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A Note on Autocamina Groups

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Abstract. In this paper we define Autocamina Group. In Camina group, conjugacy class of an element outside the commutator subgroup coincides with the coset of the commutator subgroup. Similarly we call a group an Autocamina group, if fusion class is the coset of autocommutator subgroup. We study the structure of Autocamina groups in this paper.

Keywords: Camina group, Autocamina group, Frobenius group

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1 Introduction

Let G be a group. Denote by G' and Z(G) the commutator subgroup and the center of G, respectively and by G^*, L(G) the autocommutator subgroup and the absolute center of G, respectively (see[5]). The lower autocentral series and upper autocental series have been defined in [8]. Camina groups have been studied extensively in literature [1, 3]. A.R. Camina [1] gave the structure of Camina groups in his paper. Also in [2], the authors defined Con-Cos group and gave the structure of the Con-Cos group. As in a Con-Cos group, the conjugacy class of an element outside a normal subgroup coincide with the coset, that is cl(x)=xN for all $x \in G\backslash N$, where N is a normal subgroup of G. In case of abelian group N turns out to be trivial, and for non-abelian group N turns out to be G' and this gives G is a Camina Group.
Cangelmi and Muktibodh [2] gave the structure of a group in which Camina kernel is the union of two or three conjugacy classes. Instead of taking conjugacy classes and normal subgroups we proposed to define Auto Con-Cos groups with fusion classes and characteristic subgroups. The first and the second section deals with the definition and basic properties of Auto Con-Cos groups. In the third section we study Autocamina groups with some examples.

The fourth section of this paper is devoted to the study of Autocamina Groups in which the autocommutator subgroup is the union of n- fusion classes.

Let $G^* = K(G) =< [g, \alpha] = g^{-1}\alpha(g)|g \in G, \alpha \in Aut(G) >$, be the autocommutator subgroup of G and $L(G) = \{g \in G|\alpha(g) = g, \alpha \in Aut(G)\}$, be the absolute center of G. The autocommutator of higher weights defined inductively in [8] as follows:
$[g, \alpha_1, \alpha_2, ..., \alpha_i] = [[g, \alpha_1, \alpha_2, ..., \alpha_{i-1}], \alpha_i]$, for all $\alpha_1, \alpha_2, ..., \alpha_i \in Aut(G)$, $i \geq 2$.
The autocommutator subgroup of weight i is defined as:
$K_i(G) = [G, Aut(G), Aut(G), ..., Aut(G)] =< [g, \alpha_1, \alpha_2, ..., \alpha_i]|g \in G, , \alpha_1, \alpha_2, ..., \alpha_i \in Aut(G) >$. Clearly $K_1(G) = G^*$, and $K_i(G)$ is a characteristic subgroup of G, for all $i \geq 1$.Thus, we have a descending chain of autocommutator subgroups of G as:
$G = K_0(G) \supseteq K_1(G) \supseteq K_2(G) \supseteq ... \supseteq K_i(G) \supseteq ...$, called as lower autocentral series of G. Similarly in [8], the upper autocentral series has also been defined as follows: $< 1 >= L_0(G) \subseteq L_1(G) = L(G) \subseteq ... \subseteq L_n(G) \subseteq ...$, where $\frac{L_n(G)}{L_{n-1}(G)} = L(\frac{G}{L_{n-1}(G)})$. If we take the group of inner automorphisms, we obtain the usual lower and upper central series of G.

2 Definitions and Basic Properties

Two elements a and b of a group G are said to be fused [7], if there exists an automorphism α such that $\alpha(a) = b$. We define a fusion relation in G, two elements a, b are related if they are fused. It is easy to check that fusion relation is an equivalence relation and we get fusion class in G, denoted by $\overline{cl(a)} = \{\alpha(a)|\alpha \in Aut(G)\}$, where $a \in G$. We denote the conjugacy class of $a \in G$ by $cl(a) = \{\alpha(a)|\alpha \in Inn(G)\}$
Next we write the set $F_a = \{\alpha \in Aut(G)|\alpha(a) = a\}$, clearly F_a is a subgroup of $Aut(G)$. If $a \in L(G)$, then $\alpha(a) = a$, for all $\alpha \in Aut(G)$. Thus for $a \in L(G)$, $\overline{cl(a)} = \{a\}$. We have

$$|G| = |L(G)| + \textstyle\sum_{a \notin L(G)} |\overline{cl(a)}|,$$

called the fusion class equation for a group G. It is clear that $cl(a) \subseteq \overline{cl(a)}$. Also if $b \in \overline{cl(a)}$, then $cl(b) \subseteq \overline{cl(a)}$. Thus each fusion class is the union of conjugacy classes.
Next we prove the order of $\overline{cl(a)}$ for $a \notin L(G)$ has order equal to $[Aut(G) : F_a]$. Define a map

$$\phi : cl(a) \mapsto S,$$

where S is the set of all left cosets of F_a in $Aut(G)$, by

$$\phi(\alpha(a)) = \alpha F_a$$

Clearly, ϕ is well defined and bijective map. We get $|\overline{cl(a)}| = [Aut(G) : F_a]$. Thus we have $|G| = |L(G)| + \sum_{a \notin L(G)} [Aut(G) : F_a]$.
Let G be a group and K be a proper characteristic subgroup of G. Then we have two partitions of G, one is coset partition and other is fusion class partition. If these two paritions coincide in $G \setminus K$, that is $\overline{cl(g)} = gK$, for all $g \in G\setminus K$, then we call the group G as **Auto con-cos group**.
It is easy to check that $G^* = K$. For $G^* = 1$, we have $L(G) = G$, and therefore G has order 2.

Example 1. The cyclic group C_2 of order 2 is Auto Con-Cos group.

Example 2. $D_8 =< a, b|a^4 = b^2 = 1, bab = a^{-1} >$(the dihedral group of order 8) is con-cos as well as Auto con-cos group with $G^* =< a >$.

Lemma 1. Let G be a finite group and K be a proper characteristic subgroup of G. Then the following two conditions are equivalent:

(1) $\overline{cl(g)} \subseteq gK$, for all $g \in G\setminus K$.

(2) $G^* \leq K$.

Proof is straight forward. Also it is easy to prove the below proposition.

Proposition 1. Let G be an Auto con-cos group and K be a proper characteristic subgroup of G such that $\overline{cl(g)} = gK$, for all $g \in G\setminus K$. Then $K = G^*$ and $G^* = \{[g, \alpha] | \alpha \in Aut(G)\}$, for any $g \in G\setminus G^*$.

Remark 1. $\overline{cl(g)} = gG^*$, for all $g \in G\setminus G^*$ need not to be true in general.

Lemma 2. Any two fused element of a group G belong to the same coset of G^*.

Proof. Let a and b be two fused elements of a group G. Then there exists an automorphism α of G such that $a = \alpha(b)$. Thus we have

$$a^{-1}b = (\alpha(b))^{-1}b = (b^{-1}\alpha(b))^{-1} \in G^*,$$

Thus, $G^*a = G^*b$. QED

3 Autocamina Group

3.1 Autocamina Kernel

An Autocamina kernel in a group G is a proper characteristic subgroup $1 \neq K$ which satisfies the following equivalent conditions:

(1) $K \subseteq \{[g, \alpha | \alpha \in Aut(G)\}$, for any $g \in G \backslash K$.

(2) $gK \subseteq \{\alpha(g) | \alpha \in Aut(G)\} = \overline{cl(g)}$, for all $g \in G \backslash K$.

We call (G, K) as Autocamina pair.

Lemma 3. If (G,K) is an autocamina pair, then $L(G) \leq K$.

Proof. Let $1 \neq z \in L(G)$. If $zK \subseteq cl(z) = \{z\}$, implies $K = 1$, a contradiction. QED

Definition 1 (Autocamina Group)**.** A group G is called an Autocamina group if G^* is Autocamina kernel, that is, (G, G^*) is Autocamina pair.

Remark 2. In an Autocamina group G, $L(G) \leq G^*$.

We have the following proposition

Proposition 2. Let G be a group. Then the following conditions are equivalent:

(1) G is Auto con-cos group.

(2) G is either Autocamina group or $G \cong C_2$.

(3) $xG^* = \overline{cl(x)}$, for all $x \in G \backslash G^*$.

(4) $G^* = \{[x, \alpha] | \alpha \in Aut(G)\}$, for any $x \in G \backslash G^*$.

Proposition 3. Let G be a group(possibly infinite) and $1 \neq K$ is finite Autocamina kernel of G. If N is a proper characteristic subgroup of K, then K/N is Autocamina kernel of G/N, and there exists some $i \geq 1$ for which either $L_i(G) = K$ or $L_i(G) = L_j(G)$ for all $j \geq i$, that is the upper autocentral series $\{L_i(G)\}_{i \geq 1}$ becomes stationary after a finite number of steps.

Proof. We shall prove that $K/N \subseteq \{[gN, \overline{\alpha}] | \overline{\alpha} \in Aut(G/N)\}$, for any $gN \in G/N \backslash K/N$. Let $gN \in G/N \backslash K/N$, therefore $g \notin K, g \notin N$.
Since $K \subseteq \{[g, \alpha] | \alpha \in Aut(G)\}$, therefore $k = [g, \alpha]$ for some $\alpha \in Aut(G)$. We have

$$\begin{aligned} kN &= g^{-1}\alpha(g)N, \\ &= (gN)^{-1}\overline{\alpha}(gN), \\ &= [gN, \overline{\alpha}] \end{aligned}$$

It follows that $K/N \subseteq \{[gN, \overline{\alpha}] | \alpha \in Aut(G)\} \subseteq \{[gN, \overline{\alpha}] | \overline{\alpha} \in Aut(G/N)\}$

Now, suppose the second case does not hold,that is the chain $\{L_i(G)\}_{i\geq 1}$ does not become stationary, and if $L(G) \subset K$, then the pair (G/L(G), K/L(G)) is Autocamina pair. Thus $L(G/L(G)) \leq K/L(G)$, and hence $L_2(G) \leq K$. If $L_2(G) \subset K$, then $(G/L_2(G), K/L_2(G))$ is Autocamina pair. Continuing in this way, we get $L_i(G) = K$, for some $i \geq 1$.

QED

Remark 3. If G is an Autocamina group in which the chain of upper autocentral factor is not stationary, then $G^* = L_i(G)$ for some $i \geq 1$, hence implies G is nilpotent

Now denote the centralizer of an element 'a' in G as $C(a)$ and $D(a) = \{x \in G | [x, a] \in L(G)\}$. Now denote the sets for any element 'a' in G by
$F_a = \overline{C(a)} = \{\alpha \in Aut(G) | [a, \alpha] = 1\}$ and
$\overline{D(a)} = \{\alpha \in Aut(G) | [a, \alpha] \in L(G)\}$. Thus $[Aut(G) : \overline{C(a)}] = |\overline{cl(a)}|$.
$\overline{C(a)}$ is subgroup of $\overline{D(a)}$ which is a subgroup of $Aut(G)$. We have the following deductions for the above defined subgroups

Lemma 4. Let G be a group and $\overline{C(g)}$ $\overline{D(g)}$ denote the subgroups of G defined above where $g \in G$. Then for every element 'g' in G we have

(1) $C(g)/Z(G) \cong Inn(G) \cap \overline{C(g)}$.

(2) $D(g)/Z(G) \cong Inn(G) \cap \overline{D(g)}$.

(3) $|cl(g)|$ divides $|\overline{cl(g)}|$.

Proof. It is easy to check that $x \in C(g)$ if and only if the inner automorphism induced by x denoted by f_x belongs to $\overline{C(g)}$. Thus there exists a surjective homomorphism from $C(g)$ to $Inn(G) \cap \overline{C(g)}$ which maps x to f_x. The kernel of this homomorphism is $Z(G)$. This gives (1).
Similarly, it is easy to check that $x \in D(g)$ if and only if the inner automorphism induced by x denoted by f_x belongs to $Inn(G) \cap \overline{D(g)}$. Thus there exists a surjective homomorphism from $D(g)$ to $\overline{D(g)}$ which maps x to f_x. The kernel of this homomorphism is $Z(G)$, and this proves (2)
Now to prove the part(3), we proceed as follows: .
We know that for $g \in L(G)$, $|cl(g)| = |\overline{cl(g)}| = 1$. Suppose $g \in G \backslash L(G)$. We have $[Aut(G) : \overline{C(g)}] = |\overline{cl(g)}|$.

$$\begin{aligned}\text{Therefore,}\ \overline{cl(g)} &= [Aut(G):\overline{C(g)}]\\ &= [Aut(G):Inn(G)\overline{C(g)}][Inn(G)\overline{C(g)}:\overline{C(g)}]\\ &= [Aut(G):Inn(G)\overline{C(g)}][Inn(G):Inn(G)\cap\overline{C(g)}]\\ &= [Aut(G):Inn(G)\overline{C(g)}][G/Z(G):C(g)/Z(G)]\\ &= [Aut(G):Inn(G)\overline{C(g)}][G:C(g)]\\ &= [Aut(G):Inn(G)\overline{C(g)}]|cl(g)|\end{aligned}$$

QED

Theorem 1. Let G be a group. Then the following four statements are equivalent:

(1) $gN \subseteq \overline{cl(g)}$, for all $g \in G\backslash N$.

(2) $N \subseteq \{[g,\alpha]|\alpha \in Aut(G)\}$, for $g \in G\backslash N$.

(3) $[\overline{D(g)}:\overline{C(g)}] = |N|$.

(4) $|\overline{C_{Aut(G)}(gN)}| = |\overline{C(g)}||N|$.

where $\overline{C_{Aut(G)}(gN)} = \{\overline{\alpha}|\alpha \in Aut(G), \overline{\alpha}(xN) = xN\}$, $\overline{D}(g) = \{\alpha \in Aut(G)|[g,\alpha] \in N\}$ and $\overline{C}(g) = \{\alpha \in Aut(G)|[g,\alpha] = 1\}$

Proof. (1) $\Rightarrow$ (2) is trivial.
To prove (2) $\Rightarrow$ (3) Clearly $\overline{C(g)}$ is a subgroup of $\overline{D(g)}$. Define a mapping $\phi : S \mapsto N$ by, where S is the set of right cosets of $\overline{C(g)}$ in $\overline{D(g)}$

$$\phi(\alpha\overline{C(g)}) = [g,\alpha].\ \text{Then}\ \phi\ \text{is well defined bijective map.}$$

Also it is clear that $|\overline{D(g)}| = |\overline{C_{Aut(G)}(gN)}|$ Thus $\frac{|\overline{D(g)}|}{|\overline{C(g)}|} = |N|$ this gives (3) and hence (4).
(4) $\Rightarrow$(1).
We have one to one correspondance between the set S and the set$\{[x,\alpha]|\alpha \in \overline{D(g)}\}$. Therefore, $[\overline{D(g)}:\overline{C(g)}] \leq |N|$. Equality can occur only if to each $n \in N$, there exists $\alpha \in \overline{D(g)}$ such that $[g,\alpha] = n$. QED

Remark 4. It is easy to check that the group $\overline{D(g)}/\overline{C(g)}$ is abelian.

Lemma 5. (See [4] lemma 1.1) Let N be normal subgroup of a group G, $x \in N$. Then $|cl_N(x)|/|cl(x)|$.

Theorem 2. If N is a characteristic subgroup of a group G such that $N \cap L(G) = 1$, and for all $1 \neq x \in N$, $|\overline{cl(x)}| = m$, then N is abelian.

Proof. Let N be the union of $k+1$ fusion classes. Then $|N| = 1 + km$, implies $(|N|, m) = 1$. Now $|cl_N(x)|/|cl(x)|/|\overline{cl(x)}|$ and $|\overline{cl(x)}| = m$. Also $|cl_N(x)|/|N|$. Therefore $|cl_N(x)| = 1$, hence N is abelian. $\boxed{QED}$

Proposition 4. Let G be an Autocamina group. Then every characteristic subgroup outside G^* is abelian.

Proof. The statement easily follows as the characteristic subgroups outside G^* have same order. $\boxed{QED}$

Theorem 3. If G is a cyclic Autocamina group, then G is a 2-group.

Proof. Let G be a cyclic Autocamina group. Suppose $G =< a >$, if $|G|$ is odd number, then $G = G^*$, therefore $|G|$ is even. Let $|G| = 2^k m$, where m is odd. Suppose $G =< a, b | O(a) = 2^k, O(b) = m, [a, b] = 1 >$. Then it is easy to check that $G^* =< a^2, b >$, and $L(G)$ is not contained in G^*, a contradiction. Thus G must be a 2-group. $\boxed{QED}$

Theorem 4. If G is non cyclic abelian Autocamina group, then G is 2-group.

Proof. Let $G = S_{p_1} \times S_{p_2} \times ...S_{p_n}$, where $S_{p_i}s$ are Sylow p_i-subgroups of G. Then $Aut(G) = Aut(S_{p_1}) \times Aut(S_{p_2}) \times ...Aut(S_{p_n})$. WLOG suppose G is a p-group. Write G as the direct product of cyclic subgroups. If any one of the direct factor of G is of odd order, then $G = G^*$, therefore G must be 2-group. $\boxed{QED}$

Now we shall find the structure of non abelian Autocamina groups.

Theorem 5. If (G, K) is an Autocamina pair with $K \leq Z(G)$, then G is a p-group, and if $G\backslash K$ has an element of order p, then K is an elementary abelian p-group.

Proof. Since $K \subseteq \{[g, \alpha] | \alpha \in Aut(G)\}$, for $g \in G\backslash K$, therefore to each element $x \in K$ and $g \in G\backslash K$, there exists an automorphism say α, such that $\alpha(g) = gx$. Therefore, $|g| = |gx|$ for each $x \in K, g \in G\backslash K$. Thus $|x|$ divides $|g|$ for each $x \in K, g \in G\backslash K$. Now suppose p and q are two distinct prime divisors of order of G. If K is a p-group or q-group then we can choose an element g in $G\backslash N$ of q-power or p-power order respectively. We get a contradiction. Therefore suppose both prime p and q divides the order of K. Now at least one of the Sylow p-subgroup or Sylow q-subgroup is not contained in N. WLOG suppose N is a Sylow p-subgroup of G not contained in K. Now we can choose an element g in N not in K of p-power order and x in K of q-power order. Again we have a contradiction. Thus G is a p-group.

QED

Theorem 6. If (G,K) is Autocamina pair and $|K|$ and $[G : K]$ are of coprime order, then G is Frobenius group with kernel K.

Proof. The proof can be obtained arguing as in [1] Prop.1, and by using the Lemma 2.4 . To prove the result Camina used the fact that order of element and its conjugate are same. We also have the fact that the order of element and its fusion element are same.

QED

Corollary 1. If G is an Autocamina group such that $G^* \leq Z(G)$, then G is a p-group.

Proposition 5. Let G be an Autocamina group. Then G^* intersects with every non trivial characteristic subgroup of G.

Proof. Let H be a non trivial characteristic subgroup of G. As G is an Autocamina group, therefore $L(G) \leq G^*$. Thus we may suppose that $H \cap L(G) = 1$. Assume on contrary $H \cap G^* = 1$. Let $1 \neq a \in H$. It follows that there exists an automorhism say α such that $a^{-1}\alpha(a) \in G^*$. Also $a^{-1}\alpha(a) \in H$. We have $H \cap G^* \neq 1$, a contradiction. QED

Remark 1. If G is an Autocamina group such that G^* is a π- group, then G can not have a $\pi^{'}$ characteristic subgroup.

Remark 2. It is easy to see that for a group G having $Aut(G)$ same as $Aut_c(G)$, where $Aut_c(G)$ is the group of all class preserving automorphisms of G. G is Camina if and only if G is Autocamina as $cl(x) = \overline{cl(x)}$, for all $x \notin G' = G^*$

3.2 Examples of Autocamina Groups

[1] $D_{2n} =< a, b|a^n = b^2 = 1, b^{-1}ab = a^{-1} >$ (the dihedral group of order 2n):here $G^* =< a >$.

[2] $C_{2^n} =< a >$(cyclic group of order 2^n): here $G^* =< a^2 >$.

[3] S_3(Permutation group of degree 3): here $G^* = G' = A_3$.

[4] $D_{2n} \times C_2 =< a, b, c|a^n = b^2 = c^2 = 1, b^{-1}ab = a^{-1}, [a, c] = [b, c] = 1 >$: here $G^* =< a, c|a^n = c^2 = 1, [a, c] = 1 >= C_n \times C_2$.

[5] $D_{2n} \times C_{2^2} =< a,b,c|a^3 = b^2 = c^4 = 1, b^{-1}ab = a^{-1}, [a,c] = [b,c] = 1 >$:here $G^* =< a, c^2|a^n = c^4 = 1, [a,c] = 1 >= C_n \times C_2$.

[6] $S_3 \times C_2 =< a,b,c|a^n = b^2 = c^2 = 1, b^{-1}ab = a^{-1}, [a,c] = [b,c] = 1 >$:here $G^* =< a, c|a^3 = c^2 = 1, [a,c] = 1 >= C_3 \times C_2 \cong C_6$.

[7] $Z_3 \rtimes Z_4 =< a,b|a^3 = b^4 = 1, b^{-1}ab = a^2 >$ (Semi direct product of cyclic groups of order 3 and 4):here $G^* =< a, b^2 >$ is a cylic group of order 6 (This group is not Camina group).

4 n-Autocamina

As we know any two fused element belongs to the same coset of G^*. Thus G^* is the union of complete fusion classes. In [2], the authors defined n-Con-Cos group. On the similar line, we define in this section n-Auto Con-Cos and n-Autocamina group and characterize the properties of such groups.

Definition 2. An n-Autocamina(n-Auto Con Cos) group, where n $\geq$ 2 is an Autocamina(Auto Con-Cos) group in which G^* is the union of n-fusion classes.

Proposition 6. Let G be an n-Autocamina group. Then $|L(G)| \leq n$. If $|L(G)| = n$, then $L(G) = G^*$ and for each $a \in G^*, cl(a) = \overline{cl(a)} = \{a\}$.

Proof. Let $G^* = 1 \cup \overline{cla_2} \cup \overline{cla_3} \cup .... \cup \overline{cla_n}$. Since $L(G) \leq G^*$, and for each $x \in L(G)$ we have $\overline{cl(x)} = \{x\} = cl(x)$, therefore $|L(G)| \leq n$. If $|L(G)| = n$, then $G^* = L(G)$.

QED

Example 3. (1) A cyclic group of order 2^n is n-Autocamina.
(2) D_8, the dihedral group of order 8 is 3- Autocamina.
(3) S_3, the permutation group of degree 3 is 2-Autocamina.
(4) $S_3 \times C_2$ is 4-Autocamina.
(5) $C_4 \times C_2$ is 2-Autocamina.

Now we study the properties of 2-Autocamina groups.

Theorem 7. If G is a finite 2-Autocamina group, then the following statements are true:

(1) G^* is minimal characteristic subgroup of G, hence G^* is characteristically simple. Therefore $G' = G^*$, when G is non abelian.

(2) G^* is elementary abelian p-group.

(3) If $|L(G)| = 1$, then a subgroup H of G is characteristic subgroup of G if and only if $G^* \leq H$.

(4) If $|L(G)| = 1$, then G^* is unique minimal characteristic subgroup of G

(5) If $Z(G) \neq 1$, then $G^* \leq Z(G)$, implies G is a p-group

(6) $|G|$ is a composite number

(7) $|Aut(G)|$ is divisible by two consequtive integers.

(8) If $|G^*| > 2$, then $|L(G)| = 1$.

Proof. (1) Since order of each non identity element in same fusion class is same. Therefore each non identity element in G^* has same order. Therefore each element in G^* has prime order say p. Thus G^* is a p-group. Let $1 \neq H$ be any characteristic subgroup of G such that $1 \neq H \leq G^*$. Choose $1 \neq x \in H \leq G^*$, then $\overline{cl(x)} \subseteq H \leq G^*$, thus $\overline{cl(x)} = \overline{cl(a)}$, we get $H = G^*$.

(2) Since $G^{*'}$ is proper characteristic subgroup of G, by above part $G^{*'} = 1$, hence G^* is abelian, and therefore elementary abelian.

(3) Suppose H is characteristic subgroup of G other than G^*. Then $G^* \cap H$ is also characteristic subgroup of G contained in G^*. We claim $G^* \cap H \neq 1$. Suppose $1 \neq g \in H$, then there exists an automorphism of G, say α such that $\alpha(g) \neq g$. Thus $1 \neq g^{-1}\alpha(g) \in G^* \cap H$. Now the result follows from the part(1). Conversely, suppose that H is a subgroup of G such that $G^* \leq H$. Let $h \in H$. Then $h^{-1}\alpha(h) \in G^* \leq H$, for all $\alpha \in Aut(G)$. Thus $\alpha(h) \in H$, for all $\alpha \in Aut(G)$. Hence H is characteristic in G.

(4) Proof follows from above part.

(5) If $|L(G)| = 2$, then $G^* = L(G) \leq Z(G)$. Suppose $|L(G)| = 1$, and result follows from the above part.

(6) Suppose $|G^*| = m, [G : G^*] = k$, suppose $G^* = 1 \cup \overline{cl(a)}, a \in G$. Number of fusion classes are k+1. $|G| = km =$ (number of fusion classes -1)$|G^*|$. Hence

$|G|$ is a composite number.

(7) Suppose $G^* = 1 \cup \overline{cl(a)}, a \in G$. As $|G^*| = |G^*x| = |\overline{cl(x)}|, x \in G\backslash G^*$ divides $|Aut(G)|$, and $|G^*| - 1 = |\overline{cl(a)}|$ divides $|Aut(G)|$.

(8) It is holds trivially. $\boxed{QED}$

Proposition 7. If G is a 2-Autocamina group with $|Z(G)| = 1$. Then G^* is elementary abelian p-group and hence any Sylow q-subgroup is not characteristic $(p \neq q)$.

Proof. Let Q be a Sylow $q-$ subgroup of G. If Q is characteristic, then $G^* \leq Q$, this is not possible. Thus Q can not be characteristic. $\boxed{QED}$

S_3 is an example where the above proposition can be verified.

Proposition 8. Let G be a 2-Camina group in which $G' = G^*$, then G is 2-Autocamina.

Proof. Proof is straight forward. $\boxed{QED}$

Proposition 9. If G is a non abelian group such that $G^{'} = G^* = L(G) \cong C_2$, then G is 2-Autocamina 2-group.

Proof. Since $\overline{cl(x)} \subseteq G^*x$, for all $x \in G\backslash G^*$, thus $|\overline{cl(x)}| = 2 = |G^*|$. In fact G is 2-Camina 2-group. $\boxed{QED}$

Remark 3. The dihedral group of order 8 is 2-group in which $G' = L(G) = C_2$, but it is not 2-Autocamina, in fact, it is 3-Autocamina.

Theorem 8. Let G be a finite non abelian 2-Autocamina group. Then G is one of the following types:
(1) G is 2-group.
(2) G is a metabelian p-group.
(3) G is Frobenius group with kernel G^* an elementary abelian group and complement is abelian.

Proof. Suppose $|L(G)| \neq 1$. Then $L(G) = G^* \cong C_2$. Thus G is a 2-group. Therefore, suppose $|L(G)|$=1. Now suppose $|Z(G)| \neq 1$, then by theorem 7, G is a p-group. Also $G' = G^*$, and G/G^* is abelian. Thus G is a metabelian p-group.
Now, for $|Z(G)| = 1$, $|G|$ has atleast two prime divisors. If $(|G^*|, |G/G^*|) = 1$, then G is a Frobenius group with kernel G^* as an elementary abelian group and complement is abelian. Now suppose p, q be two prime divisors of $|G|$. Suppose p divides $|G^*|$, and p, q both divides $|G/G^*|$. Let Q/G^* be a Sylow

q-subgroup of G/G^*. Thus there exists a Sylow q- subgroup Q' of G such that $Q = Q'G^* = Q' \times G^*$, since $G^* \subset Q$, therefore Q is characteristic subgroup of G, implies Q' is characteristic subgroup of G, a contradiction by Prop. 7. Therefore $(|G^*|, |G/G^*|) = 1$ $\boxed{QED}$

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