Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **35** (2015) no. 2, 17–37. doi:10.1285/i15900932v35n2p17

(m,p)-hyperexpansive mappings on metric spaces

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Received: 22.12.2014; accepted: 2.4.2015.

Abstract. In the present paper, we define the concept of (m,p)-hyperexpansive mappings in metric space,which are the extension of (m,p)-isometric mappings recently introduced in [13]. We give a first approach of the general theory of these maps.

Keywords: metric space, (m,p)-isometric, expansive maps , hyperexpansive maps.

MSC 2010 classification: primary 54E40, secondary 47B99.

1 Introduction and notations

The introduction of the concept of m-isometric transformation in Hilbert spaces by Agler and Stankus yielded a flow of papers generalizing this concept both in Hilbert and Banach spaces, for example (see [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 21, 22]).

An operator T acting on a Hilbert space $\mathcal{H}$ is called m-isometric for some integer $m \geq 1$ if

$$\sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} T^{*k}T^k = 0 \tag{1.1}$$

where $\binom{m}{k}$ be the binomial coefficient. A simple manipulation proves that (1.1) is equivalent to

$$\sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0, \quad \text{for all } x \in \mathcal{H} \tag{1.2}$$

Evidently, an isometric operator (i.e., a 1-isometric operator) is an m-isometric for all integers $m \geq 1$. Indeed the class of m-isometric operators is a generalization of the class of isometric operators and a detailed study of this class and in particular 2-isometric operators on a Hilbert space has been the object of some intensive study, especially by J.Agler and M. Stankus in [2], [3] and [4], but also by S.M. Patel [23]. B.P.Duggal [15, 16] studied when the tensor product of operators is an m-isometry.

A generalization of m-isometries to operators on general Banach spaces has been presented by several authors in the last years. Botelho [14] and Sid Ahmed [21] discussed operators defined via (1.2) on (complex) Banach spaces. Bayart introduced in [8] the notion of (m, p)-isometries on general (real or complex) Banach spaces. An operator T on a Banach space X into itself is called an (m, p)-isometry if there exists an integer $m \geq 1$ and a $p \in [1, \infty)$, with

$$\forall\, x \in X, \quad \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|T^{m-k}x\|^p = 0 \tag{1.3}$$

It is easy to see that, if $X = \mathcal{H}$ is a Hilbert space and $p = 2$, this definition coincides with the original definition (1.1) of m-isometries. In [19] the authors took off the restriction $p \geq 1$ and defined (m, p)-isometries for all $p > 0$. They studied when an (m, p)-isometry is an (μ, q)-isometry for some pair (μ, q). In particular, for any positive real number p they gave an example of an operator T that is a $(2, p)$-isometry, but is not a $(2, q)$-isometry for any q different from p. In [9, 10] it is proven that the powers of an m-isometry are m-isometries and some products of m-isometries are again m-isometries.

The authors, O.A.M. Sid Ahmed and A. Saddi introduced the concept of (A, m)-isometric operators. They gave several generalizations of well known facts on m-isometric operators according to semi-Hilbertian space structures.We refer the reader to [22] for more details about (A, m)-isometric operators. Recently, B.P. Duggal has introduced the concept of an $A(m, p)$-isometry of a Banach space, following a definition of Bayart in the Banach space.

Definition 1.1. ([17]) Let T and $A \in \mathcal{B}(X)$ (the set of bounded linear operators from X into itself), m is a positive integer and $p > 0$ a real number. We say that T is an $A(m,\ p)$-isometry if, for every $x \in X$

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|AT^{m-k}x\|^p = 0. \tag{1.4}$$

For any $T \in \mathcal{B}(\mathcal{H})$ we let

$$\theta_m(T) := \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*k}T^k. \tag{1.5}$$

The Concept of completely hyperexpansive operators on Hilbert space has attracted much attention of various authors. In [1], J. Agler characterized subnormality with the positivity of $\theta_m(T)$ in (1.5) and also extended his inequalities to the concept of m-isometry (cf. [2−4]). On the other hand, A. Athavale considered completely hyperexpansive operators in [5]. In further studies, mainy authors have studied k-hyperexpansive (cf. [7, 18]). The concept of (A, m)-expansive operators on Hilbert space was introduced in [20].

Definition 1.2. $\big([18]\big)$ An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be

(i) m-isometry $(m \geq 1)$ if $\theta_m(T) = 0$.

(ii) m-expansive $(m \geq 1)$ if $\theta_m(T) \leq 0$.

(iii) m-hyperexpansive $(m \geq 1)$, if $\theta_k(T) \leq 0$ for $k = 1, 2, ..., m$.

(iv) Completely hyperexpansive if $\theta_m(T) \leq 0$ for all m.

We refer the reader to [6, 7, 18] for recent articles concerning this subject.

In [8] the author defined $\beta_k^{(p)}(T,\ .) : X \longrightarrow \mathbb{R} : \quad x \longmapsto \beta_k^{(p)}(T,\ x)$ by

$$\beta_k^{(p)}(T,\ x) = \frac{1}{k!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} \|T^j x\|^p, \ \forall\ x \in X \tag{1.6}$$

For $k, n \in \mathbb{N}$ denote the (descending Pochhammer) symbol by $n^{(k)}$, i.e.

$$n^{(k)} = \begin{cases} 0, \ \text{ if } \ n = 0 \\ 0 \text{ if } n > 0 \ \text{ and } \ k > n \\ \binom{n}{k} k! \ \text{ if } \ n > 0 \ \text{ and } \ k \leq n. \end{cases}$$

Then for $n > 0,\ k > 0$ and $k \leq n$ we have

$$n^{(k)} = n(n-1)...(n-k+1).$$

It was proved in [8, Proposition 2.1] that

$$\|T^n x\|^p = \sum_{0 \leq k \leq m-1} n^{(k)} \beta_k^{(p)}(T,\ x) \tag{1.7}$$

for all integers $n \geq 0$ and $x \in X$. In particular,

$$\beta_{m-1}^{(p)}(T,\ x) = \lim_{n \longrightarrow \infty} \frac{\|T^n x\|^p}{\binom{n}{m-1}(m-1)!} \geq 0$$

with equality if and only if T is $(m-1;\ p)$-isometric.

In recent work T. Bermúdez, A. Martinôn and V. Müller introduced the concept of (m, p)-isometric maps on metric spaces (see [13]).

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.3. ([13]) Let E be a metric space. A map $T : E \longrightarrow E$ is called an (m,p)-isometry,($m \geq 1$ integer and $p > 0$) if, for all $x, y \in E$

$$\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}d\big(T^{m-k}x, T^{m-k}y\big)^p = 0. \tag{1.8}$$

For $m \geq 2$, T is a strict (m,p)-isometry if it is an (m,p)-isometry, but is not an $(m-1,p)$-isometry.

For any $p > 0$; $(1,\ p)$-isometry coincide with isometry, that is $d\big(Tx, Ty\big) = d\big(x, y\big)$ for all $x, y \in E$. Every isometry is an $(m,\ p)$-isometry for all $m \geq 1$ and $p > 0$. Many results known in the Banach space setting are established in [13] for metric spaces. For example, an (m,p)- isometry is an $(m+1,p)$-isometry and any power of (m,p)-isometry is again an (m,p)-isometry.

Let $T : E \longrightarrow E$ is an $(m,\ p)$-isometry. In [13] the authors defined $f_T(h,\ p,\ x, y)$ for $h \in \mathbb{N}$, a positive real number p and $x, y \in E$ by :

$$f_T(h,\ p,\ x, y) = \sum_{0\leq k\leq h}(-1)^{h-k}\binom{h}{k}d\big(T^kx, T^ky\big)^p. \tag{1.9}$$

We have from (1.9) that

$$d\big(T^nx, T^ny\big)^p = \sum_{0\leq k\leq m-1}\binom{n}{k}f_T(k,p;\ x, y). \tag{1.10}$$

for all $n \geq 0$ and $x, y \in X$ (see [13]).

Definition 1.4. ([5]) A real- valued function Ψ on $\mathbb{N}_0$ is said to be

(1) completely monotone if $\Psi \geq 0$ and $\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\Psi(n+k) \geq 0$, $\forall\ n \geq 0$ and $m \geq 1$.

(2) completely alternating if $\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\Psi(n+k) \leq 0$,$\forall\ n \geq 0$ and $m \geq 1$.

The content of this paper is as follows. In Section one we set up notation and terminology. Furthermore, we collect some facts about (m,p)-isometries. In Section two, we introduce and study the concept of (m,p)-expansive and hyperexpansive mappings on a metric space and we investigate various structural properties of this classes of mappings.We prove that $(2,p)$-hyperexpansive mappings which are (m,p)-expansive must be $(m-1,p)$-expansive for $m \geq 2$. Recall that if T is an m-isometry (resp. k-expansive or (A,m)-expansive) operator, then so are all its power T^n; for $n \geq 1$ (cf [9, 18, 20]). It turns out that the same assertion remains true for $(2,p)$-hyperexpansive and completely p-hyperexpansive mapping (Theorem 2.3 and Theorem 2.4). Moreover, we prove that the intersection of the class of completely p-hyperexpansive mapping and the class of (m,p)-isometries for $m \geq 2$ is the class of $(2,p)$-isometries (Proposition 2.10). The section three of this paper is an attempt to develop some properties of the class of (m,p)-expansive mappings in seminormed spaces parallel to those of m-isometries.

2 (m,p)-Hyperexpansive maps in metric spaces

In this section, let (X,d) be a metric space, $T : X \longrightarrow X$ is a map, $m \in \mathbb{N}$ and $p > 0$ is a real number. We define the quantity

$$\Theta_m^{(p)}(d,T;x,y) := \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} d\big(T^k x, T^k y\big)^p,$$

for all $x,y \in X$ and we give several results on (m,p) expansive and hyperexpansive mappings on a metric space.

In the following definition, $\Theta_m^{(p)}(d,\ T;\ x,y) \leq 0$ $\big($resp. $\Theta_m^{(p)}(d,T;\ x,y) \geq 0\big)$ really means $\Theta_m^{(p)}(d,T;\ x,y) \leq 0$ for all $x,y \in X$ $\big($resp. $\Theta_m^{(p)}(d,T;\ x,y) \geq 0$ for all $x,y \in X\big)$.

Definition 2.1. Let $T : X \longrightarrow X$ be a map. We say that

(i) T is (m,p)-expansive if $\Theta_m^{(p)}(d,\ T;\ x,y) \leq 0$.

(ii) T is (m,p)-hyperexpansive if $\Theta_k^{(p)}(d,T;\ x,y) \leq 0$ for $k = 1,2,...,m$.

(iii) T is completely p-hyperexpansive if T is (k,p)-expansive for all $k \in \mathbb{N}$.

(iv) T is (m,p)-contractive if $\Theta_m^{(p)}(d,T;x,y) \geq 0$.

(v) T is (m,p)-hypercontractive if $\Theta_k^{(p)}(d,\ T;\ x,y) \geq 0$ for $k = 1,2,...,m$.

(vi) T is completely p-hypercontractive if T is (k,p)-contractive for all $k \in \mathbb{N}$.

For any $p > 0$, $(1,p)$-expansive coincide with expansive; that is, maps T satisfying $d(Tx,Ty) \geq d(x,y)$, for all $x,y \in X$.

For any $p > 0$, $(1,p)$-contractive coincide with contractive; that is, maps T satisfying $d(Tx,Ty) \leq d(x,y)$, for all $x,y \in X$. (m,p)-isometries maps are special cases of the class of (m,p)-expansive and contractive maps.

We consider the following examples of (m,p)-expansive map and (m,p)-contractive map which are not (m,p)-isometric map.

Example 2.1. Let $X=\mathbb{R}$ be equipped with the Euclidean metric $d\big(x,y\big)=|x-y|$ for all $x,y\in X$. Define $T:X\longrightarrow X$ by $Tx=2x$. Clearly $\Theta_m^{(p)}(d,\ T;\ x,y)=\big(1-2^p\big)^m|x-y|^p$. So we can say that T is neither (m,p) -isometric, for all $m\geq 1$ and $p>0$. However, one can easily verify that T is (m,p) expansive map for positive odd integer m and (m,p)-contractive map for positive even integer m.

Remark 2.1. Every (m,p)-expansive map T is injective. In fact if $Tx=Ty$ then $T^kx=T^ky$ for $k=1,2,...,m$ and from (i) of Definition 2.1 we obtain $d(x,y)\leq 0$ i.e $x=y$. Hence T is an injective map.

We not that an (m,p)-expansive map is in general not an $(m+1,p)$-expansive, as we shown in the following example.

Example 2.2. Consider the usual metric $d(x,y)=|x-y|$ on $\mathbb{R}$. Let $T:(\mathbb{R},d)\longrightarrow(\mathbb{R},d)$ defined by $Tx=1+2x$.Then it is easy to see that $d(Tx,Ty)\geq d(x,y)$ and

$$d(T^2x,T^2y)^p-2d(Tx,Ty)^p+d(x,y)^p=(2^p-1)^2|x-y|^p\nleq 0.$$

Clearly T is $(1,p)$-expansive which is not $(2,p)$-expansive.

Remark 2.2. We note the following:

(1) $\Theta_m^{(p)}(d,T,\ x,y)\leq 0\Longleftrightarrow\Theta_m^{(p)}(d,T,\ T^nx;T^ny)\leq 0,\ \forall\, x,y\in X,\ \forall\, n\in\mathbb{N}_0.$

(2) $\Theta_m^{(p)}(d,T,\ x,y)\geq 0\Longleftrightarrow\Theta_m^{(p)}(d,T,\ T^nx;T^ny)\geq 0,\ \forall\, x,y\in X,\ \forall\, n\in\mathbb{N}_0.$

Remark 2.3. We deduce from ([5], Proposition 1 and Proposition 2) the following characterizations of completely p-hyperexpansive and completely p-hypercontractive maps.

(1) A map $T:X\longrightarrow X$ is completely p-hyperexpansive if and only if for every $x,y\in X$, the map $n\longmapsto\Psi_{(T,\,p,\,x,y)}(n)=d\big(T^nx,T^ny\big)^p$ is completely alternating.

(2) A map $T:X\longrightarrow X$ is completely p-hypecontractive if and only for every $x,y\in X$, the map $n\longmapsto\Psi_{(T,\,p,\,x,y)}(n)=d\big(T^nx,T^ny\big)^p$ is completely monotone.

In the next proposition we invoke the following relation which plays an important role in the proof of main results.

Proposition 2.1. For a map $T:X\longrightarrow X$, $m\in\mathbb{N}$, real number $p>0$ and $x,y\in X$, we have that

$$\Theta_m^{(p)}(d,T;\ x,y)=\Theta_{m-1}^{(p)}(d,T;\ x,y)-\Theta_{m-1}^{(p)}(d,T;\ Tx,Ty).\qquad(2.1)$$

Proof. By the standard formula $\binom{m}{j} = \binom{m-1}{j} + \binom{m-1}{j-1}$ for binomial coefficients we have the equalities

$$\begin{aligned}
& \Theta_m^{(p)}(d,T;\ x,y) \\
= & \sum_{0\leq k\leq m} (-1)^k \binom{m}{j} d\big(T^k x, T^k y\big)^p \\
= & d\big(x,y\big)^p + \sum_{1\leq k\leq m-1} (-1)^k \binom{m}{k} d\big(T^k x, T^k y\big)^p + (-1)^m d\big(T^m x, T^m y\big)^p \\
= & d\big(x,y\big)^p + \sum_{1\leq k\leq m-1} (-1)^k \Big(\binom{m-1}{k} + \binom{m-1}{k-1}\Big) d\big(T^k x, T^k y\big)^p + \\
& +(-1)^m d\big(T^m x, T^m y\big)^p \\
= & \Theta_{m-1}^{(p)}(d,T;\ x,y) - \Theta_{m-1}^{(p)}(d,T;\ Tx,Ty).
\end{aligned}$$

QED

Remark 2.4. We note the following equivalences:

(1) T is $(m,p)-$ expansive $\iff \forall\, x,y \in X$

$$\sum_{\substack{0\leq k\leq m \\ k \text{ even}}} \binom{m}{k} d\big(T^k x, T^k y\big)^p \leq \sum_{\substack{0\leq k\leq m \\ k \text{ odd}}} \binom{m}{k} d\big(T^k x, T^k y\big)^p$$

(2) T is $(m,p)-$ contractive $\iff \forall\, x,y \in X$

$$\sum_{\substack{0\leq k\leq m \\ k \text{ even}}} \binom{m}{k} d\big(T^k x, T^k y\big)^p \geq \sum_{\substack{0\leq k\leq m \\ k \text{ odd}}} \binom{m}{k} d\big(T^k x, T^k y\big)^p$$

Lemma 2.1. Let $T : X \longrightarrow X$ be an $(2,\ p)$-expansive mapping. Then the following properties hold

(1) $d\big(Tx,Ty\big)^p \geq \dfrac{n-1}{n} d\big(x,y\big)^p,\ \ n\geq 1,\ x,y\in X.$

(2) $d\big(Tx,Ty\big)^p \geq d\big(x,y\big)^p$ for all $x,y\in X.$

(3) $d\big(T^n x,T^n y\big)^p + (n-1)d\big(x,y\big)^p \leq n.d\big(Tx,Ty\big)^p,\ x,y\in X,\ n=0,1,2,...$

(4) $d\big(Tx,Ty\big) \leq 2^{\frac{1}{p}} d\big(x,y\big)\ \ \forall\, x,y\in \mathcal{R}(T)$ (the range of T).

Proof. Using the fact that T is $(2,p)$-expansive map , we get

$$d\big(T^2x,T^2y\big)^p - d\big(Tx,Ty\big)^p \leq d\big(Tx,Ty\big)^p - d\big(x,y\big)^p.$$

Replacing x by T^kx and y by T^ky leads to

$$d\big(T^{k+2}x,T^{k+2}y\big)^p - d\big(T^{k+1}x,T^{k+1}y\big)^p \leq d\big(T^{k+1}x,T^{k+1}y\big)^p - d\big(T^kx,T^ky\big)^p,$$

for $k \geq 0$. Hence

$$\begin{array}{rcl} d\big(T^nx,T^ny\big)^p & = & \displaystyle\sum_{1\leq k\leq n}\Big(d\big(T^kx,T^ky\big)^p - d\big(T^{k-1}x,T^{k-1}y\big)^p\Big) + d\big(x,y\big)^p \\ & \leq & n\Big(d\big(Tx,Ty\big)^p - d\big(x,y\big)^p\Big) + d\big(x,y\big)^p \\ & \leq & nd\big(Tx,Ty\big)^p + (1-n)d\big(x,y\big)^p. \end{array}$$

Which implies 1. and 3. Letting $n \longrightarrow \infty$ in 1. yields 2.

4. The $(2,p)$-expansivity of T implies that

$$d\big(T^2x,T^2y\big)^p \leq 2d\big(Tx,Ty\big)^p - d\big(x,y\big)^p \leq 2d\big(Tx,Ty\big)^p.$$

Thus,

$$d\big(T^2x,T^2y\big) \leq 2^{\frac{1}{p}}d\big(Tx,Ty\big).$$

QED

Remark 2.5. We make the following remarks:

(1) $(2,p)$-isometric is completely p-hyperexpansive.

(2) Every $(k+1,p)$-hyperexpansive is (k,p)-hyperexpansive for $k = 1,2,....$

Lemma 2.2. Let $T : X \longrightarrow X$ be an $(2,\ p)$-expansive map, then for all integer $k \geq 2$ and $x,y \in X$, we have

$$d\big(T^kx,T^ky\big)^p - d\big(T^{k-1}x,T^{k-1}y\big)^p \leq d\big(Tx,Ty\big)^p - d\big(x,y\big)^p.$$

Proof. We prove the assertion by induction on k. Since T is an $(2,\ p)$-expansive the result is true for $k = 2$. Now assume that the result is true for k i.e.; for all $x,y \in X$,

$$d\big(T^kx,T^ky\big)^p - d\big(T^{k-1}x,T^{k-1}y\big)^p \leq d\big(Tx,Ty\big)^p - d\big(x,y\big)^p, \qquad (2.2)$$

and let us prove it of $k+1$. From (2.2) we obtain the following inequalities

$$\begin{array}{rcl} d\big(T^{k+1}x,T^{k+1}y\big)^p - d\big(T^kx,T^ky\big)^p & \leq & d\big(T^2x,T^2y\big)^p - d\big(Tx,Ty\big)^p \\ & \leq & d\big(Tx,Ty\big)^p - d\big(x,y\big)^p. \end{array}$$

QED

Proposition 2.2. Let $T : X \longrightarrow X$ be a $(2,p)$-expansive map.Then the following statements hold.

(1) T is $(2,p)$-hyperexpansive map.

(2) $d\big(Tx,Ty\big)^{2p} \geq d\big(x,y\big)^p d\big(T^2x,T^2y\big)^p$ for all $x,y \in X$.

(3) For each n and $x,y \in X$ such that $x \neq y$,the sequence

$$\left(\frac{d\big(T^{n+1}x,T^{n+1}y\big)^p}{d\big(T^nx,T^ny\big)^p}\right)_{n\geq 0} \tag{2.3}$$

is monotonically decreasing to 1.

Proof. (1) Follows from part (2) of Lemma 2.1.

(2) Since from (1) T is $(2,p)$-hyperexpansive map, we have that

$$\begin{aligned} d\big(Tx,Ty\big)^{2p} &\geq \left(\frac{d\big(x,y\big)^p + d\big(T^2x,T^2y\big)^p}{2}\right)^2 \\ &\geq \left(d\big(x,y\big)^{\frac{p}{2}} d\big(T^2x,T^2y\big)^{\frac{p}{2}}\right)^2 \\ &\geq d\big(x,y\big)^p d\big(T^2x,T^2y\big)^p. \end{aligned}$$

(3) Observe that the $(2,p)$-expansivity of T implies that

$$d\big(T^{n+1}x,T^{n+1}y\big)^p - 2d\big(T^nx,T^ny\big)^p + d\big(T^{n-1}x,T^{n-1}y\big)^p \leq 0. \tag{2.4}$$

On the other hand, since

$$\left(d\big(T^{n-1}x,T^{n-1}y\big)^{\frac{p}{2}} - d\big(T^{n+1}x,T^{n+1}y\big)^{\frac{p}{2}}\right)^2 \geq 0$$

it follows that

$$\begin{aligned} & d\big(T^{n-1}x,T^{n-1}y\big)^{\frac{p}{2}} d\big(T^{n+1}x,T^{n+1}y\big)^{\frac{p}{2}} \\ \leq\ & \frac{d\big(T^{n+1}x,T^{n+1}y\big)^p + d\big(T^{n-1}x,T^{n-1}y\big)^p}{2} \\ \leq\ & d\big(T^nx,T^ny\big)^p \qquad \text{(by} \quad (2.4)). \end{aligned}$$

Thus,

$$d\big(T^{n-1}x,T^{n-1}y\big)^p d\big(T^{n+1}x,T^{n+1}y\big)^p \leq d\big(T^nx,T^ny\big)^{2p}$$

and hence,

$$\frac{d\big(T^{n+1}x,T^{n+1}y\big)^p}{d\big(T^nx,T^ny\big)^p}\le\frac{d\big(T^nx,T^ny\big)^p}{d\big(T^{n-1}x,T^{n-1}y\big)^p},$$

so the sequence (2.3) is monotonically decreasing. To calculate its limit in view of part (2) of Lemma 2.1, divide (2.4) by $d\big(T^{n-1}x,T^{n-1}y\big)^p$ to get

$$1-2\frac{d\big(T^nx,T^ny\big)^p}{d\big(T^{n-1}x,T^{n-1}y\big)^p}+\frac{d\big(T^{n+1}x,T^{n+1}y\big)^p}{d\big(T^nx,T^ny\big)^p}\frac{d\big(T^nx,T^ny\big)^p}{d\big(T^{n-1}x,T^{n-1}y\big)^p}\le 0.$$

and let n tend to infinity we obtain that

$$\frac{d\big(T^nx,T^ny\big)^p}{d\big(T^{n-1}x,T^{n-1}y\big)^p}\longrightarrow 1\quad\text{as }\ n\longrightarrow\infty.$$

. QED

The following theorem gives a sufficient condition for (m,p)-expansive map to be $(m-1,p)$-expansive map for $m\ge 3$.

Theorem 2.1. Let $T:X\longrightarrow X$ be an (m,p)-expansive map for $m\ge 3$. If T is $(2,p)$-expansive,then T is $(m-1,p)$-expansive.

Proof. The conditions $d\big(x,y\big)^p-d\big(Tx,Ty\big)^p\le 0$ and

$$d\big(x,y\big)^p-2d\big(Tx,Ty\big)^p+d\big(T^2x,T^2y\big)^p\le 0$$

guarantee that the sequence $\Big(d\big(T^{n+1}x,T^{n+1}y\big)^p-d\big(T^nx,T^ny\big)^p\Big)_{n\ge 0}$ is monotonically non-increasing and bounded, so that is converges.Thus there exists a constant C such that

$$d\big(T^{n+1}x,T^{n+1}y\big)^p-d\big(T^nx,T^ny\big)^p\longrightarrow C\ \text{ as }\ n\longrightarrow\infty.$$

Since $\Theta_m^{(p)}(d,T;\ x,y)\le 0$ with $m\ge 2$. By Proposition 2.1

$$\Theta_m^{(p)}(d,T;\ x,y)=\Theta_{m-1}^{(p)}(d,T;x,y)-\Theta_{m-1}^{(p)}(d,T;\ Tx;Ty),$$

we have that

$$\Theta_{m-1}^{(p)}(d,T;\ x,y)\le\Theta_{m-1}^{(p)}(d,T;\ Tx,Ty).$$

An induction argument shows that

$$\Theta_{m-1}^{(p)}(d,T;\ x,y)\le\Theta_{m-1}^{(p)}(d,T;\ T^nx,T^ny),\ n\ge 1.$$

Thus,it suffices to show that

$$\Theta_{m-1}^{(p)}(d, T;\ T^n x, T^n y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Note that

$$\Theta_{m-1}^{(p)}(d, T;\ x, y) = \Theta_{m-2}^{(p)}(d, T;\ x, y) - \Theta_{m-2}^{(p)}(d, T;\ Tx, Ty),$$

so that

$$\Theta_{m-1}^{(p)}(d, T;\ T^n x, T^n y) = \sum_{0 \le j \le m-2} (-1)^j \binom{m-2}{j} \big[d\big(T^{n+j}x, T^{n+j}y\big)^p - d\big(T^{n+1+j}x, T^{n+1+j}y\big)^p\big].$$

Letting $n \longrightarrow \infty$ in the preceding equality leads to

$$\Theta_{m-1}^{(p)}(d, T;\ T^n x, T^n y) \longrightarrow \sum_{0 \le j \le m-2} (-1)^j \binom{m-2}{j} C = 0.$$

This completes the proof. QED

Example 2.3. Let us consider again Example 2.2. $T : \mathbb{R} \longrightarrow \mathbb{R}$, $Tx = 1 + 2x$. This example shows that T is $(5, p)$-expansive,but not $(4, p)$-expansive, so the assumption for T to be $(2, p)$-expansive in Theorem 2.1 below is necessary.

The following criterion for (m, p)-hyperexpansivity follows from Theorem 2.1

Corollary 2.1. Let T be (m, p)-expansive and $(2, p)$-expansive mapping. Then T is (m, p)-hyperexpansive.

Proposition 2.3. Let $T : X \longrightarrow X$ be an $(2, p)$-expansive map and assume that T is a (m, p)-isometric for some $m \ge 2$. Then T is a $(2, p)$-isometric.

Proof. Assume that $\Theta_m^{(p)}(d, T;\ x, y) = 0$ for all $x, y \in X$. Since

$$\Theta_m^{(p)}(d, T;\ x, y) = \Theta_{m-1}^{(p)}(d, T;\ x, y) - \Theta_{m-1}^{(p)}(d, T;\ Tx, Ty)$$

we deduce that

$$\Theta_{m-1}^{(p)}(d, T;\ x, y) = \Theta_{m-1}^{(p)}(d, T;\ Tx, Ty) = \Theta_{m-1}^{(p)}(d, T;\ T^n x, T^n y);$$

for $n = 1, 2, ...$ In the same way as in the proof of Theorem 2.1 we obtain that $\Theta_{m-1}^{(p)}(d, T;\ x, y) = 0$. Applying the corresponding results of the $(m-1, p)$-isometric, we have that $\Theta_{m-2}^{(p)}(d, T;\ x, y) = 0$. Continue the above process to get $\Theta_2^{(p)}(d, T;\ x, y) = 0$ and so T is $(2, p)$-isometric. QED

In the following lemma we generalize Lemma 1.3 in [14] and Proposition 2.11 in [13].

Lemma 2.3. Let $T : X \longrightarrow X$ be a contractive mapping. If T is an (m,p)-isometry then T is an $(m-1,p)$-isometry.

Proof. Since T is contractive,we have the following inequality

$d\big(T^{n+1}x, T^{n+1}y\big)^p \leq d\big(T^n x, T^n y\big)^p$ for all $x, y \in X$ and $n \in \mathbb{N}_0$. This means that $\big(d\big(T^n x, T^n y\big)^p\big)_{n\in\mathbb{N}_0}$ is deceasing sequence, so convergent.
Using the fact that T is an (m,p)-isometry and together (2.1), we obtain

$$\Theta^{(p)}_{m-1}(d,T;x,y) = \Theta^{(p)}_{m-1}(d,T;Tx,Ty) = ... = \Theta^{(p)}_{m-1}(d,T;T^n x,T^n y).$$

Note that

$$\Theta^{(p)}_{m-1}(d,T;T^n x,T^n y) = \Theta^{(p)}_{m-2}(d,T;T^n x,T^n y) - \Theta^{(p)}_{m-2}(d,T;T^{n+1}x,T^{n+1}y),$$

so that

$$\begin{aligned} &\Theta^{(p)}_{m-1}(d,T;T^n x,T^n y) \\ = &\sum_{j=0}^{m-2}(-1)^j\binom{m-2}{j}\big[d\big(T^{n+j}x,T^{n+j}y\big)^p - d\big(T^{n+1+j}x,T^{n+1+j}y\big)^p\big]. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the preceding equality leads to

$$\Theta^{(p)}_{m-1}(d,T;T^n x,T^n y) \longrightarrow 0.$$

Thus, $\Theta^{(p)}_{m-1}(d,T;x,y) = 0$ and hence, T is an $(m-1,p)$-isometry. QED

As a consequence of the lemma, we have the following proposition.

Proposition 2.4. If T is a contractive mapping on X, then T is an (m,p)-isometry if and only if T is an isometry.

Proposition 2.5. Let $T : X \longrightarrow X$ be a map for which T^2 is isometric , then the following properties hold

(i) T is (m,p)-expansive map if and only if T is expansive.

(ii) T is (m,p)-contractive if and only if T is contractive.

Proof. (i) If we assume that m is odd integer i.e., $m = 2q+1$ we have by the assumption that

$$\begin{aligned}
&\Theta^{(p)}_{2q+1}(d,T;\ x,y)\\
=&\sum_{0\le j\le q}\Big[\binom{2q+1}{2j}d\big(T^{2j}x,T^{2j}y\big)^p-\binom{2q+1}{2j+1}d\big(T^{2j+1}x,T^{2j+1}y\big)^p\Big]\\
=&\sum_{0\le j\le q}\Big[\binom{2q+1}{2j}d\big(x,y\big)^p-\binom{2q+1}{2j+1}d\big(Tx,Ty\big)^p\Big]
\end{aligned}$$

Since $\sum_{0\le k\le 2q+1}(-1)^k\binom{2q+1}{k}=0$, it follows that

$$\sum_{0\le j\le q}\binom{2q+1}{2j+1}=\sum_{0\le j\le q}\binom{2q+1}{2j}$$

and we deduce that

$$\Theta^{(p)}_{2q+1}(d,T;\ x,y)=\sum_{0\le j\le q}\binom{2q+1}{2j}\big(d\big(x,y\big)^p-d\big(Tx,Ty\big)^p\big).$$

Similarly if m is even integer i.e.,$m = 2q$ we have

$$\begin{aligned}
&\Theta^{(p)}_{2q}(d,T;\ x,y)\\
=&\sum_{0\le j\le q}\binom{2q}{2j}d\big(T^{2j}x,T^{2j}y\big)^p-\sum_{j=1}^{q}\binom{2q}{2j-1}d\big(T^{2j-1}x,T^{2j-1}y\big)^p\\
=&\sum_{0\le j\le q}\binom{2q}{2j}d\big(x,y\big)^p-\sum_{1\le j\le q}\binom{2q}{2j-1}d\big(Tx,Ty\big)^p
\end{aligned}$$

Since $\sum_{0\le k\le 2q}(-1)^k\binom{2q}{k}=0$, we have that $\sum_{1\le j\le q}\binom{2q}{2j-1}=\sum_{0\le j\le q}\binom{2q}{2j}$ and hence

$$\Theta^{(p)}_{2q}(d,T;\ x,y)=\sum_{0\le j\le q}\binom{2q}{2j}\Big(d\big(x,y\big)^p-d\big(Tx,Ty\big)^p\Big).$$

Therefore,we conclude that (i) and (ii) hold and this establishes the proposition.

QED

Proposition 2.6. If $T : X \longrightarrow X$ be an map satisfies $T^2 = T$, then the following properties hold

(i) T is (m,p)-expansive if and only if T is expansive.

(ii) T is (m,p)-contractive if and only if T is contractive.

Proof. By the assumption on T , we have that

$$\Theta_m^{(p)}(d,T;\ x,y) = d\big(x,y\big)^p - d\big(Tx,Ty\big)^p,\ \ \forall\ \ x,y \in X.$$

It is clear from the foregoing that a sufficient condition for T to be (m,p)-expansive (resp. (m,p)-contractive) is that T is expansive (resp. contractive). QED

Proposition 2.7. ([13]) If T is a bijective (m,p)-isometry, then T^{-1} is also an (m,p)-isometry.

We have the following result about bijective (m,p)-expansive and contractive maps.

Proposition 2.8. Let $T : X \longrightarrow X$ be an bijective map, we have the following properties

(1) If T is (m,p)-expansive, then

(i) for m even , T^{-1} is (m,p)-expansive.

(ii) for m odd , T^{-1} is (m,p)-contractive.

(2) If T is (m,p)-contractive, then

(i) for m even , T^{-1} is (m,p)-contractive.

(ii) for m odd , T^{-1} is (m,p)-expansive.

Proof. (1) Assume that $\Theta_m^{(p)}(d,T;\ x,y) \leq 0\ \ \forall\, x,y \in X$ and for positive integer m. By a computation stemming essentially from the formula

$$\binom{m}{j} = \binom{m}{m-j}; \text{for } j = 0,1,...,m,$$

we deduce that

$$\Theta_m^{(p)}(d,T^{-1};\ x,y) = (-1)^m\Theta_m^{(p)}(d,T;\ T^{-m}x,T^{-m}y).$$

It follows that $\Theta_m^{(p)}(d,T^{-1};\ x,y) \leq 0$ for even integer m i.e., T^{-1} is (m,p)-expansive, and $\Theta_m^{(p)}(d,T^{-1};\ x,y) \geq 0$ for odd integer m i.e., T^{-1} is (m,p)-contractive.

(2) The proof is similar. QED

Proposition 2.9. Let $T : X \longrightarrow X$ be an bijective $(2,p)$-expansive map, then T is $(1,p)$-isometric.

Proof. Since T is $(2,p)$-expansive, we have by Lemma 2.1 that $d\big(Tx,Ty\big)^p \geq d\big(x,y\big)^p$.This means that T is $(1,\ p)$-expansive. Moreover if T is bijective $(2,p)$-expansive, then T^{-1} is $(2,p)$-expansive, hence $d\big(T^{-1}u,T^{-1}v\big)^p \geq d\big(u,v\big)^p$ for all $u,v \in X$. Letting $u = Tx$ and $v = Ty$, this implies

$$d\big(Tx,Ty\big)^p = d\big(x,y\big)^p$$

for all $x,y \in X$. This means that T is $(1,\ p)$-isometric. QED

Theorem 2.2. Let $T; S : X \longrightarrow X$ two maps such that $ST = I_X$ (the identity mapping). Assume that there exists an integer $m \geq 1$ such that $\Theta_m^{(p)}(d,S;\ x,y) \leq 0$ for all $x,y \in \mathcal{R}(T^m)$,the following statements hold.

(i) If m is even, then T is (m,p)-expansive map.

(ii) If m is odd, then T is (m,p)-contractive map.

Proof. Since $\Theta_m^{(p)}(d,S;\ T^mu,T^mv) \leq 0$ for all $u,v \in X$, we have that

$$\begin{aligned} 0 &\geq \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}d\big(S^kT^mu,S^kT^mv\big)^p \\ &\geq \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}d\big(T^{m-k}u,T^{m-k}v\big)^p \\ &\geq (-1)^m\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}d\big(T^ku,T^kv\big)^p \end{aligned}$$

QED

Note that every power of k-expansive (resp. (A,m)-expansive) operators on a Hilbert space is k-expansive (resp. (A,m)-expansive). See ([18], Theorem 2.3) and ([20], Proposition 3.9).

In the following theorem we investigate the powers of $(2,p)$-expansive maps as well as $(2,p)$-expansive maps by using Lemma 2.2.

Theorem 2.3. Let $T : X \longrightarrow X$ be an $(2,p)$-expansive map. Then for any positive integer n , T^n is $(2,p)$-expansive map.

Proof. We will induct on n, the result obviously holds for $n=1$. Suppose then the assertion holds for $n \geq 2$, i.e

$$d\big(T^{2n}x, T^{2n}y\big)^p - 2d\big(T^n x, T^n y\big)^p + d\big(x,y\big)^p \leq 0,\ \forall\, x,y \in X.$$

Then

$$\begin{aligned}
& d\big(T^{2n+2}x, T^{2n+2}y\big)^p - 2d\big(T^{n+1}x, T^{n+1}y\big)^p + d\big(x,y\big)^p \\
= \;& d\big(T^2T^{2n}x, T^2T^{2n}y\big)^p - 2d\big(T^{n+1}x, T^{n+1}y\big)^p + d\big(x,y\big)^p \\
\leq \;& 2d\big(T^{2n+1}x, T^{2n+1}y\big)^p - d\big(T^{2n}x, T^{2n}y\big)^p \\
& -2d\big(T^{n+1}x, T^{n+1}y\big)^p + d\big(x,y\big)^p \\
\leq \;& 2(2d\big(T^{n+1}x, T^{n+1}y\big)^p - d\big(Tx,Ty\big)^p) - d\big(T^{2n}x, T^{2n}y\big)^p \\
& -2d\big(T^{n+1}x, T^{n+1}y\big)^p + d\big(x,y\big)^p \\
\leq \;& 2d\big(T^{n+1}x, T^{n+1}y\big)^p - d\big(T^{2n}x, T^{2n}y\big)^p \\
& -2d\big(Tx,Ty\big)^p + d\big(x,y\big)^p \\
\leq \;& 2d\big(T^{n+1}x, T^{n+1}y\big)^p - (2d\big(T^n x, T^n y\big)^p - d\big(x,y\big)^p) \\
& -2d\big(Tx,Ty\big)^p + d\big(x,y\big)^p \\
\leq \;& 2d\big(T^{n+1}x, T^{n+1}y\big)^2 - 2d\big(T^n x, T^n y\big)^p - 2d\big(Tx,Ty\big)^p + 2d\big(x,y\big)^p \\
\leq \;& 2\big(d\big(Tx,Ty\big)^p - d\big(x,y\big)^p\big) - 2d\big(Tx,Ty\big)^p + 2d\big(x,y\big)^p \quad \text{(by } \textit{Lemma } 2.2\text{)}. \\
\leq \;& 0.
\end{aligned}$$

Thus means that T^n is $(2,p)$-expansive map. QED

In the following theorem we investigate the powers of completely p-hyperexpansive mapping as well as completely p-hyperexpansive mapping.

According to [5, Remark 1.] for every completely p-hyperexpansive map, the condition that $n \longmapsto d\big(T^n x, T^n y\big)^p$ be completely alternating on $\mathbb{N}$ implies the representation, for every $x, y \in X$,

$$d\big(T^n x, T^n y\big)^p = d(x,y)^p + n\mu_{x,y}(\{1\}) + \int_{[0,1)} (1-t^n)\frac{d\mu_{x,y}(t)}{1-t}, \qquad (2.5)$$

where $\mu_{x,y}$ is a positive regular Borel measure on $[0;\ 1]$ (for more details see [5]).

Theorem 2.4. Any positive integral power of a completely p-hyperexpansive mapping is completely p-hyperexpansive.

Proof. Let T be a completely p-hyperexpansive map and let $k \geq 1$. In view of (2.5) we have that

$$\begin{aligned} & d\big((T^k)^n x, (T^k)^n y\big)^p = d\big(T^{nk}x, T^{nk}y\big)^p \\ = \;& d\big(x,y\big)^p + nk\mu_{x,y}(\{1\}) + \int_{[0,1)} (1-t^{nk}) \frac{d\mu_{x,y}(t)}{1-t} \\ = \;& d\big(x,y\big)^p + n(k\mu_{x,}(\{1\})) + \int_{[0,1)} (1-s^n) \frac{d\mu'_{x,y}(s)}{1-s^{\frac{1}{k}}}. \end{aligned}$$

Therefore the map $n \longmapsto d\big(T^{nk}x, T^{nk}y\big)^p$ is completely alternating and so that T^k is completely p-hyperexpansive. QED

The next proposition describes the intersection of the class of completely p-hyperexpansive maps with the class of (m,p)-isometries.

Proposition 2.10. Let T be a mapping on metric space X into itself. If T is completely p-hyperexpansive as well as (m,p)-isometric ($m \geq 2$), then T is a $(2,p)$-isometric.

Proof. First, if T is isometric, then T is a $(2,p)$-isometric. Assume that T is a (m,p)-isometric with $m \geq 2$, then we have that $\Theta_m^{(p)}(d,T;\ x,y) = 0$ and from (2.5) it follows that

$$\begin{aligned} 0 \;=\; & \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} k\mu_{x,y}(\{1\}) \\ & + \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \int_{[0,\,1)} (1-t^k) \frac{d\mu_{x,y}(t)}{1-t} \\ =\; & -\int_{[0,\,1)} (1-t)^{m-1} d\mu_{x,y}(t) \end{aligned}$$

Now $\displaystyle\int_{[0,\,1)} (1-t)^{m-1} d\mu_{x,y}(t) = 0$ gives that

$$d\big(T^k x, T^k y\big)^p = d\big(x,y\big)^p + k\mu_{x,y}(\{1\}) \quad \text{for all } \; k$$

and therefore

$$\Theta_2^{(p)}(d,T;\ x,y) = 0.$$

QED

Example 2.4. Consider the map $T : (\mathbb{R}, d) \longrightarrow (\mathbb{R}, d)$ defined by

$$Tx = \begin{cases} 2x - 1, & \text{for } x \leq 0 \\ \\ 2x + 1 & \text{for } x > 0. \end{cases}$$

It is easy to verify that T is a $(1, p)$-expansive, but T is neither continuous nor linear.

3 (m, p)-hyperexpansive maps in seminormed space

Let X be a linear vector space and considering a seminorm s on X, we may define the quantity

$$\Theta_m^{(p)}(s, T; x) := \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} s\big(T^k x\big)^p$$

for all $x \in X$, and introducing a concept similar to that of (m, p)-expansive maps.

Definition 3.1. Let $T : X \longrightarrow X$ be a map , $m \in \mathbb{N}$ and $p > 0$. We say that

(1) T is $s(m, p)$-isometric if $\Theta_m^{(p)}(s,\ T;\ x) = 0$ for all $x \in X$.

(2) T is $s(m, p)$-expansive if $\Theta_m^{(p)}(s, T;\ x) \leq 0\ \ \forall\, x \in X$

(3) T is $s(m, p)$-hyperexpansive if $\Theta_k^{(p)}(s, T;\ x) \leq 0$ for $k = 1, ..., m$ and $x \in X$.

(4) T is completely s-hyperexpansive if T is $s(k, p)$-expansive for all $k \in \mathbb{N}$.

(5) T is $s(m, p)$-contractive if $\Theta_m^{(p)}(s, T; x) \geq 0\ \forall\, x \in X$.

(6) T is $s(m, p)$-hypercontractive if $\Theta_k^{(p)}(s, T;\ x) \geq 0$ for $k = 1, 2, ..., m$ and $x \in X$.

(7) T is completely s- hypercontractive if T is $s(k, p)$-contractive for all $k \in \mathbb{N}$.

For any $p > 0$, $s(1, p)$-expansive coincide with s-expansive; that is, maps T satisfying $s(Tx) \geq s(x)$, for all $x \in X$. Every s-isometry is an $s(m, p)$-isometry for all $m \geq 1$ and $p > 0$. $s(m, p)$-isometries maps are special cases of the class of s(m , p)-expansive maps.

If X is a normed space with norm $\|.\|$ and $T : X \longrightarrow X$ we have that

$$\Theta_m^{(p)}(\|.\|, T; x) = \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|T^k x\|^p.$$

Clearly m-hyperexpansivity on Hilbert spaces agree with $(m,2)$-hyperexpansivity.

Example 3.1. Let $\mathbb{D}$ denote the open unit disk in the complex plane. An analytic function f on $\mathbb{D}$ is said to be Bloch if

$$s(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The mapping $f \longmapsto s(f)$ is a semi-norm on the space $\mathcal{B}$ of Bloch functions, called the Bloch space. See [24] for some additional details. Consider for $\lambda \in \mathbb{C}$ the map $T_\lambda : \mathcal{B} \longrightarrow \mathcal{B} : T_\lambda(f) = \lambda f$.
A simple computation shows that

$$\begin{aligned}
&\sum_{0\le k\le m}(-1)^k\binom{m}{k}s(T_\lambda^k f)\\
=\;&\sum_{0\le k\le m}(-1)^k\binom{m}{k}|\lambda|^k\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|\\
=\;&\big(1-|\lambda|\big)^m\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|
\end{aligned}$$

and it follows that

$$\begin{cases}
\Theta_m^{(p)}(s,T_\lambda,f) \le 0, \text{ if } |\lambda|>1, \text{ for odd } m.\\
\Theta_m^{(p)}(s,T_\lambda,f) = 0 \text{ if } |\lambda|=1 \text{ and for all } m\\
\Theta_m^{(p)}(s,T_\lambda,f) > 0 \text{ if } |\lambda|<1 \text{ and for all } m.
\end{cases}$$

Setting

$$\beta_k^{(p)}(s,\ T;\ x) := \frac{1}{k!}\sum_{0\le j\le k}(-1)^{k-j}\binom{k}{j}s\big(T^jx\big)^p,\ \forall\ x\in X. \tag{3.1}$$

In the following theorem, we generalized the identities (1.7) and (1.10) to seminormed space. We omit the proof which is very similar to [13, Theorem 2.5] and [8, Proposition 2.1].

Theorem 3.1. Let (X,s) be seminormed space and $T : X \longrightarrow X$ be a map, we have that

(i) $s\big(T^nx\big)^p = \sum_{0\le j\le n} n^{(j)}\beta_j^{(p)}(s,\ T;\ x);\ \forall\, n\in\mathbb{N}.$

(ii) T is an $s(m,p)$-isometry if and only if

$$s\big(T^nx\big)^p = \sum_{0\le j\le m-1} n^{(j)}\beta_j^{(p)}(s,\ T;\ x);\ \forall\, n\in\mathbb{N}.$$

Proposition 3.1. Let (X, s) be a seminormed space and $T : X \longrightarrow X$ be a map. The following are true

(i) T is $s(m,p)$-expansive if and only if, λT is $s(m,p)$-expansive for all $\lambda \in \mathbb{C}$: $|\lambda| = 1$,

(ii) If T is $s(2,p)$-expansive, then

(1) λT is $s(2,p)$-expansive for $|\lambda| < 1$, if λT^2 is s-expansive.

(2) λT is $s(2,p)$-expansive for $|\lambda| > 1$, if λT^2 is s-contractive.

Proof. (i) Note that, for all $p > 0$, $\lambda \in \mathbb{C}$, and all $x \in X$, we have

$$\Theta_m^{(p)}(s, T\ x) = \Theta_m^{(p)}(s, \lambda T;\ x),\ |\lambda| = 1.$$

(ii) If T is $s(2,p)$-expansive, then

$$-2|\lambda|^p s\big(Tx\big)^p \leq |\lambda|^p\big[-s\big(T^2x\big)^p - s\big(x\big)^p\big] \text{ for every } \lambda \in \mathbb{C}.$$

So we have for every $\lambda \in \mathbb{C}$

$$|\lambda|^{2p}s\big(T^2x\big)^p - 2|\lambda|^p s\big(Tx\big)^p + s\big(x\big)^p \leq \big(|\lambda|^p - 1\big)\big(|\lambda|^p s\big(T^2x\big)^p - s\big(x\big)^p\big)$$

This finishes the proof. $\boxed{QED}$

Acknowledgements. The author would like to thank the referee for a number of helpful comments and suggestions that improved the presentation of the paper.

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