorheological fluids, biological systems where such solution modelizes a process depending on the average of itself. See, for example, [12, 13, 27, 30] and its references.

Such equation is a general version of the Kirchhoff equation in [22]. Accurately, he introduced a model

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ_0, ρ, L and h are constants associated to the effects of the changes in the length of strings during the vibrations. It is an extension of the classical D'Alembert's wave equation.

These type of problems have attracted the attention of many researchers like [2, 3, 4, 6, 11, 21, 25] after the pioneer work of Lions [24]. We may learn some early research of Kirchhoff equations from [10] and [26].

Recently, Autuori, Pucci and Salvatori [7] have investigated the Kirchhoff type equation involving the $p(x)-$Laplacian of the form

$$u_{tt} - M(\int_\Omega \frac{1}{p(x)} \mid \nabla u \mid^{p(x)} dx)\Delta_{p(x)}u + Q(t,x,u,u_t) + f(x,u) = 0.$$

They have introduced the asymptotic stability, as time tends to infinity. This kind of Kirchhoff problems with stationary process has received considerable attention by several researchers; we just quote [1, 5, 7, 8, 9, 14, 15, 18].

Motivated by the above mentioned papers, we consider the following nonlocal problem

$$\begin{gathered} h\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right)\left(-\Delta_{p(x)}u + |u|^{p(x)-2}u\right) = f(x,u) \quad \text{in } \Omega, \\ u = constant \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} d\sigma_x = 0, \end{gathered} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and the operator $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$Laplacian with $p \in C(\overline{\Omega})$ satisfies

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < \infty$$

and $N \geq 1, q \in C_+(\overline{\Omega})$, with $C_+(\overline{\Omega})$ is defined by

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \ \ and \ \ ess \inf_{x\in\overline{\Omega}} p(x) > 1\}.$$

We introduce the following assumptions in order to state the main result of this paper,

(h_1) $h : (0,+\infty) \to (0,+\infty)$ continuous for $t > 0, h_0 = \inf_\Omega h(x) > 0$.

(h_2) There is a positive constant α such that

$$\limsup_{t\to 0^+} \frac{H(t)}{t^\alpha} < +\infty,$$

where $H(t) = \int_0^t h(s)ds$.

(f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function and there exists $1 < q(x) < p^*(x)$, such that

$$|f(x,t)| \leq C_1 + C_2|t|^{q(x)-1}$$

for a.e $x \in \Omega, t \in \mathbb{R}$, with C_1 and C_2 are positives constants and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & if \;\; p(x) < N, \\ +\infty & if \;\; p(x) \geq N. \end{cases}$$

(f_1) $\limsup_{|t|\to+\infty} \dfrac{p(x)F(x,t)}{\mid t \mid^{p(x)}} < h_0\lambda_1$, uniformly for a.e $x \in \Omega$ with

$$F(x,t) = \int_0^t f(x,s)ds,$$

$$\lambda_1 = \inf_{(W_0^{1,p(x)}(\Omega)\oplus\mathbb{R})\backslash\{0\}} \frac{\int_\Omega \frac{1}{p(x)}(\mid \nabla u \mid^{p(x)} + \mid u \mid^{p(x)})dx}{\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx} > 0.$$

(f_2) There exist $a_0 > 0$ and $\delta > 0$ such that

$$F(x,t) \geq a_0|t|^{q_0(x)}, \;\forall x \in \Omega, \;|t| < \delta,$$

where $q_0 \in C(\overline{\Omega})$ with $q_0^+ < \alpha p^-$.

(f_3) f is odd with respect to the second argument, i.e

$$f(x,t) = -f(x,-t)$$

for $x \in \Omega$ and $t \in \mathbb{R}$.

(f_4) $f(x,t) = O(t^{p^+-1})$ as $t \to 0$ uniformly a.e. $x \in \Omega$.

Now, we present our main results,

Theorem 1. *(a) Under the assumptions* $(f_0)-(f_2)$, $(h_1),(h_2)$, *the problem* (0.1) *has at least a nontrivial solution in* $W_0^{1,p(x)}(\Omega) \oplus \mathbb{R}$.

(b) If in addition f satisfies the condition (f_3), *then the problem* (0.1) *has a sequence of solutions* $\{\pm u_n \;:\; n = 1,2...\}$ *such that* $\phi(\pm u_n) < \infty$.

Theorem 2. *Suppose that the Carathéodory function f satisfies* (f_0) *with* $\inf_\Omega\{q(x)-p(x)\} < 0$, *and* (h_1) *holds, then the problem* (0.1) *has a weak solution.*

Theorem 3. *Assume that the function f is of class* C^1 *and satisfies* $(f_0),(f_4)$ *with* $q^- > p^+$, *and* (h_1) *holds, then the problem* (0.1) *has a weak solution .*

The problems studied in the present paper involve a variable exponent. The $p(x)-$Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous ([14, 19, 20]). We restrict to the case of the No-flux boundary value condition.

As far as we are aware, there are very few contributions dealt with the non-local problem involving No flux boundary condition with a variable exponent. Further, we provide sufficient conditions for the existence of solutions of (0.1), since the present results extend those considered withe classical conditions to the best (for instance, here we are more closing to the exponent $p(x)$).

This paper is organized as follows. In Section 2, we recall some preliminary and facts on variable exponent spaces. In Section 3, we give the proof of result and we establish the existence of solutions via variational structure and Leray-Schauder degree.

1 Preliminaries

We introduce the setting of our problem with some auxiliary results. For convenience, we only recall some basic facts which will be used later, we refer to [16, 19, 23] for more details. Let $p \in C_+(\overline{\Omega})$ and define

$$L^{p(x)}(\Omega) = \{\text{u is measurable} : \int_\Omega |u(x)|^{p(x)}\,dx < +\infty\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\| = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 1. *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2. *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$. For any $u \in L^{p(x)}(\Omega)$, then*

(1) for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$.

(2) $|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$.

(3) if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$.

(4) if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$.

(5) $\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$.

(6) $\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.

Proposition 3. *If $q \in C_+(\overline{\Omega})$ and $q(x) \leq p^*(x)$ ($q(x) < p^*(x)$) for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Proposition 4. *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$ holds a.e. in Ω. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)}.$$

Lemma 1. *(cf. [20]) If $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and*

$$|f(x,s)| \leq a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \quad \forall (x,s) \in \overline{\Omega} \times \mathbb{R},$$

where $p_1(x), p_2(x) \in C(\overline{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $p_1(x) > 1$, $p_2(x) > 1$, $a(x) \geq 0$ and $b \geq 0$ is a constant, then the Nemytskii operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by

$$N_f(u)(x) = f(x, u(x))$$

is a continuous and bounded operator.

Note that $W_0^{1,p(x)}(\Omega) \subset W_0^{1,p(x)}(\Omega) \oplus \mathbb{R} \subset W^{1,p(x)}(\Omega)$.

Let $E = W_0^{1,p(x)}(\Omega) \oplus \mathbb{R}$ with the norm $\| \,.\, \|$, and E^* is the dual space of E.

Definition 1. We say that $u \in E$ is a weak solution of (0.1), if

$$h\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)\right)\int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv)dx - \int_\Omega f(x,u)v dx = 0,$$

$\forall v \in E$.

In order to study (0.1) by means of variational methods, we introduce the functional associated

$$\phi(u) = H(I_1(u)) - \int_\Omega F(x,u)dx,$$

for $u \in E$, where

$$I_1(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx,$$

we set,

$$J(u) = H(I_1(u))$$

and

$$\psi(u) = \int_\Omega F(x,u)dx.$$

2 Proof of the main result

Firstly, we give the following result.

Proposition 5. *(i)* $H \in C^1([0,+\infty))$, $H(0) = 0$, $H'(t) = h(t)$ *for any* $t > 0$.

(ii) The functionals $J(u)$ *and* $\psi(u)$ *are sequentially weakly lower semi continuous and thus* ϕ *is sequentially weakly lower semi continuous.*

(iii) The mapping ϕ' *is bounded on each bounded, and is of type* (S_+), *namely*

$$u_n \rightharpoonup u \quad \textit{and} \quad \limsup_{n\to\infty} \phi'(u_n)(u_n - u) \leq 0 \quad \textit{implies} \quad u_n \to u.$$

Proof. The first assertion is immediate. It is obvious that ϕ is continuously Gâteau differentiable, whose Gâteau derivative at $u \in E$,

$$\phi'(u).v = h(I_1(u))\left(\int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla v dx + \int_\Omega |u|^{p(x)-2} uv dx\right) - \int_\Omega f(x,u)v dx,$$

$\forall v \in E$.

Because the functional I_1 is sequentially weakly lower semi-continuous and the fact that H is increasing, so the functional J is sequentially weakly lower semi-continuous and we can see that ψ and ψ' are completely continuous. These mappings are sequentially weakly continuous, then ϕ is sequentially weakly lower semi-continuous.

It is obvious that the mappings J' and ψ' are bounded on each bounded subset. Since ϕ' is the sum of a (S_+) type map J' and ψ' which is weakly-strongly continuous, so ϕ' is also of (S_+) type. $\boxed{QED}$

Proof of Theorem 1. (a) Let us prove that ϕ is coercive, for $\| u \| > 1$, by (h_1) and (f_1) we reach

$$\begin{aligned}
\phi(u) &= H(I_1(u)) - \int_\Omega F(x,u)dx \\
&\geq h_0 \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - h_0(\lambda_1 - \epsilon)\int_\Omega \frac{|u|^{p(x)}}{p(x)} dx + c \\
&\geq h_0 \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \frac{h_0(\lambda_1 - \epsilon)}{\lambda_1}\int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx + c \\
&\geq \frac{h_0}{p^+}\left(1 - \frac{(\lambda_1 - \epsilon)}{\lambda_1}\right) \| u \|^{p^-} + c.
\end{aligned}$$

Hence, ϕ is coercive, by Proposition 1 ϕ has a global minimizer u_1 which is non trivial. Indeed, fix $v_0 \in E \setminus \{0\}$ and $t > 0$ is small enough, so from (h_2) and (f_2) we have that

$$\begin{aligned}\phi(tv_0) &\leq C_2\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|v_0|^{p(x)}dx\right)^\alpha - \int_\Omega F(x,tv_0)dx\\ &\leq C_3 t^{\alpha p^-} - C_4 t^{q_0^+} < 0, \qquad (2.1)\end{aligned}$$

because $q_0^+ < \alpha p^-$.

Remark 1. Assume that $h \in L^\infty(\Omega)$. By using the Mountain Pass Theorem and the fact that ϕ is coercive, we construct a continuous curve

$$\gamma: \ [0,1] \to E, \quad \gamma(0) = 0, \ \gamma(1) = u_1.$$

From this point of view, we can show the existence of an other critical point $u_2 \in E$ of ϕ such that $u_2 \neq u_1$ and u_2 is nontrivial.

(b) Now, it is clear that ϕ is an even functional in E. From the coercivity of ϕ and by proposition 5, to verify that ϕ satisfies (P.S) condition on E, it is enough to verify that any (P.S) sequence is bounded. Hence ϕ satisfies the (P.S) condition in E. Denote by $\gamma(K)$ the genus of K, see [28]. Set

$$\Gamma = \{K \subset E \setminus \{0\} \ : \ K \quad \text{is compact and symetric}\},$$

$$\Gamma_n = \{K \in \Gamma \ : \ \ \gamma(K) \geq n\},$$

$$c_n = \inf_{K\in\Gamma_n} \sup_{u\in K} \phi(u) \ n = 1, 2....$$

Thereby,

$$-\infty < c_1 \leq c_2 \leq ... \leq c_n \leq c_{n+1} \leq ...$$

We check that $c_n < 0$ for every n.

It is known that $C_0^\infty(\Omega)$ is an infinite-dimensional subspace of E, we can choose a n-dimensional linear subspace $E_n \subset C_0^\infty(\Omega) \subset E$.

Set

$$S^{(n)} = \{u \in E_n \ : \ \ \| u \| = 1\}.$$

From the compactness of $S^{(n)}$ and by the condition (f_2) there is a constant $b_n > 0$ such that

$$\int_\Omega a_0|t|^{q_0(x)} \geq b_n, \ \forall u \in S^{(n)},$$

then we can find $t \in]0,1[$ such that

$$\phi(tu) < 0, \ \forall u \in S^{(n)}.$$

Putting $K_n = tS^{(n)}$, so we entail that $\gamma(K_n) = n$ and then $c_n < 0$.

The genus theory and the Ljusternik-Schnirelman theorem show that each c_n is a critical value of the functional ϕ and then the second assertion of the theorem is achieved. $\boxed{QED}$

In the sequel, we are to prove Theorem 2, so for simplicity, we should recall this interesting proposition,

Proposition 6. *(cf. [29]) Let E be a real Banach space, $\widetilde{B}$ be a bounded open subset of E, $A : \overline{\widetilde{B}} \to E$ is compact continuous, I is the identity mapping on E, then the Leray-Schauder degree defined by $deg(I-A, \widetilde{B}, 0)$ of $I-A$ verifies the following assertions:*

(i) $deg(I, \widetilde{B}, 0) = 1$;

(ii) $deg(I-A, \widetilde{B}, 0) \neq 0$ then $Ax = x$ has a solution;

(iii) If $L : \overline{\widetilde{B}} \times [0,1] \to E$ is compact continuous mapping with $L(x, \lambda) \neq x$ for $x \in \partial\widetilde{B}$ and $\lambda \in [0,1]$ then $deg(I - L(.,\lambda), \widetilde{B}, 0)$ does not depend on the choice of λ.

We define the operators

$$\langle Au, v\rangle = h\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)}\nabla u + |u|^{p(x)}\right)dx\right)\int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv)dx$$

and

$$\langle Bu, v\rangle = \int_\Omega f(x,u)vdx, \quad \forall v \in E,$$

where

$$A, B : E \to E^*.$$

A standard argument shows that A is homomorphism and strictly monotone operator.

Lemma 2. *The operator $A^{-1} \circ B$ is compact continuous from E to E where A^{-1} is the inverse operator of A.*

Proof. Let $(u_n)_n$ be a bounded sequence of E, and then up to a subsequence denoted also by $(u_n)_n$, there exists $u \in E$ such that

$$u_n \to u \quad \text{in} \quad L^{q(x)}(\Omega).$$

Therefore, from lemma 1, we infer that Bu_n is strong convergent in E^*. Since A^{-1} is a bounded homeomorphism then $A^{-1} \circ B$ is strong convergent in E. $\boxed{QED}$

Proof Theorem 2. Denote $L_\lambda u: \ E \to E^*$ by

$$\langle L_\lambda u, v\rangle = \lambda \langle Bu, v\rangle, \quad \forall v \in E.$$

We consider the equation

$$Au = L_\lambda u. \tag{2.2}$$

The solutions of (2.2) are uniformly bounded for $\lambda \in [0,1]$, if not, then there exists a sequence of solutions $(u_n)_n$ of (2.2) such that $\| u_n \| \to +\infty$ and

$$h\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx\right)\int_\Omega (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx = \int_\Omega \lambda_n f(x,u_n)u_n dx,$$

with $(\lambda_n)_n \subset [0,1]$.
In view of (f_0) we have

$$\int_\Omega \lambda_n f(x,u_n)u_n dx \le \varepsilon \int_\Omega |u_n|^{p(x)}dx + C(\varepsilon),$$

with $q(x) << p(x)$. Taking $\varepsilon > 0$ small enough, it follows that $(u_n)_n$ is bounded, which is a contradiction.

Therefore, there exists a radius $R > 0$ which all solutions of (2.2) are in the ball $\mathcal{B}(0,R)$.
Applying the Leray-Schauder degree, proposition 6, we entail that

$$deg(I - A^{-1} \circ L_\lambda, \mathcal{B}(0,R), 0) \ = \ deg(I - A^{-1} \circ L_0, \mathcal{B}(0,R), 0),$$

where $L_0 = 0$ and I is the identity mapping on E.

We point out that $I - A^{-1} \circ L_0$ has zero as a unique solution and thus from proposition 6, we obtain

$$\begin{aligned} deg(I - A^{-1} \circ L_1, \mathcal{B}(0,R), 0) &= deg(I - A^{-1} \circ L_0, \mathcal{B}(0,R), 0) \\ &= 1, \end{aligned}$$

and consequently there exists $u \in \mathcal{B}(0,R)$ such that

$$Au - Bu = 0.$$

Once this statement is reached, the proof will be complete. $\boxed{QED}$

Proof of Theorem 3.

Lemma 3. *Under the assumptions of Theorem 3, there exist $\rho > 0$ and $\alpha > 0$ such that*
$$\phi(u) \geq \alpha > 0$$
for all $u \in E$ with $\|u\| = \rho$.

Proof. Let $\|u\| < 1$ sufficiently small, we have

$$\begin{aligned}
\phi(u) &= H(I_1(u)) - \int_\Omega F(x,u)dx \\
&\geq \frac{h_0}{p^+}\|u\|^{p^+} - \varepsilon \int_\Omega |u|^{p^+} - C(\varepsilon)|u|^{q(x)}dx \\
&\geq \frac{h_0}{p^+}\|u\|^{p^+} - \varepsilon C_1\|u\|^{p^+} - C(\varepsilon)C_2\|u\|^{q^-}.
\end{aligned}$$

Due to the fact that $q^- > p^+$, we can choose $\sigma > 0$ and $\rho > 0$ such that $\phi(u) \geq \sigma > 0$ for all $u \in E$ with $\|u\| = \rho$. $\boxed{QED}$

Define
$$\overline{B}_\rho = \{u \in E \ : \ \|u\| \leq \rho\}$$
and
$$\partial B_\rho = \{u \in X \ : \ \|u\| = \rho\}$$
endowed with distance

$$d(u,v) = \|u - v\|, \quad u, v \in \overline{B}_\rho.$$

By the previous Lemma 3, we know that

$$\phi(u)/\partial B_\rho \geq \alpha > 0.$$

It is clear that the functional ϕ is bounded from below.

Set $c_* = \min_{\overline{B}_\rho} \phi(u)$, by using a straightforward computation so we can have $c_* < 0$. By the Ekeland's variational principle in [17], we may find a sequence $(u_n)_n$ such that
$$c_* \leq \phi(u_n) \leq c_* + \frac{1}{n} \tag{2.3}$$
and
$$\phi(v) \leq \phi(u_n) - \frac{1}{n}\|u_n - v\|, \quad \forall v \in \overline{B}_\rho$$
and thus $\|u_n\| \leq \rho$ when $n > 1$ is large enough. The case $\|u_n\| = \rho$ cannot occur. In fact, from the Lemma 3 , $\phi(u_n) \geq \alpha > 0$ and passing to the limit when $n \to \infty$ and combining with (2.3) we reach $0 < \alpha \leq c_* < 0$, which is contradiction and then $\|u_n\| < \rho$.

It remains to show that $\phi'(u_n) \to 0$ in E. Let $u \in E$ such that $\|u\| = 1$, putting $\omega_n = u_n + \lambda u$. Fix $n > 1$ we have

$$\|\omega_n\| \leq \|u_n\| + \lambda < \rho$$

with $\lambda > 0$ is small enough. It yields

$$\phi(u_n + \lambda u) \geq \phi(u_n) - \frac{\lambda}{n}\|u\|$$

then

$$\frac{\phi(u_n + \lambda u) - \phi(u_n)}{\lambda} \geq -\frac{1}{n}\|u\|,$$

tending $\lambda \to 0$ so we get

$$\phi'(u_n).u \geq -\frac{1}{n}$$

that is,

$$|\phi'(u_n).u| \leq \frac{1}{n}$$

with $\|u\| = 1$. So $\phi'(u_n) \to 0$ in E^*. By a standard procedure, there exists $u_* \in E$ such that $\phi'(u_*) = 0$ and $\phi(u_*) = c_* < 0$. $\boxed{QED}$

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