Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **35** (2015) no. 1, 1–13. doi:10.1285/i15900932v35n1p1

Postulation of general unions of lines and decorated lines

Edoardo Ballico[i]
Department of Mathematics, University of Trento
ballico@science.unitn.it

Received: 2.11.2013; accepted: 3.6.2014.

Abstract. A +line $A \subset \mathbb{P}^r$, $r \geq 3$, is the scheme $A = L \cup v$ with L a line and v a tangent vector of $\mathbb{P}^r$ supported by a point of L, but not tangent to L. Here we prove that a general disjoint union of lines and +lines has the expected Hilbert function.

Keywords: postulation, Hilbert function, lines, zero-dimensional schemes

MSC 2000 classification: primary 14N05, secondary 14H99

Introduction

Fix a line $L \subset \mathbb{P}^r$, $r \geq 2$, and $P \in L$. A tangent vector of $\mathbb{P}^r$ with P as its support is a zero-dimensional scheme $Z \subset \mathbb{P}^r$ such that $\deg(Z) = 2$ and $Z_{red} = \{P\}$. The tangent vector Z is uniquely determined by P and the line $\langle Z \rangle$ spanned by Z. Conversely, for each line $D \subset \mathbb{P}^r$ with $P \in D$ there is a unique tangent vector v with $v_{red} = P$ and $\langle v \rangle = D$. A +line $M \subset \mathbb{P}^r$ supported by L and with a nilradical at P is the union $v \cup L$ of L and a tangent vector v with P as its support and spanning a line $\langle v \rangle \neq L$. The set of all +lines of $\mathbb{P}^r$ supported by L and with a nilradical at P is an irreducible variety of dimension $r-1$ (the complement of L in the $(r-1)$-dimensional projective space of all lines of $\mathbb{P}^r$ containing P). Hence the set of all +lines of $\mathbb{P}^r$ supported by L is parametrized by an irreducible variety of dimension r. Therefore the set of all +lines of $\mathbb{P}^r$ is parametrized by an irreducible variety of dimension $2(r-1) + r = 3r - 1$. Now assume $r \geq 3$. For all integers $t \geq 0$ and $c \geq 0$ let $L(r,t,c)$ be the set of all disjoint unions $X \subset \mathbb{P}^r$ of t lines and c +lines. If $(t,c) \neq (0,0)$, then $L(r,t,c)$ is an irreducible variety of dimension $(t+c)(2r-1) + cr$. Fix any $X \in L(r,t,c)$ and any integer $k > 0$. It is easy to check that $h^0(\mathcal{O}_X(k)) = (k+1)(t+c) + c$ and $h^i(\mathcal{O}_X(k)) = 0$ for all $i > 0$ (Lemma 2). A closed subscheme $E \subset \mathbb{P}^r$ is said to have *maximal rank* if for every integer $k > 0$ either $h^0(\mathcal{I}_E(k)) = 0$ or $h^1(\mathcal{I}_E(k)) = 0$, i.e. $h^0(\mathcal{I}_E(k)) = \max\{0, \binom{r+k}{r} - h^0(\mathcal{O}_E(k))\}$.

---

[i]This work is partially supported by MIUR and GNSAGA (INDAM)

Theorem 1. *Fix integers $r \geq 3$, $t \geq 0$ and $c \geq 0$ such that $(t, c) \neq (0, 0)$. If $r \geq 4$, then assume that the characteristic is zero. Then a general $X \in L(r, t, c)$ has maximal rank.*

We prove Theorem 1 for $r = 3$ in arbitrary characteristic, while we assume characteristic zero if $r \geq 4$. We also get intermediate results (e.g. $B_{r,k}$) that may be useful as a sample of lemmas which may be proved with +lines. We see +lines as a tool to prove something involving the Hilbert function of unions of curves and fat points. For an alternative approach to such disjoint unions, see Remark 1.

1 Preliminaries

Remark 1. Fix a line $L \subset \mathbb{P}^n$, $n \geq 2$, and a linear system $V \subseteq H^0(\mathcal{O}_{\mathbb{P}^n}(k))$. Let $L^{(1)}$ be the first infinitesimal neighborhood of L in $\mathbb{P}^n$, i.e. the closed subscheme of $\mathbb{P}^n$ with $(\mathcal{I}_L)^2$ as its ideal sheaf. Let A be any +line with L as its support. For any closed subscheme $B \subset \mathbb{P}^n$ set $V(-B) := \{f \in V : f_{|B} \equiv 0\}$. The +line A gives independent conditions to V with the only restriction of that L is the support of L if either $V(L) = \{0\}$ or $\dim(V(-A)) = \dim(V(-L)) - 1$. A general +lines with L as its supports does not give independent conditions to V with the only restriction that L is its support if and only if $V(-L) \neq \{0\}$ and $V(-L^{(1)}) = V(-L)$. Now assume $\dim(V(-L^{(1)})) = \dim(V(-L)) - \gamma$ for some $\gamma > 0$. The integer γ is the maximal number of tangent vectors $v_1, \dots, v_\gamma$ of $\mathbb{P}^n$ supported by points of L and imposing independent conditions to $V(-L)$ (with the restriction that their support is a point of L). So if we only need an integer $t + c \geq 2$, $t + c$ disjoint lines and $x \geq 2$ tangent vectors supported by some of these lines we may decide to put more than one tangent vector on a single line.

Lemma 1. *Let $X \subset \mathbb{P}^r$ be a closed subscheme such that the nilradical sheaf $\eta \subseteq \mathcal{O}_X$ is supported by finitely many points. Set $Y := X_{red}$ and fix $k \in \mathbb{N}$. Then:*

(1) $\chi(\mathcal{O}_X(k)) = \chi(\mathcal{O}_Y(k)) + \deg(\eta)$;

(2) $h^0(\mathcal{I}_X(k)) \leq h^0(\mathcal{I}_Y(k)) \leq h^0(\mathcal{I}_X(k)) + \deg(\eta)$;

(3) $h^1(\mathcal{I}_Y(k)) \leq h^1(\mathcal{I}_X(k)) \leq h^1(\mathcal{I}_Y(k)) + \deg(\eta)$;

(4) $h^0(\mathcal{I}_X(k)) - h^1(\mathcal{I}_X(k)) = h^0(\mathcal{I}_Y(k)) - h^1(\mathcal{I}_Y(k)) - \deg(\eta)$.

Proof. By the definition of the reduction of a scheme the sheaf η is the ideal sheaf of Y in X. We have exact sequence (respectively of $\mathcal{O}_X$-sheaves and of $\mathcal{O}_{\mathbb{P}^r}$-sheaves):

$$0 \to \eta \to \mathcal{O}_X(k) \to \mathcal{O}_Y(k) \to 0 \quad (1)$$

$$0 \to \mathcal{I}_X(k) \to \mathcal{I}_Y(k) \to \eta \to 0 \quad (2)$$

Since η is supported by finitely many points, we have $h^i(\eta) = 0$ for all $i > 0$ and $\deg(\eta) = h^0(\eta)$. Use the cohomology exact sequences of (1) and (2). QED

Remark 2. Fix integers $r \geq 3$, $t \geq 0$ and $c > 0$. Fix $A \in L(r,t,c)$, $D \in L(r,t+c,0)$ and set $B := A_{red}$.

(1) We have $B \in L(r,t+c,0)$. If A is general in $L(r,t,c)$, then B is general in $L(r,t+c,0)$.

(2) Assume that D is general in $L(r,t+c,0)$ and fix a decomposition $D = D_1 \sqcup D_2$ with $D_1 \in L(r,t,0)$ and $D_2 \in L(r,c,0)$. Let E be a general element of $L(r,0,c)$ with $E_{red} = D_2$. Then D_1 is general in $L(r,t,0)$, D_2 is general in $L(r,c,0)$ and $D_1 \cup E$ is general in $L(r,t,c)$.

Lemma 1 and Remark 2 give the following result.

Lemma 2. *Fix integers $r \geq 3$, $t \geq 0$ and $c > 0$. Fix $X \in L(r,t,c)$ and set $Y := X_{red}$. We have $Y \in L(r,t+c,0)$. If A is general in $L(r,t,c)$, then B is general in $L(r,t+c,0)$. For each integer $k > 0$ we have $h^1(\mathcal{O}_X(k)) = 0$, $h^0(\mathcal{O}_X(k)) = (t+c)(k+1)+c$, $h^0(\mathcal{I}_Y(k)) - c \leq h^0(\mathcal{I}_X(k)) \leq h^0(\mathcal{I}_Y(k))$ and $h^1(\mathcal{I}_Y(k)) \leq h^1(\mathcal{I}_X(k)) \leq h^1(\mathcal{I}_Y(k)) + c$.*

For all integers $r \geq 3$ and $k \geq 0$ let $H_{r,k}$ denote the following statement:

Assertion $H_{r,k}$, $r \geq 3$, $k \geq 0$: Fix $(t,c) \in \mathbb{N}^2 \setminus \{(0,0)\}$ and take a general $X \in L(r,t,c)$. If $(k+1)t + (k+2)c \geq \binom{r+k}{k}$, then $h^0(\mathcal{I}_X(k)) = 0$. If $(k+1)t + (k+2)c \leq \binom{r+k}{k}$, then $h^1(\mathcal{I}_X(k)) = 0$.

Lemma 3. *Fix a general $X \in L(r,t,c)$. If $2t+3c \leq r+1$, then $h^1(\mathcal{I}_X(1)) = 0$. If $2t+3c \geq r+1$, then $h^0(\mathcal{I}_X(1)) = 0$.*

Proof. Since the case $c = 0$ is obvious, we may assume $c > 0$ and use induction on c. Fix a general $Y \in L(r,t,c-1)$ and write $X = Y \sqcup A$ with A a general +line of $\mathbb{P}^r$. If $2t+3(c-1) \geq r-1$, we immediately see that $h^0(\mathcal{I}_{Y \cup A_{red}}(1)) = 0$. Hence $h^0(\mathcal{I}_X(1)) = 0$. Hence we may assume $2t+3(c-1) \leq r-2$. Let $M \subset \mathbb{P}^r$ be the $(2t+3c-4)$-dimensional linear subspace spanned by Y. Since A is general, it spans a plane N such that $M \cap N = \emptyset$. Hence $h^0(\mathcal{I}_{Y \cup A}(1)) = h^0(\mathcal{I}_Y(1)) - 3 = r+1-2t-3c$. QED

Remark 3. Fix an integer $r \geq 3$. By the definition of maximal rank and the irreducibility of each $L(r,t,c)$ Theorem 1 is true for the integer r if and only if all $H_{r,k}$ are true. Since $H_{r,0}$ is obviously true, to prove Theorem 1 in $\mathbb{P}^r$ it is sufficient to prove $H_{r,k}$ for all $k > 0$. Lemma 3 says that $H_{r,1}$ is true.

Remark 4. Fix integers $r \geq 3$ and $k > 0$ and suppose you want to prove $H_{r,k}$. Fix $(t, c) \in \mathbb{N}^2 \setminus \{(0, 0)\}$. First assume $t > 0$ and $(k+1)(t+c)+c < \binom{r+k}{r}$. Hence $(k+1)(t+c)+(c+1) \leq \binom{r+k}{r}$. Suppose that $h^1(\mathcal{I}_X(k)) = 0$ for a general $X \in L(r, t-1, c+1)$. Then $h^1(\mathcal{I}_Y(k)) = 0$ for a general $Y \in L(r, t, c)$ (Lemma 1). Now assume $c > 0$ and $(k+1)(t+c)+c > \binom{r+k}{r}$ and so $(k+1)(t+c)+(c-1) \geq \binom{r+k}{r}$. Suppose that $h^0(\mathcal{I}_A(k)) = 0$ for a general $A \in L(r, t+1, c-1)$. Then $h^0(\mathcal{I}_B(k)) = 0$ for a general $B \in L(r, t, c)$. Therefore to prove $H_{r,k}$ it is sufficient to test all (t, c) such that either $(k+1)(t+c)+c = \binom{k+r}{r}$ or $t = 0$ and $(k+2)c < \binom{k+r}{r}$ or $c = 0$ and $(k+1)t > \binom{r+k}{r}$. We do not need to test the pairs $(t, 0)$ by [6]. Among the pairs $(0, c)$ with $(k+2)c \leq \binom{r+k}{r}$ it is sufficient to test the ones with $\binom{r+k}{r} - k - 1 \leq (k+2)c \leq \binom{r+k}{r}$.

For all integers $r \geq 3$ and $k \geq 0$ define the integers $m_{r,k}$ and $n_{r,k}$ by the relations

$$(k+1)m_{r,k} + n_{r,k} = \binom{r+k}{k}, \ 0 \leq n_{r,k} \leq k \tag{3}$$

Remark 5. Fix integers $r \geq 3$ and $k > 0$. Since $m_{r,k} \leq k$ and $k(k+1) \leq \binom{k+3}{3} \leq \binom{r+k}{r}$, we get $m_{r,k} \geq n_{r,k}$.

For all integers $r \geq 3$ and $k \geq 0$ set $u_{r,k} := \lceil \binom{r+k}{r}/(k+2) \rceil$ and $v_{r,k} := (k+2)u_{r,k} - \binom{r+k}{r}$.

Notice that

$$(k+2)(u_{r,k} - v_{r,k}) + (k+1)v_{r,k} = \binom{r+k}{r} \tag{4}$$

and that $0 \leq v_{r,k} \leq k+1$.

For all integers $k > 0$ let $A_{r,k}$ denote the following assertion:

Assertion $A_{r,k}$, $k > 0$: Let $X \subset \mathbb{P}^r$ be a general union of $v_{r,k}$ lines and $u_{r,k} - v_{r,k}$ +lines. Then $h^0(\mathcal{I}_X(k)) = 0$.

2 The proof in $\mathbb{P}^3$

In this section we prove the case $r = 3$ of Theorem 1.

Lemma 4. *Fix integers $a \geq 0$, $b \geq 0$, $y \geq 0$. Let $Z \subset Q$ be a general union of y tangent vectors. Then $h^0(\mathcal{I}_Z(a, b)) = \max\{0, (a+1)(b+1) - 2y\}$ and $h^1(\mathcal{I}_Z(a, b)) = \max\{0, 2y - (a+1)(b+1)\}$.*

Proof. By the semicontinuity theorem for cohomology ([5], III.12.8) it is sufficient to find a disjoint union $W \subset Q$ of y tangent vectors such that $h^0(\mathcal{I}_W(a, b)) = \max\{0, (a+1)(b+1) - 2y\}$. It is obviously sufficient to do it for the integers

$y = \lfloor (a+1)(b+1) \rfloor$ and $y = \lceil (a+1)(b+1)/2 \rceil$. First assume a odd. Let $L_0, \dots, L_b$ be $b+1$ distinct lines of type $(0,1)$. Let $E_i \subset L_i$ be any disjoint union of $(a+1)/2$ tangent vectors. In this case we may take $W = E_1 \cup \cdots \cup E_b$. In the same way we conclude if b is odd. Hence we may assume that both a and b are even. If $b = 0$, then take y tangent vectors of L_0. Similarly we conclude if $a = 0$. Hence we may assume $a \ge 2$ and $b \ge 2$ and use induction on a. It is obviously sufficient to check the integers y such that $2y \ge (a+1)(b+1) - 1$. Fix a smooth $C \in |\mathcal{O}_Q(2,2)|$. C is a smooth elliptic curve and in particular it is irreducible. Take a general $S \subset C$ with $\sharp(S) = a+b$. Let $W \subset C$ be the union of the 2-points of C with the points of S as their support, i.e. the degree $2a+2b$ effective divisor of C in which each point of S appears with multiplicity two. Let $W' \subset Q$ be a union of $y-a-b$ general tangent vectors. Set $Z := W \cup W'$. By the inductive assumption we have $h^0(\mathcal{I}_{W'}(a-2,b-2)) = \max\{0, (a-1)(b-1) - 2y + 2a + 2b\}$, i.e. $h^0(\mathcal{I}_{W'}(a-2,b-2)) = \max\{0, (a+1)(b+1) - 2y\}$ and $h^1(\mathcal{I}_{W'}(a-2,b-2)) = \max\{0, 2y - (a+1)(b+1)\}$. There are only finitely many (four in characteristic $\neq 2$, one or two in characteristic 2) line bundles R with $R^{\otimes 2} \cong \mathcal{O}_C(a,b)$. Since C has genus > 0 for general S the line bundle $\mathcal{O}_C(S)$ is not one of them. Hence $W \notin |\mathcal{O}_C(a,b)|$. Since $\deg(W) = \deg(\mathcal{O}_C(a,b))$, Riemann-Roch gives $h^i(C, \mathcal{O}_C(a,b)(-W)) = 0$, $i = 0,1$. Since $\mathrm{Res}_C(Z) = W'$, the Castelnuovo's sequence gives $h^i(\mathcal{I}_Z(a,b)) = h^i(\mathcal{I}_{W'}(a-2,b-2))$. QED

We have $u_{3,k} := \lceil (k+3)(k+1)/6 \rceil$ and $v_{3,k} := (k+2)u_{3,k} - \binom{3+k}{3}$. Write $k = 6m + b$ with $0 \le b \le 5$. We have $u_{3,6m} = 6m^2 + 4m + 1$, $v_{3,6m} = 3m + 1$, $u_{3,6m+1} = 6m^2 + 6m + 2$, $v_{3,6m+1} = 4m + 2$, $u_{3,6m+2} = 6m^2 + 8m + 3$, $v_{3,6m+2} = 3m + 2$, $u_{3,6m+3} = 6m^2 + 10m + 4$, $v_{3,6m+3} = 0$, $u_{3,6m+4} = 6m^2 + 12m + 6$, $v_{3,6m+4} = m + 1$, $u_{3,6m+5} = 6m^2 + 14m + 8$, $v_{3,6m+5} = 0$. The construction below works (in particular Lemma 6) only because $u_{3,6m+7} - v_{3,6m+7} \ge u_{3,6m+5} - v_{3,6m+5}$ (both sides of the inequality are equal to $u_{3,6m+5} = 6m^2 + 14m + 8$). Without this inequality we would have needed a longer proof. In general we need $u_{3,k+2} - v_{3,k+2} \ge u_{3,k} - v_{3,k}$ for all $k > 0$, but only in the case $k = 6m + 5$ the right hand side is not much bigger than the left hand side.

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. We have $\mathrm{Pic}(Q) \cong \mathbb{Z}^2$ and we take two distinct, but intersecting, lines contained in Q as a basis of $\mathrm{Pic}(Q)$. We will call $|\mathcal{O}_Q(1,0)|$ and $|\mathcal{O}_Q(0,1)|$ the two rulings of Q and call any $D \in |\mathcal{O}_Q(a,b)|$ a divisor (or a curve) of type (a,b). Fix a line $L \subset Q$, $P \in L$ and let γ be the set of all +lines $A \subset \mathbb{P}^3$ with L as their support and P as the support of the nilradical sheaf of $\mathcal{O}_A$. The set γ is the complement of a point in a two-dimensional projective space (it is $\mathbb{P}(T_P\mathbb{P}^3) \setminus \mathbb{P}(T_PL)$). The set γ' of all $A \in \gamma$ contained in Q is the line $\mathbb{P}(T_PQ)$ minus the point $\mathbb{P}(T_PL)$. If $A \in \gamma'$, then $A \subset Q$ and $\mathrm{Res}_Q(A) = \emptyset$. If $A \notin \gamma'$, then $A \cap Q = L$ (as schemes) and $\mathrm{Res}_Q(A) = \{P\}$ (as schemes). Now take $A \subset Q$, see L as a divisor of Q; we have $\mathrm{Res}_L(A) = \{P\}$.

Lemma 5. *$A_{r,1}$ and $A_{3,2}$ are true.*

Proof. $A_{r,1}$ is is true by Remark 3.

We have $v_{3,2} = 2$ and $u_{3,2} - v_{3,2} = 1$. Take any $B = L_1 \sqcup L_2 \sqcup L_3 \in L(3,3,0)$. B is contained in a unique quadric surface, Q', and Q' is a smooth quadric. Let $A \subset \mathbb{P}^3$ be a general +line with L_3 as its support. We have $L_1 \cup L_2 \cup A \in L(3,2,1)$. Since $A \nsubseteq Q'$, we have $h^0(\mathcal{I}_{L_1 \cup L_2 \cup A}(2)) = 0$. QED

Of course, $A_{r,k}$ makes sense only if $u_{r,k} - v_{r,k} \geq 0$; this is the reason why we didn't defined $A_{r,0}$. By the semicontinuity theorem for cohomology ([5], III.12.8) to prove $A_{3,k}$ it is sufficient to find $A \in L(3, v_{3,k}, u_{3,k} - v_{3,k})$ such that $h^0(\mathcal{I}_A(k)) = 0$. Fix an integer $k > 0$ and let $X \subset \mathbb{P}^3$ be a general union of $v_{3,k}$ lines and $u_{3,k} - v_{3,k}$ +lines. We have $h^0(\mathcal{O}_X(k)) = (u_{3,k} - v_{3,k})(k+2) + v_{3,k}(k+1) = \binom{k+3}{3}$, the latter equality being true by the definition of the integer $v_{3,k}$. Hence $h^1(\mathcal{I}_X(k)) = h^0(\mathcal{I}_X(k))$.

Lemma 6. *$A_{3,k} \Rightarrow A_{3,k+2}$ for all $k > 0$.*

Proof. We have $0 \leq u_{3,k+2} - u_{3,k} \leq k + 2$.

(a) First assume $v_{3,k+2} \geq v_{3,k}$, i.e. $k \equiv 0, 3, 4, 5 \pmod 6$. Notice that in all cases we have $u_{3,k} - v_{3,k} \leq u_{3,k+2} - v_{3,k+2}$ (we even have equality if $k = 6m+5$, because $u_{3,6m+7} = 6m^2 + 18m + 14$, $v_{3,6m+7} = 4m+6$, $u_{3,6m+5} = 6m^2 + 14m + 8$ and $v_{3,6m+5} = 0$). Let $L_1 \subset Q$ be the union of $v_{3,k+2} - v_{3,k}$ distinct lines of type $(1,0)$ and $L_2 \subset Q$ the union of $(u_{3,k+2} - v_{3,k+2}) - (u_{3,k} - v_{3,k})$ distinct lines of type $(1,0)$ with $L_1 \cap L_2 = \emptyset$. Let $A_2 \subset Q$ be the union of $(u_{3,k+2} - v_{3,k+2}) - (u_{3,k} - v_{3,k})$ general +lines contained in Q and with the lines of L_2 as its support. Let $S_2 \subset L_2$ be the support of the nilradical of $\mathcal{O}_{A_2}$. Take a general $Y \in L(3, v_{3,k}, u_{3,k} - v_{3,k})$. For general Y we have $Y \cap (L_1 \cup L_2) = \emptyset$ and so $Y \cup A_2 \cup L_1 \in L(3, v_{3,k+2}, u_{3,k+2} - v_{3,k+2})$. By the semicontinuity theorem for cohomology ([5], III.12.8) it is sufficient to prove $h^0(\mathcal{I}_{Y \cup A_2 \cup L_1}(k)) = 0$. Since $\mathrm{Res}_Q(Y \cup A_2 \cup L_1) = Y$ and $h^0(\mathcal{I}_Y(k-2)) = 0$, it is sufficient to prove $h^0(Q, \mathcal{I}_{Q \cap (Y \cup A_2 \cup L_1)}(k+2)) = 0$. Since $L_1 \cup L_2 \subset (Y \cap Q) \cup A_2 \cup L_1$ and $L_1 \cup L_2 \in |\mathcal{O}_Q(u_{3,k+2} - u_{3,k}, 0)|$, it is sufficient to prove $h^0(Q, \mathcal{I}_{\mathrm{Res}_{L_1 \cup L_2}((Y \cap Q) \cup A_2 \cup L_1)}(k+2-u_{3,k+2}+u_{3,k}, k+2)) = 0$. We have $\mathrm{Res}_{L_1 \cup L_2}((Y \cap Q) \cup A_2 \cup L_1) = (Y \cap Q) \cup S_2$. For general A_2 the set S_2 is a set containing a general point of $(u_{3,k+2} - v_{3,k+2}) - (u_{3,k} - v_{3,k})$ general lines of type $(1,0)$ and nothing else. Hence S_2 may be considered as a general union of $(u_{3,k+2} - v_{3,k+2}) - (u_{3,k} - v_{3,k})$ points of Q. For general Y the set $Y \cap Q$ is a general subset of Q with cardinality $2u_{3,k}$. Hence it is sufficient to check that $\sharp((Y \cap Q) \cup S_2) = h^0(Q, \mathcal{O}_Q(k+2-u_{3,k+2}+u_{3,k}, k+2))$, i.e. $2u_{3,k} + (u_{3,k+2} - v_{3,k+2}) - (u_{3,k} - v_{3,k}) = (k+3-u_{3,k+2}+u_{3,k})(k+3)$, i.e. $2u_{3,k} + (k+2)(u_{3,k+2} - u_{3,k}) = (k+3)^2 + v_{3,k+2} - v_{3,k}$. Taking the difference of

(4) for the integer $k' = k+2$ from (4) and using that $\binom{k+5}{3} - \binom{k+3}{3} = (k+3)^2$ we get $2u_{3,k} + (k+2)(u_{3,k+2} - u_{3,k}) = (k+3)^2 + v_{3,k+2} - v_{3,k}$, as wanted.

(b) Now assume $v_{3,k+2} < v_{3,k}$, i.e. $k \equiv 1,2 \pmod 6$. Take a general $Y \in L(3, v_{3,k}, u_{3,k} - v_{3,k})$ and write $Y = E \cup F$ with $E \in L(3, v_{3,k+2}, u_{3,k} - v_{3,k})$ and $F \in L(3, v_{3,k} - v_{3,k+2}, 0)$. For general Y we have $h^0(\mathcal{I}_Y(k)) = 0$ (by the inductive assumption) and $Y \cap Q$ is a general subset of Q with cardinality $2u_{3,k}$. For each line $L \subseteq F$ fix one of the point $P_L \in L \cap Q$ and call v_L a general tangent vector of Q at P_L. Let $A_L = L \cup v_L \subset \mathbb{P}^3$ be the +lines with L as its reduction, P_L as the support of its nilradical and containing v_L. Set $G := \cup_{L \in F} A_L$. Let $M \subset Q$ be a union of $u_{3,k+2} - u_{3,k}$ general lines of type $(1,0)$. Let $N \subset Q$ be a general union of $u_{3,k+2} - u_{3,k}$ +lines with M as the union of their support. We have $E \cup G \cup N \in L(3, v_{3,k+2}, u_{3,k+2} - v_{3,k+2})$. Since $\mathrm{Res}_Q(E \cup G \cup N) = Y$ and $h^0(\mathcal{I}_Y(k)) = 0$, it is sufficient to prove $h^0(Q, \mathcal{I}_{Q \cap (E \cup G \cup N)}(k+2, k+2))$. Since $M \subset Q \cap (E \cup G \cup N)$, it is sufficient to prove $h^0(Q, \mathcal{I}_{\mathrm{Res}_M(Q \cap (E \cup G \cup N))}(k+2-u_{3,k+2}+u_{3,k}, k+2)) = 0$. The scheme $G \cap Q$ is a general union δ of $v_{3,k}$ tangent vectors of Q and a general union of $v_{3,k} - v_{3,k+2}$ points of Q; we do not want to use here that general tangent vectors gives the maximal possible number of conditions to any linear system, because it requires characteristic zero ([4], [1], Lemma 1.4); however, since $v_{3,k} \le 2(k+2)/3$, it is obvious that $h^1(Q, \mathcal{I}_\delta(k+2-u_{3,k+2}+u_{3,k}, k+2)) = 0$; alternatively, use Lemma 4. Set $S := \mathrm{Res}_M(N)$. The set S contains one point for each line of M and it is general with this condition. Since M is a general union of $u_{3,k+2} - u_{3,k}$ lines of type $(1,0)$, S may be considered as a general subset of Q with its cardinality. The set $E \cap Q$ is a general subset of Q with cardinality $2u_{3,k} - 2v_{3,k+2}$. Since $2(v_{3,k} - v_{3,k+2}) + (v_{3,k} - v_{3,k-2}) + (u_{3,k+2} - u_{3,k}) + 2(u_{3,k} - v_{3,k+2}) = (k+3-u_{3,k+2}+u_{3,k})(k+3)$, we are done. QED

Lemma 7. *For all integers $k > 0$ and $c > 0$ such that $c(k+2) \le \binom{k+3}{3}$ we have $h^1(\mathcal{I}_X(k)) = 0$ for a general $X \in L(3,0,c)$.*

Proof. If $k = 1$, then $c = 1$. The lemma is obvious in this case.

Now assume $k = 2$. It is sufficient to do the case $c = 2$. Take $A = A_1 \cup A_2 \in L(3,0,2)$ with L_1 and L_2 two different lines of type $(1,0)$ of Q, $A_1 \subset Q$ and general with these restrictions, $A_2 \nsubseteq Q$ and general among the +lines supported by Q. We get $h^0(Q, \mathcal{O}_{Q \cap A}(2)) = 2$ and $h^0(\mathcal{I}_{\mathrm{Res}_Q(A)}) = 0$, because $\mathrm{Res}_Q(A) \neq \emptyset$.

From now on we assume $k \ge 3$. Lemmas 5 and 6 give that $A_{3,k-2}$ and $A_{3,k}$ are true. We have $c \le u_{3,k}$ and $c \le u_{3,k} - 1$ if $v_{3,k} > 0$. If $c \le u_{3,k} - v_{3,k}$, then we may use $A_{3,k}$. In particular we are done if $v_{3,k} = 0$. Hence we may assume $v_{3,k} > 0$. In this case it is sufficient to do the case $c = u_{3,k} - 1$. Fix a general $Y \in L(3, v_{3,k-2}, u_{3,k-2} - v_{3,k-2})$. We have $h^i(\mathcal{I}_Y(k-2)) = 0$, $i = 0,1$, by $A_{3,k-2}$. We mimic part (b) of the proof of Lemma 6. Write $Y = E \sqcup F$ with $E \in L(3, 0, u_{3,k} - v_{3,k})$ and $F \in L(3, v_{3,k}, 0)$. For each line $L \subseteq F$ fix one of the point $P_L \in L \cap Q$ and call v_L a general tangent vector of Q at P_L. Let $A_L = L \cup v_L \subset \mathbb{P}^3$ be the +line with L as its reduction, P_L as the support of its nilradical and containing v_L. Set $G := \cup_{L \in F} A_L$. Let $M \subset Q$ be a union of $u_{3,k+2} - u_{3,k} - 1$ general lines of type $(1,0)$. Let $N \subset Q$ be a general union of $u_{3,k+2} - u_{3,k} - 1$ +lines with M as the union of their support. Take $X := E \cup G \cup N \in L(3, 0, u_{3,k} - 1)$. Since $v_{3,k} \le k+1$, as in part (b) of the proof of Lemma 6 we get $h^1(Q, \mathcal{I}_{X \cap Q}(k)) = 0$ and hence $h^1(\mathcal{I}_X(k)) = 0$. QED

Proof of Theorem 1 for $r = 3$: It is sufficient to prove $H_{3,k}$ for all $k \ge 2$ (Remark 3). Fix an integer $k > 0$. It is sufficient to check the Hilbert function in degree k of a general element of $L(3, t, c)$ with either $t = 0$ and $(k+2)c \le \binom{k+3}{3}$ or $(k+1)t + (k+2)c = \binom{k+3}{3}$ (Remark 4). By Lemma 7 it is sufficient to check the pairs (t, c) with $t > 0$ and $(k+1)t + (k+2)c = \binom{k+3}{3}$, i.e. (since the integers $k+1$ and $k+2$ are coprime) the pairs (t, c) with $t = v_{3,k} + (k+2)\alpha$, $c = u_{3,k} - v_{3,k} - (k+1)\alpha$ for some non-negative integer α such that $(k+1)\alpha \le u_{3,k} - v_{3,k}$. By [6] we may assume $c > 0$. By Lemma 6 we may assume $t > v_{3,k}$. Since $v_{3,2} = 2$, $u_{3,2} = 3$ and $A_{3,2}$ is true (Lemma 5), we may assume $k \ge 3$. By induction on k we may assume that $h^i(\mathcal{I}_W(k-2)) = 0$, $i = 0, 1$, for a general $W \in L(3, t', c')$ for all non-negative integers t', c' such that $(k-1)t' + kc' = \binom{k+1}{3}$. Fix a general $Y \in L(3, m_{3,k-2} - n_{3,k-2}, n_{3,k-2})$. We have $h^i(\mathcal{I}_Y(k-2)) = 0$, $i = 0, 1$.

Claim 1: We have $t + c \ge m_{3,k-2}$.

Proof of Claim 1: Assume $t + c \le m_{3,k-2} - 1$, i.e. assume $(k-1)(t+c) + k - 1 \le \binom{k+1}{3}$. Since $(k+1)t + (k+2)c = \binom{k+3}{3}$, we get $(k+2)/(k-1) > \binom{k+3}{3}/\binom{k+1}{3} = (k+3)(k+2)/k(k-1)$, a contradiction.

Claim 2: We have $c \ge n_{3,k-2}$.

Proof of Claim 2: If $k - 2 \equiv 0, 1 \pmod 3$, then $n_{3,k-2} = 0$. If $k - 2 \equiv 2 \pmod 3$, then $n_{3,k-2} = (k-1)/3$. Hence we may assume $k \equiv 4 \pmod 3$. Since $n_{3,k} = 0$, $c > 0$ and $(k+1)t + (k+2)c = \binom{k+3}{3}$, we get $c = \beta(k+1)$ for some integer $\beta > 0$. Hence $c \ge k+1 > n_{3,k-2}$.

Notice that $m_{3,k-2} \ge n_{3,k-2}$. Fix a general $Y \in L(3, m_{3,k-2} - n_{3,k-2}, n_{3,k-2})$. By the inductive assumption we have $h^i(\mathcal{I}_Y(k-2)) = 0$, $i = 0, 1$. Set $e := t + c - m_{3,k-2}$. Claim 1 gives $e \ge 0$. Take a general union $M \subset Q$ of e lines of type $(1, 0)$.

(a) Assume $c - n_{3,k-2} \le e$. Claim 2 gives $c - n_{3,k-2} \ge 0$. Write $M = M_1 \sqcup M_2$ with M_2 a union of $c - n_{3,k-2}$ lines and M_1 a union of $e - (c - n_{3,k-2})$ lines. Let $A_2 \subset Q$ be a general union of $c - n_{3,k-2}$ +lines with the lines in M_2 as their support. Let S_2 be the support of the nilpotent sheaf of A_2. Since A_2 is general, S_2 is obtained taking for each line $L \subseteq M_2$ a general point of L. Set $X := Y \cup M_1 \cup A_2$. Since $X \in L(3, t, c)$, it is sufficient to prove that

$h^1(\mathcal{I}_X(k)) = 0$. Since $\mathrm{Res}_Q(X) = Y$ and $h^1(\mathcal{I}_Y(k-2)) = 0$, it is sufficient to prove that $h^1(Q, \mathcal{I}_{X\cap Q}(k)) = 0$. Since $\mathrm{Res}_M(X\cap Q) = (Y\cap Q)\cup S_2$, it is sufficient to prove that $h^1(Q, \mathcal{I}_{(Y\cap Q)\cup S_2}(k-e,k)) = 0$. Since M_2 is general and for each line $L \subset M_2$ the set $S_2\cap L$ is a general point of L, S_2 is a general subset of Q with cardinality $c - n_{3,k-2}$. Since Y is general, the set $(Y\cap Q)\cup S_2$ is a general subset of Q of cardinality $2m_{3,k-2} + c - n_{3,k-2}$. Since $(k-1)m_{3,k-2} + n_{3,k-2} = \binom{k+1}{3}$, $(k+1)t + (k+2)c = \binom{k+3}{3}$, $\binom{k+3}{3} - \binom{k+1}{3} = (k+1)^2$ and $e = t + c - m_{3,k-2}$, we have $2m_{3,k-2} + (k+1)e + c - n_{3,k-2} = (k+1)^2$, i.e. $\sharp(S_2 \cup (Y \cap Q)) = (k+1)(k+1-e) = h^0(Q, \mathcal{O}_Q(k-e,k))$. Hence $h^i(Q, \mathcal{I}_{(Y\cap Q)\cup S_2}(k-e,k)) = 0$, $i = 0, 1$.

(b) Now assume $c - n_{3,k-2} > e$. For each line $R \subseteq M$ fix a general $O_R \in R$ and call v_R a general tangent vector of Q with O_R as its support. Set $R^+ := R \cup v_R$, $M^+ := \cup_{R\subseteq M} R^+$ and $S := \cup_{R\subseteq M} O_R$. Since M is general and each O_R is general in R, S is a general subset of Q with cardinality e. We have $M^+ \subset Q$ and hence $M^+ \cap Q = M^+$ and $\mathrm{Res}_Q(M^+) = \emptyset$. Set $g := c - n_{3,k-2} - e$. Since $e = t + c - m_{3,k-2}$, we get $m_{3,k-2} - n_{3,k-2} = g + t > t$. Write $Y = Y_1 \sqcup Y_2$ with $Y_2 \in L(3, 0, n_{3,k-2})$ and $Y_1 \in L(3, m_{3,k-2} - n_{3,k-2}, 0)$. Since $m_{3,k-2} - n_{3,k-2} = g + t \geq t$, we may write $Y_1 = Y_3 \sqcup Y_4$ with $Y_3 \in L(3, t, 0)$ and $Y_4 \in L(3, g, 0)$. For each line $L \subseteq Y_4$ fix one of the two points, say O_L, of $L \cap Q$ and let w_L be a general tangent vector of Q with O_L as its support; set $L^+ := L \cup w_L \in L(3, 0, 1)$. Set ${Y_4}^+ := \cup_{L\subseteq Y_4} L^+$ and $X' := M^+ \cup {Y_4}^+ \cup Y_2 \cup Y_3$. Since $X' \in L(3, t, c)$, it is sufficient to prove that $h^1(\mathcal{I}_{X'}(k)) = 0$. Since $\mathrm{Res}_Q(X') = Y$ and $h^1(\mathcal{I}_Y(k-2)) = 0$, by the Castelnuovo's sequence it is sufficient to prove that $h^1(Q, \mathcal{I}_{X'\cap Q}(k)) = 0$. The scheme $X'\cap Q$ is the union of M^+ and $(Y_3 \cup {Y_4}^+)\cap Q$. We have $\mathrm{Res}_M(M^+ \cup (Y_3 \cup {Y_4}^+) \cap Q) = S \cup ((Y \setminus Y_4) \cap Q) \cup \bigcup_{L\subseteq Y_4} w_L$. Since $Y \cap Q$ is a general subset of Q with cardinality $2m_{3,k-2}$, the scheme $Z := S \cup ((Y \setminus Y_4) \cap Q) \cup \bigcup_{L\subseteq Y_4} w_L$ is a general union of g tangent vectors of Q and $e + 2m_{3,k-2} - g$ points of Q. By the Castelnuovo's sequence it is sufficient to prove that $h^1(Q, \mathcal{I}_Z(k-e,k)) = 0$. By Lemma 4 it is sufficient to check that $\deg(Z) \leq (k-e+1)(k+1)$. By (3) for the integers $r = 3$ and $k' = k-2$ and the equality $(k+1)t + (k+2)c = \binom{k+3}{3}$ we get $\deg(Z) = (k+1)(k-e+1)$. QED

3 When $r > 3$

In this section we prove Theorem 1 for all integers $r \geq 4$. For numerical reasons this is easier than in the case $r = 3$ (as it was in [6] and [2]). The proof in characteristic zero is very short and we will only give it.

Lemma 8. *For all integers $r \geq 3$ and $k \geq 2$ we have $m_{r,k-1} < u_{r,k}$.*

Proof. We have $km_{r,k-1} \leq \binom{k+r-1}{r}$ and $(k+2)u_{r,k} \geq \binom{r+k}{r}$. Note that

$$\binom{r+k}{r} \Big/ \binom{r+k-1}{r} = (r+k)/k$$

and that $(r+k)k > (k+2)k$ for all $r \geq 3$. QED

We need the following assumption $B_{r,k}$:

$B_{r,k}$, $r \geq 4$, $k > 0$. Fix a hyperplane $H \subset \mathbb{P}^r$. There is $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ such that the support of the nilradical sheaf of X is contained in H and $h^0(\mathcal{I}_X(k)) = 0$.

For all $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ we have $h^0(\mathcal{O}_X(k)) = \binom{r+k}{r}$ and so $h^1(\mathcal{I}_X(k)) = h^0(\mathcal{I}_X(k))$.

Lemma 9. *For all integers $r \geq 4$ and $k \geq 2$ we have $m_{r,k} \geq m_{r,k-1}$.*

Proof. We have

$$m_{r,k-1} + (k+1)(m_{r,k} - m_{r,k-1}) + n_{r,k} - n_{r,k-1} = \binom{r+k-1}{r-1} \quad (5)$$

Assume $m_{r,k} \leq m_{r,k-1} - 1$. Since $n_{r,k} - n_{r,k-1} \leq k$, (6) gives

$$m_{r,k-1} - 1 \geq \binom{r+k-1}{r-1}$$

Since $km_{r,k-1} \leq \binom{r+k-1}{r}$ and $k\binom{r+k-1}{r-1} = r\binom{r+k-1}{r}$, we get $-k \geq (r-1)\binom{r+k-1}{r-1}$, a contradiction. QED

Lemma 10. *Fix an integer $r \geq 4$ and assume that Theorem 1 is true in $\mathbb{P}^{r-1}$. Then $B_{r,k}$ is true for all $k > 0$.*

Proof. $B_{r,1}$ is true by Remark 3. Hence we may assume $k \geq 2$ and that $B_{r,k-1}$ is true. Fix $Y \in L(r, m_{r,k-1} - n_{r,k-1}, n_{r,k-1})$ such that the support of the nilradical sheaf of Y is contained in H and $h^0(\mathcal{I}_Y(k-1)) = 0$. By the semicontinuity theorem for cohomology ([5], III.12.8) we may assume that Y is general among the elements of $L(r, m_{r,k-1} - n_{r,k-1}, n_{r,k-1})$ whose nilradical sheaf is supported by points of H. Hence we may assume that no irreducible component of Y_{red} is contained in H, that Y_{red} is a general subset of H with cardinality $m_{r,k-1}$ and that for each +line $A \subset Y$, say $A = L \cup v_L$, the tangent vector v_L of A is not contained in H. The latter assumption implies $Y_{red} \cap H = Y \cap H$ (scheme-theoretic intersection) and $\mathrm{Res}_H(Y) = Y$. We have $m_{r,k} \geq m_{r,k-1}$ (Lemma 9).

(a) In this step we assume $n_{r,k} < n_{r,k-1}$. Let $F \subset H$ be a general union of $m_{r,k} - m_{r,k-1}$ lines, with the only restriction that exactly $n_{r,k-1} - n_{r,k}$ of them contain a point of $Y_{red} \cap H$. We have $m_{r,k-1} - n_{r,k-1} - (n_{r,k-1} - n_{r,k}) + m_{r,k} - m_{r,k-1} = m_{r,k} - 2n_{r,k-1} + n_{r,k}$. The scheme $Y \cup F$ is a disjoint union of $n_{r,k-1} - n_{r,k}$ sundials, $n_{r,k}$ +lines and $m_{r,k} - 2n_{r,k-1} + n_{r,k}$ lines. Since a sundial is a flat limit of a family of elements of $L(r, 2, 0)$ ([2]), it is sufficient to prove $h^0(\mathcal{I}_{Y\cup F}(k)) = 0$. Since the set $Y_{red} \cap H$ is general in H, F may be considered as a general union of lines. Hence F has maximal rank. By (5) we have $h^1(H, \mathcal{I}_F(k)) = 0$ and $h^0(H, \mathcal{I}_F(k)) = m_{r,k-1} - n_{r,k-1} + n_{r,k}$. Since for fixed $Y_{red} \cap H \cap F$ we may deform the other components of Y so that the other $m_{r,k-1} - n_{r,k-1} + n_{r,k}$ points of $Y_{red} \cap (H \setminus F)$ are general in H, then $h^i(H, \mathcal{I}_{H\cap(Y\cup F)}(k)) = 0$. Castelnuovo's sequence gives $h^0(\mathcal{I}_{Y\cup F}(k)) = 0$.

(b) In this step we assume $n_{r,k} \geq n_{r,k-1}$ and $m_{r,k} - n_{r,k} \geq m_{r,k-1} - n_{r,k-1}$. Let $E \subset H$ be a general union of $m_{r,k} - n_{r,k} - (m_{r,k-1} - n_{r,k-1})$ lines and $n_{r,k} - n_{r,k-1}$ +lines. We have $Y \cup E \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ and the support of the nilradical sheaf of $Y \cup E$ is contained in H. By (5) we have $h^0(E, \mathcal{O}_E(k)) + \deg(Y \cap H) = \binom{r+k-1}{r-1}$. Since Theorem 1 is true in $\mathbb{P}^{r-1}$, we have $h^1(H.\mathcal{I}_E(k)) = 0$. Since $Y \cap H$ is a general union of $m_{r,k-1}$ points of H, (5) implies $h^i(H, \mathcal{I}_{(Y\cup E)\cap H}(k)) = 0$.

(c) In this step we assume $n_{r,k} \geq n_{r,k-1}$ and $m_{r,k} - n_{r,k} < m_{r,k-1} - n_{r,k-1}$. Therefore $g := n_{r,k} - n_{r,k-1} - (m_{r,k} - m_{r,k-1}) > 0$. Since $n_{r,k} \leq k$, we have $g \leq k$. Since $km_{r,k-1} + n_{r,k-1} = \binom{r+k-1}{r}$ and $n_{r,k-1} \leq k-1$, we have $g \leq m_{r,k-1} - n_{r,k-1}$. Take a general union $G \subset H$ of $m_{r,k} - m_{r,k-1}$ +lines. Since Theorem 1 is assumed to be true in $\mathbb{P}^{r-1}$, G has maximal rank. By (5) we have $h^1(H, \mathcal{I}_G(k)) = 0$ and $h^0(H, \mathcal{I}_G(k)) = m_{r,k-1} + g$. Write $Y = Y_1 \sqcup Y_2 \sqcup Y_3$ with $Y_3 \in L(r, 0, n_{r,k-1})$, $Y_1 \in L(r, m_{r,k-1} - n_{r,k-1}, 0)$ and $Y_2 \in L(r, g, 0)$. For each line $L \subseteq Y_2$ let v_L be the general tangent vector of H with $L \cap H$ as its support. Set $A_2 := \cup_{L\subseteq Y_2}(L \cup v_L)$. Since $U := Y_1 \cup A_2 \cup Y_3 \cup G \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$, it is sufficient to prove that $h^0(\mathcal{I}_U(k)) = 0$. We have $\mathrm{Res}_H(U) = Y$, because each v_L is contained in H. The scheme $U \cap H$ is the union of G, $m_{r,k-1} - n_{r,k-1} - g$ general points of H and g general tangent vectors of H. Hence $h^i(H, \mathcal{I}_{U\cap H}(k)) = 0$, $i = 0, 1$ ([1], Lemma 1.4). QED

Proof of Theorem 1 for $r > 3$: Let $H \subset \mathbb{P}^r$ be a hyperplane. We use induction on r, the starting case being the one with $r = 3$ proved in section 2. Hence we assume Theorem 1 in $H \cong \mathbb{P}^{r-1}$ for all $L(r-1, t', c')$. By Remark 3 it is sufficient to prove $H_{r,k}$ for all $k > 0$. $H_{r,1}$ is true (Remark 3). Hence we may assume $k \geq 2$ and that $H_{r,k-1}$ is true. By Remark 4 it is sufficient to prove $H_{r,k}$ for the pairs (t, c) such that either $t = 0$ and $\binom{r+k}{r} - k - 1 \leq c(k+2) \leq \binom{r+k}{r}$ or $t(k+1) + (k+2)c = \binom{r+k}{r}$ and $c > 0$. If $\binom{r+k}{r} - k - 1 \leq c(k+2) \leq$

$\binom{r+k}{r}$, then either $v_{r,k} = 0$ and $u_{r,k} = c$ or $v_{r,k} > 0$ and $c = u_{r,k} - 1$. If $t(k+1) + (k+2)c = \binom{r+k}{r}$, then $t + c \ge u_{r,k}$. Hence in both cases we have $m_{r,k-1} \le t + c$ (Lemma 8). Since $k \ge 2$, we have $m_{r,k-1} \ge n_{r,k-1}$ (Remark 5). Fix a general $Y \in L(r, m_{r,k-1} - n_{r,k-1}, n_{r,k-1})$. $H_{r,k-1}$ implies $h^i(\mathcal{I}_Y(k-1)) = 0$, $i = 0, 1$. The set $H \cap Y$ is a general subset of H with cardinality $m_{r,k-1}$. We have $\mathrm{Res}_H(Y) = Y$. We have

$$m_{r,k+1} + (k+1)(t + c - m_{r,k-1}) + c - n_{r,k-1} \le \binom{t+k-1}{r-1} \tag{6}$$

and the difference among the right hand side and the left hand side is at most $k+1$.

(a) In this step we assume $c \ge n_{r,k-1}$ (this is always the case if $t = 0$). Set $e := t + c - m_{r,k-1}$.

(a1) First assume $c - n_{r,k-1} \le e$. Let $E \subset H$ be a general union of $c - n_{r,k-1}$ +lines and $e - c + n_{r-1,k}$ lines. By the inductive assumption on r the scheme E has maximal rank in $H \cong \mathbb{P}^{r-1}$. By (6) we have $\sharp(Y \cap H) + h^0(\mathcal{O}_E(k)) \le \binom{k+r+1}{r-1}$. Since E has maximal rank in H, we get $h^1(H, \mathcal{I}_E(k)) = 0$. Since $\sharp(Y \cap H) + h^0(\mathcal{O}_E(k)) \le \binom{k+r+1}{r-1}$ and $Y \cap H$ is general in H, we get $h^1(H, \mathcal{I}_{(Y\cap H)\cup H}(k)) = 0$. Since $h^1(\mathcal{I}_Y(k-1)) = 0$, the Castelnuovo's sequence gives $h^1(\mathcal{I}_{Y\cup E}(k)) = 0$, concluding the proof in this case.

(a2) Now assume $c - n_{r,k-1} > e$. Let $F \subset H$ be a general union of e +lines of H. Set $g := c - n_{r,k-1} - e = m_{r,k-1} - n_{r,k-1} - t$. We have $t \ge 0$, $g > 0$ and $t + g = m_{r,k-1} - n_{r,k-1}$. Write $Y = Y_1 \sqcup Y_2$ with Y_1 a general element of $L(r, m_{r,k-1} - n_{r,k-1}, 0)$ and Y_2 a general element of $L(r, 0, n_{r,k-1})$. Write $Y_1 = Y_3 \sqcup Y_4$ with $Y_4 \in L(r, g, 0)$ and $Y_3 \in L(r, t, 0)$. For each line $L \subseteq Y_4$ let v_L be a general tangent vector of H with $L \cap H$ as its support. Set $L^+ := L \cup v_L$ and $Y_4{}^+ := \cup_{L\subseteq Y_4} L^+$. Set $X' := F \cup Y_2 \cup Y_4{}^+ \cup Y_3$. The scheme X' is a disjoint union of t lines and c +lines. Since $\mathrm{Res}_H(X') = Y$, by the Castelnuovo's sequence it is sufficient to prove that $h^1(H, \mathcal{I}_{X'\cap H}(k)) = 0$. The scheme $X' \cap H$ is a general union of F (i.e. of e general +lines), $\deg(Y_3) + \deg(Y_2)$ general points of H and $\deg(Y_4)$ general tangent vectors of H. Since $F \subset H$ has maximal rank in H and the tangent vectors are general in H, the scheme $X' \cap H$ has maximal rank in H ([4], [1], Lemma 1.4). We have $\deg(Y_3) + \deg(Y_2) + 2\deg(Y_4) = m_{r,k-1} + m_{r,k-1} - n_{3,k-1} - t$. By (6) we have $m_{r,k-1} + g + (k+2)e \le \binom{r+k-1}{r-1}$. Hence $h^1(H, \mathcal{I}_{X'\cap H}(k)) = 0$.

(b) Now assume $c < n_{r,k-1}$. In particular we have $n_{r,k-1} > 0$. Since $n_{r,k-1} \le k-1$, we have $t = m_{r,k} - n_{r,k}$ and $c = n_{r,k}$. Lemma 10 gives $h^i(\mathcal{I}_X(k)) = 0$, $i = 0, 1$, for a general $X \in L(r, t, c)$. QED

Acknowledgements. I want to thank the referee for very useful comments.

References

[1] A. Bernardi, M. V. Catalisano, A. Gimigliano, M. Idà: *Secant varieties to osculating varieties of Veronese embeddings of* $\mathbb{P}^n$, J. Algebra, **321**, n. 3, (2009), 982–1004.

[2] E. Carlini, M. V. Catalisano, A. V. Geramita: 3-*dimensional sundials*, Cent. Eur. J. Math., **9**, n. 5, (2011), 949–971.

[3] E. Carlini, M. V. Catalisano, A. V. Geramita: *Reduced and non-reduced linear spaces: lines and points*, arXiv: 1308.0796.

[4] C. Ciliberto, R. Miranda: *Interpolations on curvilinear schemes*, J. Algebra, **203**, n. 2, (1998), 677–678.

[5] R. Hartshorne: Algebraic Geometry, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[6] R. Hartshorne, A. Hirschowitz: *Droites en position générale dans* $\mathbb{P}^n$, Algebraic Geometry, Proceedings, La Rábida 1981, 169–188, Lect. Notes in Math. 961, Springer, Berlin, 1982.