ASPECTS OF THE METRIC THEORY OF TENSOR PRODUCTS AND OPERATOR IDEALS

Andreas Defant
and
Klaus Floret

SUMMARY:
We give an introduction to Grothendieck's metric theory of tensor products with special emphasis on normed operator ideals.

CONTENTS

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0. INTRODUCTION

0.1. In the history of Functional Analysis there are few papers which were as influential as Grothendieck's «Résumé de la théorie métrique des produits tensoriels topologiques» submitted in 1954 and published in 1956 in the Bulletin of the Mathematical Society of São Paulo. It was written without proofs (with the exception of the fundamental theorem) and it seems that there were not many people who understood it – this was also due to the fact that there was some reluctance in the functional analysis community to accept thinking in terms of tensor products. It was the famous paper of Lindenstrauss and Pełczyński «Absolutely summing operators in $\mathcal{L}_p$-spaces and applications» (Studia Mathematica 1968) which stated Grothendieck's deep «théorème fondamental de la théorie métrique des produits tensoriels» as an inequality about $n \times n$ matrices and Hilbert spaces; fascinating applications were given in a «tensor-product-free» formulation about classes of operators, mainly absolutely-p-summing operators; Banach-space-theory (which had been considered as nearly completed in the mid-sixties by some people)was reactivated in an incredible way – and many of its important results nowadays are still related with the «Résumé». It is astonishing to see that many (certainly not all!) of the ideas of the Banach-space-theory of the last 20 years are even already contained in Grothendieck's paper though sometimes in a quite hidden way. The phrase «this result is implicitly contained in the Résumé» is fashionable, but nevertheless quite often true.

0.2. It seems that tensor products appeared in Functional Analysis for the first time during the late thirties in the work of Murray and John von Neumann on Hilbert-spaces. The first systematic study of classes of norms on tensor products of Banach-spaces is due to Schatten in 1943 who continued his work in a series of papers (partly together with von Neumann). Schatten's influential monograph «A Theory of Cross-Spaces» contains what was known in 1950; the most beautiful applications of the theory were on operator ideals on Hilbert spaces [75], the Hilbert-Schmidt operators, the trace-class or more generally the Schatten-von Neumann-classes S_p. Many of the more elementary aspects of Grothendieck's theory were known to Schatten but he was not aware of the important rôle of the finite-dimensional behaviour of tensornorms, e.g., in the study of the dual norms. On the other hand, the idea of operator ideals in the study of tensor products was always present. In 1968 Pietsch and his school started a systematic investigation of the notion of operator ideals on the class of Banach spaces and, ignoring tensor products, opened this way a method of thinking in a «categorical» manner which is as powerful as thinking in terms of tensor products – but it is certainly much easier to learn the basics of operator-ideal-theory than the basics of the theory of tensornorms. The development culminated in the publication of Pietsch's book «Operator Ideals» in 1978 which contains in a nearly encyclopaedic way everything known at this time about operator ideals. Though many of the ideas and results clearly came from the Résumé, tensor products were not at all used in the book.

0.3. Parallely with this development it was obvious that the use of the projective tensornorm π and the injective ε is very useful – and there were even sporadically papers dealing with general tensornorms. A highlight is Pisier's solution of the most famous problem stated in the Résumé: There is an infinite-dimensional Banach space P such that $P \otimes_\varepsilon P = P \otimes_\pi P$ isomorphically.

Pisier's 1986-book «Factorization of Linear Operators and Geometry of Banach Spaces» centers around the question under which circumstances an operator between Banach spaces factors through a Hilbert space which leads to a solution of all of the six problems stated at the end of the Résumé with the exception that the exact constant of the Grothendieck-inequality (as the «théorème fondamental» is nowadays called) is not yet known (the approximation-problem was solved in the negative by Enflo in 1972). Reading Pisier's book, it becomes apparent that it is useful to think in terms of operator ideals *and* in terms of tensor products. Another strong indication in this direction is a trick due to Pietsch from 1983 when he used tensor products of operators in order to give a simple proof of the famous result concerning the distribution of eigenvalues of absolutely-p-summing operators due to Johnson, König, Maurey, and Retherford (see [40], [43], [61] and 11.5).

0.4. The beauty and power of «tensorial» thinking, unfortunately, only becomes clear after really getting used to it. The Résumé is very hard to read and so there have been various attempts to present the theory of tensornorms (Amemiya-Shiga [1], Lotz [55], Losert-Michor [54], Michor [56], Gilbert-Leih [20] are known to us) but there seems to exist none which is easily accessible and, at the same time, incorporates the wonderful theory of operator ideals as it is nowadays. We hope that after having read this paper the reader knows that the theory of tensornorms is much less difficult than it seems sometimes and that she or he is convinced(and the historial development gives clear evidence for this) that both theories, the theory of tensornorms and of (normed!) operator ideals (if we consider them for a moment to be really different), are better understandable and richer if one works with both. It should become obvious that certain phenomena have their natural framework in tensor products and others in operators ideals.

0.5. We will give complete proofs - with the exception of Grothendieck's inequality (there are many proofs nowadays available, even in textbooks) and with the exception of characterizations of certain types of operators ((p, q)-factorable and (p, q)-dominated ones). Though there will be many results on minimal and maximal (always normed) operator ideals, we do not need but a basic knowledge from the theory of operator ideals. Much information comes directly from the simple, but basic one-to-one correspondance between maximal operator ideals $\mathcal{A}$ and tensornorms α (which are finitely generated as we shall say) given by: $\mathcal{A}$ and α are said to be *associated* if

$$\mathcal{A}(M, N) = M' \otimes_\alpha N$$

for finite-dimensional spaces. We think that the following two theorems (see 4.3 and 7.1) are fundamental for the understanding of the interplay between operator ideals $\mathcal{A}$ and associated tensornorms α:

The representation theorem for maximal operator ideals

$$\mathcal{A}(E,F') = (E \otimes_{\alpha'} F)' \qquad \text{isometrically}$$

and the representation theorem for the minimal operator ideals

$$E'\tilde{\otimes}_{\alpha} F \to \mathcal{A}^{\min}(E,F)$$

(metric surjection), where E and F are arbitrary Banach spaces.

0.6. In view of the applications it is natural to study tensornorms α first on finite-dimensional normed spaces and then extend them to arbitrary normed or Banach spaces. There are two ways to do this – an inductive procedure

$$\overrightarrow{\alpha}(z;E,F) := \inf\{\alpha(z;M,N) \mid z \in M \otimes N;\, M,N \text{ finite dim.}\}$$

and a projective procedure

$$\overleftarrow{\alpha}(z;E,F) := \sup\{\alpha(Q_L^E \otimes Q_K^F(z); E/L, F/K) \mid E/L, F/K \text{ finite dim.}\}.$$

Both coincide if (and somehow: only if, see 3.5) both spaces have the metric approximation property. Grothendieck chose the first one and this is justified when looking at the examples. But we found it very useful in our investigations to have also the «cofinite hull» $\overleftarrow{\alpha}$ at hand and we hope that we can convince the reader that it structures very well the way of thinking and is often very useful in finding and working out the proper statements and proofs. For operator ideals the cofinite hull gains importance by the fact that

$$E'\tilde{\otimes}_{\overleftarrow{\alpha}} F \overset{1}{\hookrightarrow} \mathcal{A}(E,F)$$

holds isometrically if α and $\mathcal{A}$ are associated (see 4.4).

0.7. We shall use the common notations of Banach-space-theory; in particular B_E denotes the closed unit ball of the normed space E (over the real or complex scalar field). Concerning operator ideals we follow Pietsch's book. If $T : E \to F$ is an operator, we indicate that it is a metric injection ($\|Tx\| = \|x\|$) by writing

$$T : E \overset{1}{\hookrightarrow} F$$

and that it is a metric surjection (F has the quotient norm of E via T) by

$$T : E \overset{1}{\twoheadrightarrow} F.$$

If $G \subset E$ is a subspace $I_G^E : G \overset{1}{\hookrightarrow} E$ denotes the canonical metric injection and $Q_G^E : E \overset{1}{\twoheadrightarrow} E/G$ (if G is closed) the canonical metric surjection.

If E and F are normed spaces, the projective tensornorm π on $E \otimes F$ is defined by

$$\pi(z; E, F) := \inf \{\sum_{i=1}^{n} \|x_i\| \|y_i\| \mid z = \sum_{i=1}^{n} x_i \otimes y_i\}$$

(this implies $\overset{\circ}{B}_{E \otimes_\pi F} = \Gamma \overset{\circ}{B}_E \otimes \overset{\circ}{B}_F$ for the open unit ball) and the injective tensornorm ε by

$$\varepsilon(z; E, F) := \sup\{|\langle \varphi \otimes \psi, z\rangle| \mid \varphi \in B_{E'}, \psi \in B_{F'}\}.$$

We assume the reader to be familiar with the basics of the tensornorms ε and π as they are presented e.g. in [37] or [45].

The universal property of the projective norm π says that

$$(E \otimes_\pi F)'$$

is, isometrically, the space if continuous bilinear forms on $E \times F$ and therefore again isometrically, the space of continuous linear operators $E \to F'$:

$$\begin{aligned} (E \otimes_\pi F)' &= \mathcal{L}(E, F') \qquad \text{isometrically} \\ \varphi &\rightsquigarrow L_\varphi \\ B_T &\leftsquigarrow T. \end{aligned}$$

Clearly

$$\langle L_\varphi x, y\rangle = \langle \varphi, x \otimes y\rangle \qquad \text{and} \qquad \langle B_T, x \otimes y\rangle = \langle Tx, y\rangle.$$

0.8. The trace tr_E on a normed space E is the linearization of the duality bracket

$$\begin{array}{ccc} E' \times E & \to & \mathbb{K} \\ (\varphi, x) & \rightsquigarrow & \langle \varphi, x\rangle \end{array}$$

whence

$$\begin{array}{ccc} \operatorname{tr}_E : E' \otimes E & \to & \mathbb{K} \\ \sum_{n=1}^N \varphi_n \otimes y_n & \rightsquigarrow & \sum_{n=1}^N \langle \varphi_n, y_n\rangle. \end{array}$$

For finite-dimensional spaces E this is the usual trace of operators in $E' \otimes E = \mathcal{L}(E,E)$. Clearly tr_E is continuous on $E' \otimes_\pi E$ and $\|\operatorname{tr}_E\| = 1$. The extension of

$$E' \otimes F = \mathcal{F}(E,F) \hookrightarrow \mathcal{L}(E,F)$$

($\mathcal{F}$ for the ideal of finite-dimensional operators) to the completion gives a metric surjection onto the nuclear operators $\mathcal{N}(E,F)$:

$$E'\tilde{\otimes}_\pi F \overset{1}{\twoheadrightarrow} \mathcal{N}(E,F).$$

It is well-known ([37], p. 406) that for a Banach space E

$$\tilde{\operatorname{tr}}_E : E'\tilde{\otimes}_\pi E \to \mathbb{K}$$

factors through $\mathcal{N}(E,E)$ (i.e.: the trace is defined for nuclear operators) if and only if E has the approximation property – and this again is equivalent to the injectivity of

$$F'\tilde{\otimes}_\pi E \to \mathcal{N}(F,E)$$

for all Banach spaces F.

Note di Matematica Vol. VIII - n. 2, 189-201(1988)

1. TENSORNORMS

1.1. A *tensornorm* α on the class *NORM* of all normed spaces assigns to each pair (E, F) of normed spaces a norm $\alpha(\cdot; E, F)$ on the algebraic tensor product $E \otimes F$ (shorthand: $E \otimes_\alpha F$ and $E\tilde{\otimes}_\alpha F$ for the completion) such that the following two conditions are satisfied: [(1)]

(1) α *is reasonable:* $\varepsilon \leq \alpha \leq \pi$
(2) α *satisfies the metric mapping property: If* $T_i \in \mathcal{L}(E_i, F_i)$, then

$$||T_1 \otimes T_2 : E_1 \otimes_\alpha E_2 \to F_1 \otimes_\alpha F_2|| \leq ||T_1||\,||T_2||$$

Clearly, the same detinition holds for subclasses of normed spaces: for the class *FIN* of all finite-dimensional spaces, for the class *BAN* of all Banach spaces or for the class *NORM x BAN* of pairs *(E, F)* where *E* is a normed and *F* a Banach space.

It can happen that all tensomorms are equivalent on $E \otimes F$: Pisier [63] has constructed an infinite-dimensionai Banach space *P* such that

$$P \otimes_\varepsilon P = P \otimes_\pi P$$

holds isomorphically; this celebrated example solved various other problems in Banach-space-theory.

The following *CRITERION* (it will be formulated only for *NORM)* is easy to check:

α is a tensornorm on NORM if and only if
(1) *$\alpha(\cdot; E, F)$ is a seminorm on $E \otimes F$ for all pairs (E, F) of normed spaces*
(2) $\alpha(1 \otimes 1; \mathbb{K}, \mathbb{K}) = 1$
(3) α satisfies the metric mapping property.

Though it is simple, it saves much work in many situations. Clearly

$$\alpha(x \otimes y; E, F) = ||x||\,||y||.$$

If $G \subset F$ is a subspace, then, by the mapping property,

$$\alpha(z; E, F) \leq \alpha(z; E, G) \qquad z \in E \otimes G.$$

For $\alpha = \varepsilon$ there is equality («ε respects subspaces») but for $\alpha = \pi$ the space $E \otimes_\pi G$ is in general not a topologica1 subspace of $E \otimes_\pi F$ since there is no general Hahn-Banach-theorem fo operators; if $E = L_1(\mu)$, then $E \otimes_\pi G \overset{1}{\hookrightarrow} E \otimes_\pi F$ and this characterizes L_1-spaces by a resul

(1) Schatten called a tensomorm «uniform cross-norm».

of Grothendieck's ([26]; the fact that $E \otimes_\pi G$ is always a topological subspace of $E \otimes_\pi F$ characterizes the $\mathcal{L}_1$-spaces, see 8.14). If $P : F \to G$ is a projection, then

$$\alpha(z; E, F) \leq \alpha(z; E, G) \leq ||P||\alpha(z; E, F) \qquad z \in \otimes G$$

and whence

$$E \otimes_\alpha G \overset{1}{\hookrightarrow} E \otimes_\alpha F$$

if G is 1-complemented in F.

1.2. If α is a tensornorm, then α^t

$$\alpha^t(\sum_{i=1}^n x_i \otimes y_i; E, F) := \alpha(\sum_{i=1}^n y_i \otimes x_i; F, E)$$

is a well-defined tensornorm, the *transposed tensornorm* of α. Obviously

$$\begin{array}{ccc} E \otimes_{\alpha^t} F & = & F \otimes_\alpha E \\ x \otimes y & & y \otimes x \end{array}$$

is an isometry.

1.3. If α is a tensornorm on the class *FIN* of all finite-dimensional normed spaces (same definition as in 1.1 by replacing $NORM$ by FIN), men there are two natural ways to extend it to the class of all normed spaces. For this, define for normed spaces E

$$FIN(E) := \{M \subset E \mid M \in FIN\}$$
$$COFIN(E) := \{L \subset E \mid E/L \in FIN\}$$

and

$$\overrightarrow{\alpha}(z; E, F) := \inf\left\{\alpha(z; M, N) \,\middle|\, \begin{array}{l} M \in FIN(E) \\ N \in FIN(F) \end{array} ; z \in M \otimes N\right\}$$

$$\overleftarrow{\alpha}(z; E, F) := \sup\left\{\alpha(Q_K^E \otimes Q_L^F(z); E/K, F/L) \,\middle|\, \begin{array}{l} K \in COFIN(E) \\ L \in COFIN(F) \end{array}\right\}$$

(the arrows come from me fact that the first procedure is inductive, the second projective). Obviously, it is enough to take cofinally many M, N and K, L, respectively, in the definitions. It is easy to see that *thefznite hull* $\overrightarrow{\alpha}$ and the *cofinite hull* $\overleftarrow{\alpha}$ are tensomorms such that

$$\varepsilon \leq \overleftarrow{\alpha} \leq \overrightarrow{\alpha} \leq \pi, \qquad \overleftarrow{\alpha}|_{FIN} = \overrightarrow{\alpha}|_{FIN} = \alpha$$

and

$$\overleftarrow{\alpha} \leq \alpha \leq \overrightarrow{\alpha}$$

if α was defined on *NORM.* Since ε respects subspaces: $\varepsilon = \overrightarrow{\varepsilon}$ and whence $= \overleftarrow{\varepsilon}$. The definition of the projective norm shows $\pi = \overrightarrow{\pi}$ but it will be shown in 3.5 that $\pi \neq \overleftarrow{\pi}$. A tensomorm α on *NORM* is called *finitely generated* if $\alpha = \overrightarrow{\alpha}$ and *cofinitely generated if* $\alpha = \overleftarrow{\alpha}$. Though the usual tensomonns are all finitely generated we find that the cofinite hull $\overleftarrow{\alpha}$ of a tensomorm is natural as well and its consequent use is structuring well the theory, helps understanding better various ideas and simplifies many proofs; we hope that the reader is convinced about this point after the study of this paper. This is why we adopted a more general notion of a tensomorm that Grothendieck did in his Résumé; there, all tensomorms are finitely generated by definition (but see 3.4). Grothendieck had a reason not to worry too much about cofinitely generated tensomorms:

$$\overrightarrow{\alpha}(\cdot; E, F) = \overleftarrow{\alpha}(\cdot; E, F)$$

if both spaces E and F have the metric approximation property (see 2.2 and below) and it was only in 1972 that Enflo discovered Banach spaces without the metric approximation property.

It is obvious but it is good to have it always in mind that two finitely generated (or two cofinitely generated) tensomorms are equa1 for finite-dimensiona1 spaces.

1.4. If M and F are normed spaces, M finite-dimensional, then

$$\mathcal{L}(M, F)' = (M' \otimes_\varepsilon F)' \overset{1}{=} M \otimes_\pi F'$$

by the basic duality relation between the injective tensomorm ε and the projective tensomorm π (see [45], p. 246), whence

$$\mathcal{L}(M, F)'' = (M \otimes_\pi F')' = \mathcal{L}(M, F'')$$

isometrically. Helly's lemma ([60], p. 383) on the density of $G := \mathcal{L}(M, F)$ in $G'' = \mathcal{L}(M, F'')$ with respect to the subspace

$$M \otimes N \subset M \otimes_\pi F' = G'$$

gives the

Weak principle of local reflexivity. *Let M and F be normed spaces, M jinite dimensiona/ and $S \in \mathcal{L}(M, F'')$. Then for every $\varepsilon > 0$ and $N \in FIN(F')$ there is an $R \in \mathcal{L}(M, F)$ such that*

$$\|R\| \leq (1 + \varepsilon) \|S\|$$

and

$$\langle Sx, y'\rangle_{F'',F'} = \langle Rx, y'\rangle_{F,F'}$$

for all $(x, y') \in M \times N$.

This will be basic for many investigations on tensomorms. The stronger version (R can be chosen such that $Rx = Sx$ whenever $x \in S^{-1}(F)$, see e.g. [60], p. 384) will not be needed.

1.5. Many of the interesting tensomorms can be obtained from the ones introduced by Lapresté [49] generalizing those of Saphar [66], Chevet [6] and Cohen [8]. First some notations: let E be normed, $x_1, \dots, x_n \in E$, and $p \in [1, \infty]$, then

$$\ell_p(x_i; E) := \ell_p(x_i) := \|(\|x_i\|_E)_{i=1,\dots,n}\|_{\ell_p^n} \qquad \textit{strong } \ell_p\textit{-norm}$$

$$w_p(x_i; E) := w_p(x_i) := \sup_{\varphi \in B_{E'}} \|(\langle \varphi, x_i\rangle)_{i=1,\dots,n}\|_{\ell_p^n} \qquad \textit{weak } \ell_p\textit{-norm}$$

It is easy to see that in the definition of the weak ℓ_p-norm the unit ball $B_{E'}$ can be replaced by any norming subset of $B_{E'}$.

For $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} \geq 1$ define $r \in [1, \infty]$ by

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \qquad \text{or, equivalently,} \qquad 1 = \frac{1}{r} + \frac{1}{p'} + \frac{1}{q'}$$

and for normed spaces E and F

$$\alpha_{p,q}(z; E, F) := \inf \{\ell_r(\lambda_i) w_{q'}(x_i) w_{p'}(y_i) \mid z = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i\}$$

Obviously $\alpha_{1,1} = \pi$.

Proposition.

(1) $\alpha_{p,q}$ *is a finitely generated tensornorm on* $NORM$.

(2) $\alpha_{p_2,q_2} \leq \alpha_{p_1,q_1}$ *if* $p_1 \leq p_2$ *and* $q_1 \leq q_2$

(3) $\alpha_{p,q}^t = \alpha_{q,p}$

Proof :

(1) Using criterion 1.1 only the triangle inequality is not obvious: Take $z_1, z_2 \in E \otimes F$ and $\varepsilon > 0$, choose representations

$$z_j = \sum_{i=1}^{n} \lambda_{ij} x_{ij} \otimes y_{ij} \qquad j = 1,2$$

such that

$$\ell_r(\lambda_{ij}) \leq (\alpha_{p,q}(z_j) + \varepsilon)^{\frac{1}{r}}$$
$$w_{q'}(x_{ij}) \leq (\alpha_{p,q}(z_j) + \varepsilon)^{\frac{1}{q'}}$$
$$w_{p'}(y_{ij}) \leq (\alpha_{p,q}(z_j) + \varepsilon)^{\frac{1}{p'}}$$

and whence

$$\alpha_{p,q}(z_1 + z_2) \leq \ell_r((\lambda_{ij})_{i,j}) w_{q'}((x_{ij})_{i,j}) w_{p'}((y_{ij})_{i,j}) \leq$$
$$\leq (\alpha_{p,q}(z_1) + \alpha_{p,q}(z_2) + 2\varepsilon)^{\frac{1}{r}+\frac{1}{q'}+\frac{1}{p'}}.$$

(2) There is nothing to prove for $\frac{1}{p_1} + \frac{1}{q_1} = 1$, whence assume $r_1 < \infty$ and define

$$\frac{1}{p} := \frac{1}{p_1} - \frac{1}{p_2} \quad , \quad \frac{1}{q} := \frac{1}{q_1} - \frac{1}{q_2}$$

which implies

$$\frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{p} + \frac{1}{q}.$$

Take $z \in E \otimes F$ and, for $\varepsilon > 0$, a representation

$$z = \sum_i \lambda_i x_i \otimes y_i \qquad \lambda_i \geq 0$$

with

$$\ell_{r_1}(\lambda_i) w_{q_1'}(x_i) w_{p_1'}(y_i) \leq (1+\varepsilon) \alpha_{p_1,q_1}(z).$$

Now

$$z = \sum_i \lambda_i^{r_1/r_2} (\lambda_i^{r_1/q} x_i) \otimes (\lambda_i^{r_1/p} y_i)$$

and (by Hölder's inequality)

$$\ell_{r_2}(\lambda_i^{r_1/r_2}) = [\ell_{r_1}(\lambda_i)]^{r_1/r_2}$$
$$w_{q_2'}(\lambda_i^{r_1/q} x_i) \leq [\ell_{r_1}(\lambda_i)]^{r_1/q} w_{q_1'}(x_i)$$
$$w_{p_2'}(\lambda_i^{r_1/p} y_i) \leq [\ell_{r_1}(\lambda_i)]^{r_1/p} w_{p_1'}(y_i)$$

whence

$$\alpha_{p_2,q_2}(z) \leq \ldots \leq \ell_{r_1}(\lambda_i)^{r_1/r_2 + r_1/q + r_1/p} w_{q_1'}(x_i) w_{p_1'}(y_i) \leq$$
$$\leq (1+\varepsilon) \alpha_{p_1,q_1}(z)$$

(3) is trivial.

1.6. To describe the completion of $E\otimes_{\alpha_{p,q}} F$ infinite sums will be involved. The definition of the strong and weak ℓ_p-norm of a sequence (x_i) is obvious.

Proposition.

(1) *If* $(\lambda_n)\in\ell_r$ *(in* c_0 *if* $r=\infty$*),* $w_{q'}(x_n)<\infty$ *and* $w_{p'}(y_n)<\infty$*, then the series*

$$\sum(\lambda_n x_n\otimes y_n)$$

converges unconditionally in $E\tilde{\otimes}_{\alpha_{p,q}} F$.

(2) For every $z\in E\tilde{\otimes}_{\alpha_{p,q}} F$ *there is a series as in* (1) *with*

$$z=\sum_{n=1}^{\infty}\lambda_n x_n\otimes y_n$$

Moreover:

$$\alpha_{p,q}(z;E,F)=\inf\ \ell_r(\lambda_i)\,w_{q'}(x_i)\,w_{p'}(y_i)$$

where the infimum is taken over all (finite or infinite) such representations.

Proof :

(1) is easy since the fact that $(\lambda_i)\in\ell_r$ (or c_0) forces the series to be a $\alpha_{p,q}$-Cauchy-series.

To prove (2) take for $z\in E\tilde{\otimes}_{\alpha_{p,q}} F$ and $\varepsilon>0$ elements $z_n\in E\otimes F$ with $z=\sum_{n=1}^{\infty} z_n$ and

$$\sum_{n=1}^{\infty}\alpha_{p,q}(z_n)\le(1+\varepsilon)\alpha_{p,q}(z)$$

Choose (λ_i^n), (x_i^n) and (y_i^n) (finite) with

$$z_n=\sum\lambda_i^n x_i^n\otimes y_i^n$$

and

$$\ell_r((x_i^n)_i)\le(\alpha_{p,q}(z_n)(1+\varepsilon))^{1/r}$$

$$w_{q'}((x_i^n)_i)\le(\alpha_{p,q}(z_n)(1+\varepsilon))^{1/q'}$$

$$w_{p'}((y_i^n)_i)\le(\alpha_{p,q}(z_n)(1+\varepsilon))^{1/p'}$$

Then

$$\ell_r((\lambda_i^n)_{i,n})\,w_{q'}((x_i^n)_{i,n})\,w_{p'}((y_i^n)_{i,n})\le\alpha_{p,q}(z)(1+\varepsilon)^2$$

and, by (1)

$$z = \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \sum_{i} \lambda_i^n x_i^n \otimes y_i^n .$$

In particular, if β denotes the seminorm defined by the infimum in the statement of (2):

$$\beta(z) \le \alpha_{p,q}(z) \qquad \text{for all} \qquad z \in E\tilde{\otimes}_{\alpha_{p,q}} F$$

Conversely, if $z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$ and

$$z^N := \sum_{n=1}^{N} \lambda_n x_n \otimes y_n$$

then

$$\ell_r(\lambda_n) w_{q'}(x_n) w_{p'}(y_n) \ge \ell_r((x_n)_{n=1}^N) w_{q'}((x_n)_{n=1}^N) w_{p'}((y_n)_{n=1}^N) \ge$$
$$\ge \alpha_{p,q}(z^N) \to \alpha_{p,q}(z);$$

this implies $\beta(z) \ge \alpha_{p,q}(z)$.

1.7. Special cases of $\alpha_{p,q}$-tensornorms are ($1 \le p \le \infty$)

$g_p := \alpha_{p,1}$ (g for «gauche»)
$d_p := \alpha_{1\,p}$ (d for «droite»)
$w_p := \alpha_{p,p'}$ (w for «weak»)

and therefore

$$g_1 = d_1 = \pi, \quad w_1 = d_\infty, \quad w_\infty = g_1, \quad g_p = d_p^t, \quad w_p = w_{p'}^t$$

and $w_p \le g_p$, $w_{p'} \le d_p$.

It is very simple to see that

$$g_p(z; E, F) = \inf \{\ell_p(x_i) w_{p'}(y_i) \mid z = \sum_{i=1}^{n} x_i \otimes y_i\}$$

$$d_p(z; E, F) = \inf \{w_p(x_i) \ell_p(y_i) \mid z = \sum_{i=1}^{n} x_i \otimes y_i\}$$

$$w_p(z;E,F) = \inf\{w_p(x_i)w_{p'}(y_i) \mid z = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

Clearly, a result in the spirit of 1.6 with representations

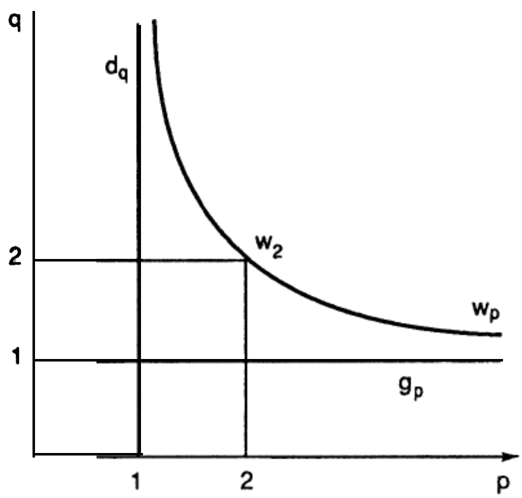
$$z = \sum_{n=1}^{\infty} x_n \otimes y_n$$

holds for g_p and d_p as well. The case w_p for $1 < p <$ CO reads as follows: If $w_p(x_i) < \infty$, $w_{p'}(y_i) < \infty$ and

$$w_p((x_i)_{i=N}^{\infty}) \underset{N\to\infty}{\to} 0$$

then the series $\sum(x_n \otimes$ y,) converges unconditionally in $E\tilde{\otimes}_{w_p}F$.

1.8. The following picture illustrates the situation:



Proposition. *For* $p, q \in]1,\infty[$ *there are constants* $c_{p,q} \geq 1$ *such that*

$$\alpha_{p,q} \leq c_{p,q} w_2 .$$

In particular,

$$w_2 \leq \alpha_{p,q} \leq c_{p,q} w_2$$

for all $p, q \in]1,2]$.

The proof will make use of the Khintchine inequality: For this take

$D_n := \{-1,1\}^n$

$\varepsilon_i : D_n \to \{-1,1\}$ i-th projection and μ_n the measure defined by $\mu_n(\{t\}) = 2^{-n}$ for all $t \in D_n$ (which is the normalized Haar measure). It follows easily that

$$\int_{D_n} \varepsilon_i \varepsilon_j \mathrm{d}\mu_n = \delta_{ij}$$

The KHINTCHINE INEQUALITY says: *For* $1 \leq r < \infty$ *there are constants* $a_r \geq 1$ *and* $b_r \geq 1$ *such that*

$$a_r^{-1}(\sum_{k=1}^n |\xi_k|^2)^{1/2} \leq (\int_{D_n} |\sum_{k=1}^n \xi_k \varepsilon_k(t)|^r \mu_n(\mathrm{d}t))^{1/r} \leq b_r(\sum_{k=1}^n |\xi_k|^2)^{1/2}$$

for all $n \in N$ *and* $\xi_1, \ldots, \xi_n \in \mathbb{K}$. For an easy proof see [43] p. 45. For the constants one can take

$a_r = \sqrt{2}$	$1 \leq r \leq 2$	
$a_r = 1$	$2 \leq r$	(obvious)
$b_r = 1$	$1 \leq r \leq 2$	(obvious)
$b_r = 5\sqrt{r}$	$2 \leq r$.	

The best constants were calculated by Haagerup [28] in 1982; they are the same for the rea1 and the complex field.

Proof of the proposition:

For $z = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$ the biorthogonality of the ε_i gives a new representation:

$$z = \sum_{i,j} \int_{D_n} \varepsilon_i \varepsilon_j \mathrm{d}\mu_n x_i \otimes y_i = \sum_{t \in D_n} \frac{1}{2^n}(\sum_{i=1}^n \varepsilon_i(t) x_i) \otimes (\sum_{j=1}^n \varepsilon_j(t) y_j).$$

Now

$$w_{q'}((\sum_{i=1}^n \varepsilon_i(t) x_i)_{t \in D_n}) = \sup_{\|x'\| \leq 1} (\sum_{t \in D_n} |\sum_{i=1}^n \varepsilon_i(t) \langle x_i, x' \rangle|^{q'})^{1/q'} \leq$$

$$= 2^{n/q'} \sup_{\|x'\| \leq 1} (\int_{D_n} |\sum_{i=1}^n \langle x_i, x' \rangle \varepsilon_i(t)|^{q'} \mu_n(dt))^{1/q'} \leq$$

$$\leq 2^{n/q'} b_{q'} w_2((x_i)_{i=1,\ldots,n}).$$

Consequently,

$$\alpha_{p,q}(z;\ E, F) \leq \frac{1}{2^n}(2^n)^{1/r+1/q'+1/p'} b_{q'} b_{p'} w_2(x_i) w_2(y_i)$$

and therefore

$$\alpha_{p,q} \leq b_{q'} b_{p'} w_2.$$

The tensornorms g_p and d_p cannot be estimated by w_2 : this will follow easily from the identification of $(E \otimes_\alpha F)'$ with a space of operators (by 4.9 the inequality $w_\infty \leq g_p \leq c w_2$ would imply that Hilbert spaces are $\mathcal{L}_\infty$-spaces, see §6).

1.9. Take $x_1, \ldots, x_n \in E$ then for $1 \le p \le \infty$

$$w_p(x_i) = \sup\{ |\sum_{i=1}^{n} \xi_i \langle x_i, x' \rangle| \; | x' \in B_{E'}, (\xi_i) \in B_{\ell_{p'}^n} \} =$$
$$= \varepsilon(\sum_{i=1}^{n} x_i \otimes e_i; E, \ell_p^n)$$

(e_i the unit-vectors in ℓ_p^n). Since $w_{p'}(e_i) = 1$ it follows the

Remark. *For every normed space E and $1 \le p \le \infty$*

$$\varepsilon(\sum_{i=1}^{n} x_i \otimes e_i; E, \ell_p^n) = w_p(\sum_{i=1}^{n} x_i \otimes e_i; E, \ell_p^n) = w_p(x_i; E)$$

for $x_i \in E$. In particular: $\varepsilon = w_p$ on $E \otimes \ell_p^n$.

1.10. One of the most striking tools in the theory of tensor-norms and the operator theory is Grothendieck's «théorème fondamental de la théorie métrique des produits tensoriels» which, since the work of Lindenstrauss and Pełczyński [51], is known in an equivalent form as GROTHENDIECK INEQUALITY: *There is a universal constant K_G such that for all $n \in N$, all matrices $(a_{ij}) \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^n)$ and all Hilbert spaces H*

$$\sup\{|\sum_{i,j=1}^{n} a_{ij} \langle x_i, y_i \rangle_H| \; | x_i, y_i \in B_H\} \le K_G \sup\{|\sum_{i,j=1}^{n} a_{ij} s_i t_j| \; | s_i, t_i \in B_{\mathbb{K}}\}$$

For a simple proof see e.g. [12]. K_G can be chosen ≤ 2. The best constants (the one for the complex case is strictly smaller than that for the real case) are not yet known.

One of the direct consequences of the inequality is that every operator $\ell_1(\Gamma) \to H$ is absolutely-1-summing (see 6.5). The same proof gives that every operator $\ell_1 \to F$ is absolutely-1-summing if F satisfies the Grothendieck-inequality as above (with the duality bracket instead of the scalar-product; $x_i \in B_F$ and $y_i \in B_{F'}$); whence the natural quotient map

$$\ell_1(B_F) \twoheadrightarrow F$$

factors through a Hilbert space and F is isomorphic to a Hilbert space: Up to isomorphy only the Hilbert spaces satisfy Grothendieck's inequality ([51]; p. 289).

1.11. For $\varphi \in (\ell_\infty^n \otimes_\pi \ell_\infty^n)' = B(\ell_\infty^n, \ell_\infty^n)$ (bilinear forms) with representing matrix

$$a_{ij} := \langle \varphi, e_i \otimes e_j \rangle$$

the norm is given by

$$||\varphi||_{(\ell_\infty^n \otimes_\pi \ell_\infty^n)'} = \sup\{|\sum_{i,j=1}^n a_{ij} s_i t_j| \,|\, (s_i), (t_j) \in B_{\ell_\infty^n}\}.$$

This implies for x_i, y_j in the unit ball B_H of a Hilbert space H and

$$z := \sum_{i,j=1}^n \langle x_i, y_j \rangle_H e_i \otimes e_j \in \ell_\infty^n \otimes \ell_\infty^n$$

that

$$\pi(z; \ell_\infty^n, \ell_\infty^n) = \sup\{|\langle \varphi, z \rangle| \,|\, \varphi \in (\ell_\infty^n \otimes_\pi \ell_\infty^n)', \ ||\varphi||_{...} \leq 1\} \leq K_G$$

by Grothendieck's inequality, whence

Corollary. *Let H be a Hilbert space. Then*

$$\pi(\sum_{i,j=1}^n \langle x_i, y_j \rangle_H e_i \otimes e_j; \ell_\infty^n, \ell_\infty^n) \leq K_G \max_i ||x_i|| \max_j ||y_i||$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in H$.

Everything is prepared for the

Theorem (Grothendieck's inequality in tensorial form). *For every $n \in \mathbb{N}$*

$$w_2 \leq \pi \leq K_G w_2 \qquad on \qquad \ell_\infty^n \otimes \ell_\infty^n$$

In particular: all $\alpha_{p,q}$ (for $1 \leq p, q \leq 2$) are equivalent on $\ell_\infty^n \otimes \ell_\infty^n$ (with constants independent from n).

Proof . Take $H := \ell_2^n$ and equip H^n with the sup-norm, then

$$\begin{array}{ccc} H^n & = & \ell_\infty^n \otimes_\varepsilon H \\ (x_1, \ldots, x_n) & \rightsquigarrow & \sum_{i=1}^n e_i \otimes x_i \end{array}$$

isometrically. Therefore, the real bilinear map (consider the spaces as real vector spaces)

$$\begin{array}{rcl} \mathrm{Tr}: (\ell_\infty^n \otimes_\varepsilon H) \times (H \otimes_\varepsilon \ell_\infty^n) & \to & \ell_\infty^n \otimes_\pi \ell_\infty^n \\ (u \otimes x, y \otimes v) & \rightsquigarrow & \langle x, y\rangle_H u \otimes v \end{array}$$

can be written as

$$\begin{array}{rcl} H^n \times H^n & \to & \ell_\infty^n \otimes_\pi \ell_\infty^n \\ ((x_1 \dots x_n), (y_1 \dots y_n)) & \rightsquigarrow & \sum_{i,j} \langle x_i, y_j\rangle e_i \otimes e_j . \end{array}$$

The corollary gives that $\|\mathrm{Tr}\| \le K_G$. Now take $z = \sum_{i=1}^n u_i \otimes v_i \in \ell_\infty^n \otimes \ell_\infty^n$, then

$$\mathrm{Tr}\Big(\sum_{i=1}^n u_i \otimes e_i, \sum_{i=1}^n e_i \otimes v_i\Big) = z$$

and, by 1.9,

$$\varepsilon\Big(\sum_{i=1}^n u_i \otimes e_i;\ \ell_\infty^n, H\Big) = w_2(u_i)$$

$$\varepsilon\Big(\sum_{i=1}^n e_i \otimes u_i;\ H, \ell_\infty^n\Big) = w_2(v_i).$$

It follows

$$\pi(z;\ \ell_\infty^n, \ell_\infty^n) \le K_G w_2(u_i) w_2(v_i)$$

and taking the mfimum over all representations of z gives the result.

1.12. Another direct consequence of Grothendieck's inequality is the

Proposition. *For every* $n \in \mathbb{N}$

$$\pi \le K_G d_\infty \qquad \textit{on} \qquad \ell_1^n \otimes \ell_2^n .$$

Proof . If $x_i \in \ell_1^n$ with $x_i = \sum_{j=1}^n a_{ij} e_j$, then

$$\begin{aligned} w_1(x_i; \ell_1^n) &= \sup_{|t_j| \le 1} \sum_{i=1}^m |\langle x_i, (t_j)\rangle| = \\ &= \sup_{|t_j| \le 1} \sup_{|s_i| \le 1} \Big|\sum_{i,j} a_{ij} s_i t_j\Big| \end{aligned}$$

whence

$$\pi(\sum_{i=1}^{m} x_i \otimes y_i) = \pi(\sum_{j=1}^{n} e_j \otimes \sum_{i=1}^{m} a_{ij} y_i) \leq$$

$$\leq \sum_{j=1}^{n} \| \sum_{i=1}^{m} a_{ij} y_i \|_{\ell_2^n} =$$

$$= \sup_{z_j \in B_{\ell_2^n}} |\sum_{i,j} a_{ij} \langle [\ell_\infty(y_k)]^{-1} y_i, z_j \rangle| \cdot \ell_\infty(y_k) \leq$$

$$\leq K_G w_1(x_i) \ell_\infty(y_i)$$

and, passing to the infimum over all representations,

$$\pi \leq K_G d_\infty$$

on $\ell_1^n \otimes \ell_2^n$.

Note di Matematica Vol. VIII - n. 2, 203-207(1988)

2. THE FOUR BASIC LEMMAS

2.1. This paragraph contains four lemmas which are basic for the understanding and use of tensornorms: the approximation, extension, embedding, and density lemma. The power and importance of these devices will become clear while working with them.

2.2. Recall that a normed space E has the *λ-approximation property* if there is a net (T_η) of finite-dimensional operators $E \to E$ with $\|T_\eta\| \leq \lambda$ and $T_\eta(x) \to x$ for all $x \in E$. If $\lambda = 1$, the space has *the metric approximation property; if* a space has the *X-approximation property* for some λ it is said to have the *bounded approximationproperty.*

Approximation lemma. *Let α and β be tensornorms (on NORM), E, F normed spaces, $c \geq 1$ and*

$$\alpha \leq c\beta \quad on \quad E \otimes N$$

for cofinally many $N \in$ FIN(F). If F has the X-approximation property, then

$$\alpha \leq \lambda c\beta \quad on \quad E \otimes F.$$

Proof. It is easy to see that

$$\mathrm{id}_E \otimes T_\eta(z) \to z$$

for the projective norm π and whence for all tensomorms. If η is such that

$$\alpha(z - \mathrm{id}_E \otimes T_\eta(z);\ E, F) \leq \varepsilon$$

and N as in the hypothesis with $T_\eta(F)$ c N then, by the metric mapping property of tensornorms,

$$\begin{aligned}\alpha(z;\ E,F) &\leq \alpha(z - \mathrm{id}_E \otimes T_\eta(z);\ E,F) + \alpha(\mathrm{id}_E \otimes T_\eta(z);\ E,F) \leq \\ &\leq \varepsilon + \alpha(\mathrm{id}_E \otimes T_\eta(z);\ E,N) \leq \\ &\leq \varepsilon + c\beta(\mathrm{id}_E \otimes T_\eta(z);\ E,N) \leq \\ &\leq \varepsilon + c\|T_\eta\|\beta(z;E,F)\end{aligned}$$

which implies the statement. .

This lemma (and its transposed version) gives for the finite and cofinite hull of a tensomorm the

Proposition. *If α is a tensornorm (on FIN), E and F have the bounded approximation property with constants λ_E and λ_F, respectively, then*

$$\overleftarrow{\alpha} \leq \overrightarrow{\alpha} \leq \lambda_E\lambda_F\overleftarrow{\alpha} \quad \text{on} \quad E \otimes F$$

In particular: $\overleftarrow{\alpha} = \overrightarrow{\alpha}$ on $E \otimes F$, if both spaces have the metric approximation property.

2.3. If $\varphi \in (E \otimes_\pi F)' = \mathcal{L}(E, F')$ and L_φ is its associated operator

$$\langle \varphi, x \otimes y \rangle = \langle L_\varphi x, y \rangle_{F',F}$$

then

$$\langle \varphi, x \otimes \text{Y”)} := \langle L_\varphi x, y'' \rangle_{F',F''}$$

(for x $\in F$ and y” $\in F''$) defines a linear form $\varphi^\wedge$ on $E \otimes F$” which is clearly continuous:

$$||\varphi|| = ||L_\varphi|| = ||\varphi^\wedge||$$

The associated bilinear form is the *unique* $\sigma(E, E') - \sigma(F'', F')$ separately continuous extension of φ to *Ex F”*. $\varphi^\wedge$ is called the *right canonica1 extension* of φ to $E \otimes F$”. Similarly *the left canonica1 extension* $^\wedge\varphi$ on $E'' \otimes F$ is defined by ($\kappa_F : F \hookrightarrow F$” the canonical embedding)

$$\langle {}^\wedge\varphi, x'' \otimes y \rangle := \langle L'_\varphi \circ \kappa_F(y), x'' \rangle_{E',E''}.$$

It is not difficult to see that

$$(\text{“cp)”} = ^\wedge (\varphi^\wedge) \qquad \text{on} \qquad E\text{”} \otimes F''$$

if and only if L_φ is weakly compact.

Extension lemma. *Let* $\varphi \in (E \otimes_\pi F)'$ *and* α *be a finitely generated tensornorm on NORM. Then:*

$$\varphi \in (E \otimes_\alpha F)' \qquad \textit{if and only if} \qquad \varphi^\wedge \in (E \otimes_\alpha F\text{”})'$$

In this case: $||\varphi||_{(E\otimes_\alpha F)'} = ||\varphi^\wedge||_{(E\otimes_\alpha F'')'}$.

Proof . The metric mapping property

$$||E \otimes_\alpha F \hookrightarrow E \otimes_\alpha F''|| \leq 1$$

implies

$$\text{IMI...} \leq ||\varphi^\wedge||...$$

Conversely, take $M \in FIN(E)$ and $N \in FIN(F\text{”})$. Then the weak principle of loca1 reflexivity (1.4) gives for every $\varepsilon > 0$ an $R \in \mathcal{L}(N, F)$ with $||R|| \leq 1 + \varepsilon$ such that for all $y'' \in N$ and $x \in M$

$$\langle y'', L_\varphi x \rangle_{F'',F'} = \langle R y'', L_\varphi x \rangle_{F,F'}.$$

This means

$$\langle \varphi^\wedge, x \otimes y'' \rangle = \langle \varphi, (\mathrm{id} \otimes R)(x \otimes y'') \rangle$$

and

$$\langle \varphi^\wedge, z \rangle = \langle \varphi, \mathrm{id}_E \otimes R(z) \rangle$$

for all $z \in M \otimes N$ and whence

$$|\langle \varphi^\wedge, z \rangle| \leq \|\varphi\| \, \|R\| \alpha(z; E, N) \leq \|\varphi\| (1 + \varepsilon) \alpha(z; E, N)$$

which implies the result, since α is finitely generated.

Sometimes the relation $(\star)$ is helpful.

Problem 1. *Does the extension lemma hold for cofinitely generated tensornorms?*

Problem *2. There are two «canonical» embeddings*

$$I_j : E'' \otimes F'' \hookrightarrow (E \otimes_\alpha F)''$$

defined by

$$\langle I_1(x'' \otimes y''), \varphi \rangle := \langle {}^\wedge(\varphi^\wedge), x'' \otimes y'' \rangle$$
$$\langle I_2(x'' \otimes y''), \varphi \rangle := \langle ({}^\wedge\varphi)^\wedge, x'' \otimes y'' \rangle$$

What are the norms induced on $E'' \otimes F''$?

If the induced norm were α in reasonable situations, this would solve easily the problem of the bidual mappings which will be treated in 5.8.

2.4. Tensornorms do not respect subspaces (see 1.1) but the embedding to the bidual usually is respected:

Embedding lemma. *If α is a finitely or cofinitely generated tensornorm (on NORM), then*

$$\mathrm{id}_E \otimes \kappa_F : E \otimes_\alpha F \overset{1}{\hookrightarrow} E \otimes_\alpha F''$$

is an isometry for all normed spaces E and F.

Proof. The mapping property implies that

$$\alpha(z; E, F'') \leq \alpha(z; E, F) \qquad z \in E \otimes F$$

holds always (the map $\mathrm{id}_E \otimes \kappa_F$ will not be written).

(1) Let α be finitely generated. Then, by the extension lemma

$$\begin{aligned}\alpha(z;\,E,F) &= \sup\{|\langle\varphi,z\rangle|\ |\varphi\in(E\otimes_\alpha F)',\|\varphi\|\le 1\} =\\ &= \sup\{|\langle\varphi^\wedge,z\rangle|\ |\varphi\in(E\otimes_\alpha F)',\|\varphi\|\le 1\}\le\\ &\le \sup\{|\langle\psi,z\rangle|\ |\psi\in(E\otimes_\alpha F'')',\|\psi\|\le 1\} =\\ &= \alpha(z;\,E,F'')\end{aligned}$$

which is the reverse inequality.

(2) If α is colinitely generated, $K\in COFIN(E)$ and $L\in COFIN(F)$, then the canonical diagram (L^{∞} formed in F'')

$$\begin{array}{ccc} F & \overset{\kappa_F}{\hookrightarrow} & F'' \\ Q_L^F\downarrow & & \downarrow Q_{L^{\infty}}^{F''} \\ F/L & \overset{=}{\rightarrow} & F''/L^{\infty}\end{array}$$

commutes and the lower map is an isometry. It follows that

$$\begin{aligned}\alpha(Q_K^E\otimes Q_L^F(z);\,E/K,F/L) &= \alpha((Q_K^E\otimes Q_{L^{\infty}}^{F''})\circ(id_E\otimes\kappa_F)(z);\,E/K,\,F''/L^{\infty})\le\\ &\le \overleftarrow{\alpha}(z;\,E,F'') = \alpha(z;\,E,F'').\end{aligned}$$

Taking the supremum for $\overleftarrow{\alpha}$ gives the missing inequality. ∎

The calculation in (1) (or the extension lemma directly) and the bipolar theorem give the

Corollary. *If α is finitely generated, then the unit ball $B_{E\otimes_\alpha F}$ is $\sigma(E\otimes F'',(E\otimes_\alpha F)')$-dense in the unit ball $B_{E\otimes_\alpha F''}$.*

2.5. Since the completion $\tilde{F}$ of F and F have the same biduals the embedding lemma gives that

$$E\otimes_\alpha F\overset{1}{\hookrightarrow}E\otimes_\alpha\tilde{F}$$

is an isometric (dense) subspace, whenever α is finitely or cofinitely generated.

Density lemma. *Let α be a finitely or cofinitely generated tensornorm, E and F normed spaces, E_0 and F_0 dense subspaces of E and F, respectively. If G is a locally convex space and $T\in\mathcal{L}(E\otimes_\pi F,G)$ such that*

$$T|_{E_0\otimes F_0}\in\mathcal{L}(E_0\otimes_\alpha F_0,G)$$

then

$$T\in\mathcal{L}(E\otimes_\alpha F,G).$$

Proof . Since $E \otimes_\alpha F$ is normed and whence a Mackey space it is enough to take $G = \mathbb{K}$ and $\varphi \in (E \otimes_\pi F)'$. The space $E_0 \otimes_\alpha F_0$ is a dense isometric subspace of $E \otimes_\alpha F$ therefore

$$\psi := \widetilde{\varphi|_{E_0 \otimes F_0}} \in (E \otimes_\alpha F)' \hookrightarrow (E \otimes_\pi F)'$$

and $\varphi = \psi$ on $E_0 \otimes F_0$, and whence $\varphi = \psi$ on $E \otimes_\pi F$.

A particularly interesting special case is given in the

Corollary. *Let α and β be tensornorms, α finitely or cofinitely generated. If $T_i \in \mathcal{L}(E_i, F_i)$ and $G_i \subset E_i$ are dense subspaces such that*

$$T_1 \otimes T_2|_{G_1 \otimes G_2} \in \mathcal{L}(G_1 \otimes_\alpha G_2, F_1 \otimes_\beta F_2)$$

then

$$T_1 \otimes T_2 \in \mathcal{L}(E_1 \otimes_\alpha E_2, F_1 \otimes_\beta F_2).$$

Since

$$T_1 \otimes T_2 : E_1 \otimes_\pi E_2 \to F_1 \otimes_\pi F_2 \to F_1 \otimes_\beta F_2 =: G$$

is continuous, the proof is obvious.

Note di Matematica Vol. VIII - n. 2, 209-213(1988)

3. DUAL TENSORNORMS

3.1. Given two (separating) dual pairings $\langle E_i, F_i\rangle$, then

$$\begin{array}{ccc} (E_1 \otimes E_2) \times (F_1 \times F_2) & \rightarrow & \mathbb{K} \\ (\sum_n x_n^1 \otimes x_n^2, \sum_m y_m^1 \otimes y_m^2) & \rightsquigarrow & \sum_{n,m} \langle x_n^1, y_m^1\rangle\langle x_n^2, y_m^2\rangle \end{array}$$

gives a dual (separating) pairing. This simplc and natural pairing is sometimes called truce *duality* for the following rcason: for normcd spaces G the tracc tr_G is defined on the finite-dimensional opcrators

$$\begin{array}{ccc} G' \otimes G & = & \mathcal{F}(G,G) \\ z & \rightsquigarrow & L_z \end{array}$$

(see 0.8). Take now M and N finite-dimensional normed spaces, $u \in M \otimes N$ and $v \in M' \otimes N'$, thcn the associated lincar operators satisfy

$$L_u \in \mathcal{L}(M', N), \quad L'_u = L_{u^t} \in \mathcal{L}(N', M),$$
$$L_v \in \mathcal{L}(M, N'), \quad L'_v = L_{v^t} \in \mathcal{L}(N, M')$$

and

$$\langle u, v\rangle = \mathrm{tr}_M(L_{u^t} . L'') = \mathrm{tr}_N(L_u . L_{v^t}) =$$
$$= \mathrm{tr}_{M'}(L_{v^t} L_u) = \mathrm{tr}_{N'}(L_v . L_{u^t})$$

(this need only be checked on elcmentary tcnsors). Note that *transposing* u means gomg to *the dual* of L_u.

3.2. The purpose of this paragraph is to study the embeddings

$$E \otimes F \hookrightarrow (E' \otimes_\varepsilon F')' \hookrightarrow (E' \otimes_\beta F')'$$
$$E' \otimes F' \hookrightarrow (E \otimes_\varepsilon F)' \hookrightarrow (E \otimes_\beta F \quad)' ,$$

given by the natura1 pairing, i.e. the trace duality. For this, dual tcnsornorms will be introduced – and first constructed on finite-dimensional tcnsor products $M \otimes N$; note that

$$M \otimes N = (M' \otimes_\alpha N')' \qquad M, N \in FIN .$$

Proposition. *Let α be a tensornorm on FIN. Then α defined by*

$$\alpha'(z;\ M,N) := \sup\{|\langle z, u\rangle| \,|\, \alpha(u;\ M',N') \leq 1\}$$

for $z \in M \otimes N$ is a tensornorm on FIN.

Proof . To apply the criterion in 1.1 (for *FIN*), observe first that α' is a norm, (2) follows from $\varepsilon = \alpha = \pi$ on $\mathbb{K} \otimes \mathbb{K}$ and (3) from

$$\langle (T_1 \otimes T_2) z, u\rangle = \langle z, (T'_1 \otimes T'_2) u\rangle. \qquad \blacksquare$$

In other words:

$$M \otimes_{\alpha'} N := (M' \otimes_\alpha N')' \qquad \text{(isometrically)}$$

The finite hull $\overrightarrow{\alpha'}$ of α' on *NORM* will bc callcd the *dual tensornorm* α' (on *NORM*) (the tcnsornorm α (on *FIN or NORM*).

3.3. The following properties are obvious:

(1) *If* $\alpha \leq c\beta$, *then* $\beta' \leq c\alpha'$.
(2) $\alpha = \alpha''$ on FIN and $\overrightarrow{\alpha} = \alpha''$.
(3) $\alpha = \alpha''$ *on NORM if and only if* α *is finitely generated.*

The relation $\varepsilon \leq$ cr' $\leq \pi$ implies for $\alpha = \varepsilon$ by dualization

$$\varepsilon \leq \pi' \leq \varepsilon'' = \varepsilon$$

and whence

$$\pi' = \varepsilon \qquad \text{and} \qquad \varepsilon' = \pi$$

This is part of the duality rclation between the projective and the injective tensornorms mentioned in 1.4.

3.4. Clearly, it is highly desirable to.know whether the following isometric relation for finite-dimensional M and N

$$M' \otimes_\alpha N' \hookrightarrow (M \otimes_{\alpha'} N)'$$

holds also for infinite-dimensional normed spaces. The answer is given by the *duality theorem.*

Theorem. *Let* α *be a tensornorm (on FIN). Then for all normed spaces E and F the following natural mappings are isometries:*

(1) $E' \otimes_{\overleftarrow{\alpha}} F' \overset{1}{\hookrightarrow} (E \otimes_{\alpha'} F)'$
(2) $E' \otimes_{\overleftarrow{\alpha}} F \overset{1}{\hookrightarrow} (E \otimes_{\alpha'} F')'$
(1) $E \otimes_{\overleftarrow{\alpha}} F \overset{1}{\hookrightarrow} (E' \otimes_{\alpha'} F')'$

Proof . To prove (3), observe first that

$$FIN(E') = \{K^0 \mid K \in COFIN(E)\}$$

and, for $(K, L) \in COFIN(E) \times COFIN(F)$,

$$\langle z, u\rangle = \langle Q_K^E \otimes Q_L^F(z), u\rangle$$

if $z \in E \otimes F$ and $u \in K^0 \otimes L^0 \subset E' \otimes F'$. Now, by the valid duality rclation for finitc-dimcnsional spaccs

$$\overleftarrow{\alpha}(z; E, F) = \sup_{K,L} \alpha(Q_K^E \otimes Q_L^F(z); E/K, F/L)$$
$$= \sup_{K\ L} \sup_{\alpha'(u;\, K^0, L^0) < 1} |\langle Q_K^E \otimes Q_L^F(z), u\rangle| =$$
$$= \sup_{\alpha'(u;\, E', F') < 1} |\langle z, u\rangle|$$

and this is (3). The commutative diagram and thc cxtcnsion lcmma

$$\begin{array}{ccccc} E' \otimes_{\overleftarrow{\alpha}} F & \overset{1}{\hookrightarrow} & (E'' \otimes_{\alpha'} F')' & \ni & \hat{}\varphi \\ & \searrow & \uparrow 1 & & \wr \\ & & (E \otimes_{\alpha'} F')' & \ni & \varphi \end{array}$$

imply (2) and (1) follows thc samc way.

The proof shows that the rcsult is, more or lcss, a rcformulation of the dchnition of the cotinitc hull. Thc thcorcm indicatcs that thc use of $\overleftarrow{\alpha}$ is a hclpful device. Since $\overleftarrow{\alpha} \leq \alpha$, it follows that all mappings $\otimes_\alpha \rightarrow$ . . in thc theorcm ($\otimes_{\overleftarrow{\alpha}}$ rcplaced by $\otimes_\alpha$) are continuous and of norm 1. (Note that, by thc thcorcm, thc cofinitc hull $\overleftarrow{\alpha}$ is identica1 with Grothcndicck's norm $\|\,.\,\|_\alpha$; sec [27], p. 11).

3.5. Having this rcsult and $\pi' = \varepsilon$ in mind thc usual proofs of the characterization of the X-approximation propcrty by thc cmbcdding

$$E \otimes_\pi F \hookrightarrow (E' \otimes_\varepsilon F')'$$

show (sec e.g. [37], p. 409 or [45], p. 315 for $\lambda = 1$):

Corollary. *For every normcd space E and $\lambda \geq 1$ are equivalent.*

(1) *E has the X-approximation property.*

(2) For every normed space F (or only $F = E'$)

$$\pi(\cdot;\ E, F) \leq \lambda \overleftarrow{\pi}(\cdot;\ E, F)$$

In particular: $\pi = \overleftarrow{\pi}$ on $E \otimes E'$ if and only if E has the metric approximation property.

3.6. For every tensomorm α on $NORM$ the relation $\overleftarrow{\alpha} \le \alpha \le \overrightarrow{\alpha}$ holds. α is called ***right-accessible*** (shortly (r)-accessible) if

$$\overleftarrow{\alpha}(\cdot;\ M,F) = \overrightarrow{\alpha}(\cdot;\ M,F)$$

whenever $(M, F) \in FIN \times NORM$ *left-accessible* (= (ℓ) -accessible) if α^t is right-accessible and ***accessible*** if it is right- and left- accessible. α is called ***totally*** accessible, if

$$\overleftarrow{\alpha} = \overrightarrow{\alpha}$$

i.e. if α is finitely and cofinitely generated. ε is totally accessible (this was already mentioned in 1.3) and π is accessible: This follows from the isometries

$$M \otimes_\pi E \overset{1}{\hookrightarrow} (M \otimes_\pi E)'' \overset{1}{=} (\mathcal{L}(M,E'))' \overset{1}{=} (M' \otimes_\varepsilon E')'$$

and the duality theorem 3.4; but π is not totally accessible by 3.5. It will be shown in §9 that all $\alpha_{p\,q}$ are accessible and all $\alpha'_{p\,q}$ are totally accessible.

Problem. *Is every finitely generated tensornorm accessible?*

This problem seems to be hard, since, by the approximation lemma, the non-accessibility of a tensornorm appears only on spaces without the metric approximation property. (In view of this problem it is suange to define *right* -accessible tensomorms; we do this in order to make some results «smoother» and since there are parallel notions for Banach-operator ideals, see §9).

Proposition. *Let α be a tensornorm on $NORM$*

(1) *α is right-accessible if and only if α' is right-accessible.*

(2) *If α is accessible, then the transposed tensornorm α^t, the dual tensornorm α' and the* adjoint *(or* contragradient) *tensornorm $\alpha^* := (\alpha^t)' = (\alpha')^t$ are accessible.*

If α is totally accessible, α' is accessible, but not necessarily totally accessible (for example $\alpha = \varepsilon$).

Proof . Clearly only (1) has to bc shown: Since, by theorem 3.4

$$M' \otimes_{\overrightarrow{\alpha}} F' \overset{1}{=} M' \otimes_{\overleftarrow{\alpha}} F' = (M \otimes_{\alpha'} F)'$$

for finite-dimensiona1 M. it follows that

$$M \otimes_{\overrightarrow{\alpha'}} F = M \otimes_{\alpha'} F \overset{1}{\hookrightarrow} (M \otimes_{\alpha'} F)'' = (M' \otimes_{\alpha''} F')'$$

holds isometrically; whence $\overrightarrow{\alpha'} = \overleftarrow{\alpha'}$ on $M \otimes F$ by 3.4.

3.7. Summarizing the dchnitions and rcsults of this paragraph (and using the approximation lemma) the relations

$$E \otimes_{\alpha} F = E \otimes_{\overleftarrow{\alpha}} F \qquad \text{and} \qquad E \tilde{\otimes}_{\alpha} F \overset{1}{\hookrightarrow} (E' \otimes_{\alpha'} F')'$$

hold isomctrically in each of the following three cases:

(1) E and F have the mctric approximation propcrty.

(2) α is right-acccssible and E has the mctric approximation propcrty.

(2') α is left-acccssible and F has the mctric approximation propcrty.

(3) α is totally acccssiblc.

So, «two ingredicnts» are nccessary to have the «good» relation bctwccn α and α'. For the bounded approximation propcrty the rclations would hold isomorphically.

Note di Matematica Vol. VIII - n. 2. 215-223(1988)

4. TENSORNORMS AND OPERATOR IDEALS

4.1. If $[\mathcal{A}, A]$ is Banach operator idcal, thcn

$$M \otimes_\alpha N := \mathcal{A}(M', N) \qquad (\star)$$

defines a tensomorm on FIN; in other words: if $z \in M \otimes N$ and $T_z \in \mathcal{L}(M', N)$ is the associated operator, then

$$\alpha(z; M, N) := A(T_z : M' \to N)$$

The fact that α is a tensomorm on *FIN* can be checked easily: the idcal property of $\mathcal{A}$ corresponds to the metric mapping property of α

4.2. Vice-versa: if α is a tensomorm on *FIN,* define $[\mathcal{A}, A]$ for finite-dimensiona1 spaces M, N by

$$\begin{aligned} \mathcal{A}(M, N) &:= M' \otimes_\alpha N \\ A(T) &:= \alpha(z_T; M', N) \end{aligned} \qquad (\star\star)$$

and extend this to all Banach spaces E and F by defining $T \in \mathcal{A}(E, F)$ if and only if

$$A(T) := \sup\{A(Q_K^F \circ T \circ I_N^E) \mid N \in FIN(E), K \in COFIN(F)\} < \infty.$$

It is easily seen that $[\mathcal{A}, A]$ is a Banach operator ideal which, by [60], 8.7.5, is even maximal. Since maximal Banach operator ideals $[\mathcal{A}, A]$ and finitely generated tensomorms α are uniquely determined by their «behaviour» on finite-dimensional spaces the

Definition. *A maximal Banach operator idea1* $[\mathcal{A}, A]$ *and a jnitely generated tensornorm* α *on NORM are called* associated, *in symbols:*

$$[\mathcal{A}, A] \sim \alpha$$

iffor all $M, N \in FIN$

$$\mathcal{A}(M, N) = M' \otimes_\alpha N \qquad \textit{isometrically}$$

establishes (via $(\star)$ and $(\star\star)$) a *one-to-one correspondence behveen maximal Banach operator ideals and jnitely generated tensornorm* . This link between the theory of operator ideals and the metric theory of tensor products is very fruitful for both theories.

4.3. If a maximal operator ideal $[\mathcal{A}, A]$ and a finitcly gcncratcd tcnsornorm are associ- atcd, thcn

$$\mathcal{A}(M,N) = M' \otimes_\alpha N = (M \otimes_{\alpha'} N')' \qquad M,N \in FIN$$

holds isomctrically. The cxtension of this to infinite-dimcnsional spaces, the *representation thcorem for maximal opcrator ideals* is basic.

Theorem. *Let* $[\mathcal{A}, A] \sim \alpha$. *Then, for all Banach spaces E und F*

$$A(E, F') = (E \otimes_{\alpha'} F)' \qquad \textit{isometricallg}$$

und

$$\mathcal{A}(E,F) = (E \otimes_{\alpha'} F')' \cap \mathcal{L}(E,F) \qquad \textit{isometrically}$$

This shows $\varepsilon \sim \mathcal{L}$ (the idcal of all opcrators) which, of course, was already clear from the dcfinition, and $\pi \sim \mathcal{I}$, the idcal of integral opcrators (sec e.g. the dcfinitions [45], p. 304 of intcgral opcrators); the latter example explains why the opcrators in **d** are sometimes called *α-intcgral operators*.

The thcorcm is due to Lotz [55]. His approach to tensomorms was different from ours and vcry influential to the devclopmcnt of the theory of operator ideals: He took, more or less, the rcpresentation theorem as a dcfinition for tcnsomorms and pointed this way at the one-to-one corrcspondcnce between maximal normcd operator ideals and tcnsomorms.

Proof . The second formula will bc proved first, i.e. it is to show for $T \in \mathcal{L}(E,F)$ that T $\in$ d(E, F) if and only if

$$B_{\kappa_F \circ T} \in (E \otimes_{\alpha'} F')'$$

(with equa1 norms). But this is easy: $T \in$ d(E, F) and $A(T) \leq$ c iff

$$A(Q_L^F \circ T \circ I_M^E) \leq c$$

for all $(M,L) \in FIN(E) \times COFIN(F)$, iff (by $\mathcal{A}(M, F/L) = (M \otimes_{\alpha'} L^0)'$) for all $z \in M \otimes L^0$

$$|\langle B_{\kappa_F \circ T}, z\rangle| = |\langle B_{Q_L^F \circ T \circ I_M^E}, z\rangle| \leq c\alpha'(z; M, L^0).$$

This implies the result, since α' is finitcly generated. To see the first formula just look at the diagram

$$\begin{array}{ccc} \varphi \in (E \otimes_{\alpha'} F)' & \hookrightarrow & (E \otimes_\pi F)' = \mathcal{L}(E, F') \\ \updownarrow 1 & / & \updownarrow \\ \varphi^\wedge \in (E \otimes_{\alpha'} F'')' & \hookrightarrow & (E \otimes_\pi F'')' \end{array}$$

and the extcnsion lcmma.

4.4. This theorem has various direct consquences

Corollary 1. *If* $[\mathcal{A}, A] \sim \alpha$, *then*

$$E' \otimes_{\overleftarrow{\alpha}} F' \overset{1}{\hookrightarrow} \mathcal{A}(E, F') \qquad \textit{isometrically}$$

$$E \otimes_{\overleftarrow{\alpha}} F \overset{1}{\hookrightarrow} \mathcal{A}(E', F) \qquad \textit{isometrically}$$

$$E' \otimes_{\overleftarrow{\alpha}} F \overset{1}{\hookrightarrow} \mathcal{A}(E, F) \qquad \textit{isometrically}$$

This follows from the duality theorem 3.4 abour tensomorms and will be referred to as the *embedding theorem* . Looking at

$$\mathcal{A}(E, F) \hookrightarrow (E \otimes_{\alpha'} F')' = \mathcal{A}(E, F'')$$

gives the following result (which is clearly well-known from «pure» operator theory).

Corollary *2. Maximal Banach operator ideals* $[\mathcal{A}, A]$ *are regular, i.e.* $T \in \mathcal{A}(E, F)$ *ifand only if* $\kappa_F \circ T \in \mathcal{A}(E, F'')$ *. In this case:*

$$A(T) = A(\kappa_F \circ T)$$

The diagram

$$\begin{array}{ccc} T \in \mathcal{L}(E, F) & \hookrightarrow & (E \otimes_\pi F')' \ni \varphi \\ \downarrow & \not\circlearrowleft & \downarrow \\ T'' \in \mathcal{L}(E'', F'') & = & (E'' \otimes_\pi F')' \ni {}^\wedge\varphi \end{array}$$

(and the extension lemma) implies the (again well-known)

Corollary *3. Let* $[\mathcal{A}, A]$ *be a maximal Banach operator ideal, then* $T \in \mathcal{A}(E, F)$ *if and only if* $T'' \in \mathcal{A}(E'', F'')$ *. In this case:*

$$A(T) = A(T'').$$

4.5. The following diagram commutes

$$\begin{array}{ccc} T \in \mathcal{L}(E, F) & \hookrightarrow & (E \otimes_\pi F')' \ni \varphi \\ \downarrow & & 1 \\ T' \in \mathcal{L}(F', E') & = & (F' \otimes_\pi E)' \ni \varphi^t \end{array}$$

Hence, if $\alpha \sim [\mathcal{A}, A]$ and if $[\mathcal{B}, B]$ is the unique maximal-Banach operator ideal associated with α^t, then, by the representation theorem for maximal operator ideals,

$$\begin{aligned}\mathcal{B}(E,F) &= (E \otimes_{(\alpha^t)'} F')' \cap \mathcal{L}(E,F)\\ &= \{T \in \mathcal{L}(E,F) \mid B_{T'} \in (F' \otimes_{\alpha'} E)'\}\\ &= \{T \in \mathcal{L}(E,F) \mid T' \in \mathcal{A}(F', E')\}\end{aligned}$$

holds isometrically, i.e., $T \in \mathcal{B}(E, F)$ iff $T' \in \mathcal{A}(F', E')$ and $B(T) = A(T')$. This means that $[\mathcal{B}, B]$ coincides with the dual Banach ideal $[\mathcal{A}^{\text{dual}}, A^{\text{dual}}]$ of $[\mathcal{A}, A]$ defined by Pietsch [60], 8.2.1. Note that the proof included that $\mathcal{A}^{\text{dual}}$ is maximal.

If $[\mathcal{D}, D]$ is the maximal Banach ideal associated with $\alpha^* = (\alpha^t)'$, then for all $M, N \in$ FIN the trace duality gives the isometric equalities

$$\begin{array}{ccccc}\mathcal{D}(M,N) & = M' \otimes_{\alpha^*} N = (N' \otimes_\alpha M)' = & \mathcal{A}(N,M)' \\ \text{w} & & \text{w} \\ T & \rightsquigarrow & [S \rightsquigarrow \operatorname{tr}_N(TS)]\end{array}$$

Therefore, $T \in \mathcal{D}(E, F)$ iff

$$\begin{aligned}D(T) &= \sup\{D(Q_L^F T I_M^E) \mid M \in FIN(E), L \in COFIN(F)\}\\ &= \sup\{|\operatorname{tr}_{F/L}(Q_L^F T I_M^E S)| \quad M \ldots . N \ldots . A(S : F/L \to M) \leq 1\},\end{aligned}$$

which implies that $[\mathcal{D}, D]$ and the adjoint Banach idea1 $[\mathcal{A}^*, A^*]$ of $[\mathcal{A}, A]$ in the sense of Pietsch [60], 9.1 are identical.

Proposition. *If* $\alpha \sim [\mathcal{A}, A]$, *then*

(1) $\alpha^t \sim [\mathcal{A}^{\text{dual}}, A^{\text{dual}}]$; *in particular: T is α^t-integral if and only if T' is α-integral.*
(2) $\alpha^* \sim [\mathcal{A}^*, A^*]$
(3) $[\mathcal{A}^{**}, A^{**}] = [\mathcal{A}, A]$.

The last result follows form (2) and $\alpha^{**} = \alpha$. Note that $\alpha^{tt} = \alpha$ gives $(\mathcal{A}^{\text{dual}})^{\text{dual}} = \mathcal{A}$ and this is another proof of corollary 3.

4.6. Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} \geq 1$ and define $r \in [1, \infty]$ by $\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1$. It was proved in 1.6 that for ali $M, N \in FIN$ and $T \in \mathcal{L}(M, N)$

$$\alpha_{p,q}(z_T; M', N) := \inf \ell_r(\lambda_i) w_{q'}(\varphi_i) w_{p'}(y_i)$$

where the infimum is taken over all finite or infinite series representations $T = \sum_i \lambda_i \varphi_i \otimes y_i$ (convergence in $\mathcal{L}(M, N)$). Hence by [60], 18.1.1 and 18.4.1

$$M' \otimes_{\alpha_{p,q}} N = \mathcal{N}_{r,p,q}(N, M)$$

where $[\mathcal{N}_{r,p,q}, N_{r,p,q}]$ denotes the ideal of all (r, p, q)-nuclear operators. By detinition (see [60], 19.4.1) the maximal Banach ideal $[\mathcal{L}_{p,q}, L_{p,q}]$ of all *(p, q)-factorable operators* coincides with $\mathcal{N}_{r,p,q}$ on finite-dimensiona1 Banach spaces and whence is the unique maximal idea1 associated with $\alpha_{p,q}$, i.e.,

$$[\mathcal{L}_{p,q}, L_{p,q}] \sim \alpha_{p,q}$$

Special cases are

$$[\mathcal{L}_p, L_p] := [\mathcal{L}_{p,p'}, L_{p,p'}] \sim \alpha_{p,p'} = w_p$$
$$[\mathcal{I}_p, I_p] := [\mathcal{L}_{p,1}, L_{p,1}] \sim \alpha_{p,1} = g_p$$

the ideals of all *p-factorable* and *p-integral operators* (see [60], 19.2.1 and 19.3.2). $\mathcal{I}_1 = \mathcal{I} \sim \pi = g_1$ are the usual integral operators.

The following important factorization theorems are proved by ultra product techniques (see [60], 19.2.6, 19.3.7, 19.3.9 and 19.4.6):

If $\frac{1}{p} + \frac{1}{q} > 1$, *then* $T \in \mathcal{L}_{p,q}(E, F)$ *if and only if there are a probability space* (Ω, μ) *and operators* $R \in \mathcal{L}(E, L_{q'}(\mu))$ *and* $S \in \mathcal{L}(L_p(\mu), F'')$ *such that*

$$\begin{array}{ccc} E & \xrightarrow{T} F \overset{\kappa_F}{\hookrightarrow} & F'' \\ R\downarrow & & \uparrow S \\ L_{q'}(\mu) & \overset{I_{p,q}}{\hookrightarrow} & L_p(\mu) \end{array}$$

($I_{p,q}$ the canonica1 embedding)

In this case $L_{p,q}(T) = \inf \|R\|\,\|S\|$.

Note that this gives in particular the factorization theorem for the p-integral operators ($\mathcal{I}_p = \mathcal{L}_{p,1}$ if $1 \le p < \infty$). For the p-factorable operators the following factorization holds:

$T \in \mathcal{L}_p(E, F) = \mathcal{L}_{p,p'}(E, F)$ *$(1 \le p \le$ co) iff there is a (strictly localizable) measure-space* (Ω, μ) *and appropriate operators R and S with*

$$\begin{array}{ccccc} E & \xrightarrow{T} & F & \overset{\kappa_F}{\hookrightarrow} & F'' \\ & R\searrow & & \nearrow S & \\ & & L_p(\mu) & & \end{array}$$

Again: $L_{,}(T) = \inf \|R\|\,\|S\|$.

It is easy to sec that for p = 2 in these statements the operator S can be chosen $L_2 \to F$ thus avoiding the bidual. So $\mathcal{L}_2$ is the ideal of operators factoring through a Hilbert space: $\mathcal{L}_2 \sim w_2$.

4.7. For $p, q \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q} \leq 1$ define $r \in [1,\infty]$ by $\frac{1}{r} := \frac{1}{p}+\frac{1}{q}$. In particular, $\frac{1}{p'}+\frac{1}{q'} \geq 1$ and $\frac{1}{r'}+\frac{1}{p}+\frac{1}{q} = 1$. Then for every $T \in \mathcal{L}(E,F)$

$$B_{\kappa_F \circ T} \in (E \otimes_{\alpha_{q',p'}} F')'$$

iff

$$\sup\{|\sum_{i=1}^{n} \lambda_i \langle T x_i, \varphi_i \rangle| \mid \ell_{r'}(\lambda_i) \leq 1, w_p(x_i) \leq 1, w_q(\varphi_i) \leq 1\} < \infty,$$

and in this case the latter supremum equals $\| B_{\kappa_F \circ T} \|_{...}$. Hence, by the representation theorem for maximal operator ideals (and Holder's inequality), an operator $T \in \mathcal{L}(E, F)$ belongs to the maximal Banach ideal

$$[\mathcal{D}_{p,q}, D_{p,q}] \sim \alpha'_{q',p'} = \alpha^*_{p',q'}$$

if and only if there is a constant $c \geq 0$ such that for all $x_1, \ldots, x_n \in E$ and $\varphi_1, \ldots, \varphi_n \in F'$

$$\ell_r(\langle \varphi_i, T x_i \rangle) \leq c w_p(x_i) w_q(\varphi_i),$$

and moreover $D_{p,q}(T) = \inf c$. Operators satisfying such inequalities are defined in [60], 17.4.1 and called *(p, q)-dominated*. Important special cases are

$$[\mathcal{D}_p, D_p] := [\mathcal{D}_{p,p'}, D_{p,p'}] \sim \alpha^*_{p',p} = w^*_{p'} = w'_p$$

the p-dominated operators, and

$$[\mathcal{P}_p, P_p] := [\mathcal{D}_{p,\infty}, D_{p,\infty}] \sim \alpha^*_{p',1} = g^*_{p'} = d'_{p'}$$

that *absolutely-p-summing* operators (note $\mathcal{P}_\infty = \mathcal{L}$).

By Proposition 4.5 it is obvious that

Proposition. *If* $\frac{1}{p}+\frac{1}{q} \geq 1$, *then*

$$\mathcal{L}^*_{p,q} = \mathcal{D}_{p',q'} \qquad \textit{isometrically}$$
$$\mathcal{I}^*_p = \mathcal{P}_{p'} \qquad \textit{isometrically},$$

4.8. There is an integral characterization of (p, q)-dominated operators due to Kwapien which is an extension of the Grothendieck-Pietsch-domination theorem ([60], 17.3.2)

$$T \in \mathcal{P}_p(E,F) \qquad \|Tx\|^p \leq c \int_{B_{E'}} |\langle x', x \rangle|^p \mu(\mathrm{d}\, x')$$

and basic for the applications of the theory. For bilinear forms it reads as follows:

Let $\varphi \in (E \otimes F)^*$. *Then*

$$\varphi \in (E \otimes_{\alpha_{p,q}} F)' \qquad (\text{i.e. } L_\varphi \in \mathcal{D}_{q',p'}(E, F'))$$

if and only if there are $c \geq 0$ *and Bore1 probability measures* μ *on* $B_{E'}$ *and* ν *on* $B_{F'}$ *such thatfor all* $x \in E$ *and* $y \in F$

$$|\langle \varphi, x \otimes y \rangle| \leq c\left(\int_{B_{E'}} |\langle x', x\rangle|^{q'} \mu(\mathrm{d}x')\right)^{\frac{1}{q'}} \left(\int_{B_{F'}} |\langle y', y\rangle|^{p'} \nu(\mathrm{d}y')\right)^{\frac{1}{p'}}$$

In this case $||\varphi||_{...}$ = inf c.

For $q' = \infty$ (or $p' = \infty$) the integrals have to be replaced by $||x||$ (or $||y||$); this is just the case of L_φ (or its dual) being absolutely-p'-summing (or absolutely-q'-summing). The proof of this result is the same as in [60], 17.4.2.

A relatively simple consequences of this is (see [60], 17.4.3).

Kwapien's factorization theorem. *For* $\frac{1}{p} + \frac{1}{q} \leq 1$

$$\mathcal{D}_{p,q} = \mathcal{P}_q^{\text{dual}} \circ \mathcal{P}_p \qquad \textit{isometrically}$$

4.9. It is good to have a list about the tensomorm and their associated operator ideals. Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} \geq 1$,then

(1)	$\varepsilon \sim \mathcal{L}$	all operators
	$\pi \sim \mathcal{I} = \mathcal{I}_1 = \mathcal{L}_{1,1} = \mathcal{L}^*$	integra1 operators
(2)	$\alpha_{p,q} \sim \mathcal{L}_{p,q}$	(p, q) -factorable operators
	$\alpha^*_{p,q} \sim \mathcal{D}_{p',q'} = \mathcal{L}^*_{p,q}$	(p', q') -dominated operators
(3)	$w_p \sim \mathcal{L}_p = \mathcal{L}_{p,p'}$	p-factorable operators
	$w^*_p \sim \mathcal{D}_{p'} = \mathcal{D}_{p',p} = \mathcal{L}^*_p$	p'-dominated operators
(4)	$g_p \sim \mathcal{I}_p = \mathcal{L}_{p,1}$	p-integra1 operators
	$g^*_p \sim \mathcal{P}_{p'} = \mathcal{D}_{p',\infty} = \mathcal{I}^*_p$	absolutely-p-summing operators (with $\mathcal{P}_\infty := \mathcal{L}$)

4.10. It is an essential goal of the theory to compare different tensornorms/maximal operator ideals. The very definition of [d, A] $\sim \alpha$ (by finite-dimensiona1 spaces) implies the

Remarkl. *Let* $[\mathcal{A}, A] \sim \alpha$, $[\mathcal{B}, B] \sim \beta$ **and** $c \geq 0$. *Then:*

$$\alpha \leq c\beta \qquad \textit{if and only if} \qquad A(\cdot) \leq cB(\cdot).$$

inthiscase: $\mathcal{B} \subset \mathcal{A}$.

For example, $\alpha_{p,q} \leq c_{p,q} w_2$ if p, $q \in]1, \infty[$ (see 1.8) implies

$$\mathcal{L}_2 \subset \mathcal{L}_{p,q} \qquad \text{and} \qquad \mathcal{D}_{p,q} \subset \mathcal{D}_2$$

and $\alpha_{p,q} = \alpha^t_{q,p}$ for all p, $q \in [1, \infty]$ gives, togehter with 4.5,

$$\mathcal{L}^{\text{dual}}_{p,q} = \mathcal{L}_{q,p} = \mathcal{L}_{q,p} \qquad \text{and} \qquad \mathcal{D}^{\text{dual}}_{p,q} = \mathcal{D}_{q\ p}$$

The factorization theorems for $\mathcal{I}_p$ and $\mathcal{P}_p$ imply

$$\mathcal{I}_p \subset \mathcal{P}_p \qquad \text{and} \qquad P_p(\cdot) \leq I_q(\cdot)$$
$$\mathcal{P}_2 \subset \mathcal{L}_2 \qquad \text{and} \qquad L_2(\cdot) \leq P_2(\cdot)$$

whence

$$g^*_{p'} \leq g_p \qquad \text{for } 1 \leq p \leq \infty$$
$$w_2 \leq g^*_2 \leq w^*_2$$

where the latter inequality follows from $\alpha_{2\ ,1} \geq \alpha_{2\ 2}$ which in turn implies

$$\mathcal{D}_2 \subset \mathcal{P}_2 \qquad \text{and} \qquad P_2(\cdot) \leq D_2(\cdot).$$

Very interesting phenomena occur from estimates on special Banach spaces. The representation theorem for maximal operator ideals and its corollary 1

$$E' \otimes_{\overleftarrow{\alpha}} F' \hookrightarrow \mathcal{A}(E, F') \stackrel{1}{=} (E \otimes_{\alpha'} F)'$$

imply the

Remark 2. *Let* $[\mathcal{A}, A] \sim \alpha$ *und* $[\mathcal{B}, B] \sim \beta$ *be associated*, $c \geq 0$ *and E and F Banach spaces. Consider the following conditions:*

(a) $\beta' \leq c\alpha'$ on $E \otimes F$

(b) $\mathcal{B}(E, F') \subset \mathcal{A}(E, F')$ *und* $A(\cdot) \leq cB(\cdot)$ *on* $\mathcal{B}(E, F')$

(c) $\overleftarrow{\alpha} \leq c\overleftarrow{\beta}$ *on* $E' \otimes F'$

Then

(1) (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c)

(2) If E' and F' havc the mctric approximation propcrty, or: α and β are accessiblc and E' or F' has thc mctric approximation propcrty thcn: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(2) is a conscqucnce of

$$E \otimes_{\gamma'} F \hookrightarrow (E' \otimes_{\gamma} F')' = (E' \otimes_{\overleftarrow{\gamma}} F')' \qquad \gamma = \alpha \quad \text{or} \quad \beta$$

which holds under the given conditions by thc duality rcsults of 93. Clcarly, if $\mathcal{B}(E, F') \subset \mathcal{A}(E, F')$ the closcd graph thcorcm gives a constant $c \geq 0$ satisfying (b).

Thesc two rcmarks are essential for the intcrplay bctwccn the theorics of tensornorms and operator ideals; thcy will bc refcrred to as the «transfer argumcnt». Notc that (2) includcs conditions under which the full dualization holds:

$$\alpha \leq c\beta \quad \text{on} \quad E' \otimes F' \quad \text{iff} \quad \beta' \leq c\alpha' \quad \text{on} \quad E \otimes F$$

Note di Matematica Vol. VIII - n. 2, 225-234(1988)

5. FURTHER TENSOR PRODUCT CHARACTERIZATIONS OF MAXIMAL OPERATOR IDEALS

5.1. There are very useful characterizations of a-integrai operators $T \in \mathcal{L}(E, F)$ *in* terms of tensor product mappings

$$T \otimes \mathrm{id}_G : E \otimes G \to F \otimes G$$

with appropriate tensomorms. There are three simple formulas (check on elementary tensors) which connect $T \in \mathcal{L}(E, F)$ and $T \otimes \mathrm{id}_G$ (remember the notation B_S and L_φ from 0.7).

(1) *For* $\varphi \in (F \otimes_\pi G)'$ *and* $z \in E \otimes G$

$$\langle B_{L_\varphi \circ T}, z\rangle = \langle \varphi, T \otimes \mathrm{id}_G(z)\rangle$$

(2) *For* $z \in E \otimes F'$

$$\langle B_{\kappa_F \circ T}, z\rangle = \langle \mathrm{tr}_F, T \otimes \mathrm{id}_{F'}(z)\rangle = \langle \mathrm{tr}_E, \mathrm{id}_E \otimes T'(z)\rangle$$

(3) *For* $\varphi \in (G \otimes_\pi E')'$ *and* $z \in G \otimes F'$

$$\langle B_{T''' \circ L_\varphi}, z\rangle = \langle \varphi, \mathrm{id}_G \otimes T'(z)\rangle.$$

5.2. The first of the announced characterizations is the

Theorem. *Let* $[d, A] \sim \alpha$ *and* $T \in \mathcal{L}(E, F)$ *. Then the following statements are equivaleni:*

(1) $T \in \mathcal{A}(E, F)$

(2) *For all Banach spaces G (or only G = F' or G = L with L' = F isometrically)*

$$T \otimes \mathrm{id}_G : E \otimes_{\alpha'} G \to F \otimes_\pi G$$

is continuous.

(3) *For all Banach spaces G (or only G = E)*

$$T' \otimes \mathrm{id}_G : F' \otimes_{\alpha^*} G \to E' \otimes_\pi G$$

is continuous.

In this case:

$$A(T) = \|T \otimes \mathrm{id}_{F'} : \otimes_{\alpha'} \to \otimes_\pi\| \geq \|T \otimes \mathrm{id}, : \otimes_{\alpha'} \to \otimes_\pi\|$$
$$A(T) = \|T' \otimes \mathrm{id}_E : \otimes_{\alpha^*} \to \otimes_\pi\| \geq \|T' \otimes \mathrm{id}_G : \otimes_{\alpha^*} \to \otimes_\pi\|$$

Proof :

(1) $\curvearrowright$ **(2):** If T $\in$ d(E, F) , then, by formula (1) and the representation theorem for maximal operator ideals,

$$|\langle\varphi, T\otimes \mathrm{id}_G(z)\rangle| \leq A(L_\varphi \circ T)\alpha'(z;\ E,G) \leq ||\varphi|| A(T)\alpha'(z;\ E,G)$$

for all $\varphi \in (F \otimes_\pi$ G)' which shows:

$$\pi(T \otimes \mathrm{id}_G(z);\ F,G) \leq A(T)\alpha'(z; E,G).$$

(2) $\curvearrowright$ (1): Assume (2) is satisfied for G = F'. Since **d** is regular (4.4) one has to prove

$$\kappa_F \circ T \in \mathbf{d}(\ E,\ F'') = (E\ \otimes_{\alpha'}\ F')'.$$

For z $\in$ $E \otimes_{\alpha'} F'$ formula (2) gives

$$|\langle B_{\kappa_F \circ T}, z\rangle| = |\langle \mathrm{tr}_F, T\otimes \mathrm{id}_{F'}(z)\rangle| \leq ||\mathrm{tr}_F|| \pi(T\otimes \mathrm{id}_{F'}(z); F, F') \leq$$
$$\leq ||T \otimes \mathrm{id}_{F'} : \otimes_{\alpha'} \to \otimes_\pi|| \alpha'(z;\ E,\ F').$$

The proof for the predual L (if it exists) is the same.

(1) $\curvearrowright$ (3) follows from (1) $\curvearrowright$ (2) by observing that T is α-integral (i.e. T $\in$ **d)** if and only if T' is α^t-integral (see 4.5). .

Note that these are statements about the composition of operators, e.g. (3)

$$\begin{array}{ccccc} \mathcal{F}(F,G) & = F'\otimes G & \overset{T'\otimes \mathrm{id}_G}{\to} & E'\otimes G = & \mathcal{F}(E,G) \\ \mathrm{w} & & & & \\ S & & \rightsquigarrow & & S\circ T. \end{array}$$

5.3. In order to obtain characterizations with ε being involved (this is a sort of dualization as will be seen) the following natural statement is needed. Reca11 that the Johnson spaces C_p (for $1 \leq$ p $< \infty$, see [39]) are separable Banach spaces (reflexive for $1 <$ p $< \infty$) with the metric approximation property such that for every $M \in FIN$ and $\varepsilon > 0$ there is a 1-complemented subspace N c C_p and an isomorphism $S \in \mathcal{L}(\ M$, N) such that $||S||\ ||S^{-1}|| \leq 1+\varepsilon$.

Lemma. *Let β and γ be tensornorms, β jinitely generated, $c \geq 0$ and $T \in \mathcal{L}(\ E,\ F)$.*

(a) *Iffor a normed space G*

$$||T\otimes id_M : E\otimes_\beta M \to F\otimes_\gamma M|| \leq c$$

for cofinally many $M \in FIN(G)$ *, then*

$$||T \otimes id_G : E \otimes_\beta G \to F \otimes_\gamma G|| \le c$$

(b) *If (for some* $1 \le p \le \infty$*)*

$$||T \otimes \mathrm{id}_{C_p} : E \otimes_\beta C_p \to F \otimes_\gamma C_p|| \le c$$

rhen

$$||T \otimes \mathrm{id}_G : E \otimes_\beta G \to F \otimes_\gamma G|| \le c$$

for all normed spaces G.

The proof is vcry easy using the mctric mapping propcrty of tensomorms.

Corollary. *Let* α *be an accessible, finitely generated tensornorm, [d, A] the associated maximal operator idea1 and* $T \in \mathcal{L}(E, F)$*. Then the following are equivalent:*

(1) $T \in \mathcal{A}(E, F)$

(2) For all Banach spaces G (or only $G = C_p$ *for some p)*

$$T \otimes \mathrm{id}_G : E \otimes_\varepsilon G \to F \otimes_{\alpha^t} G$$

is continuous.

(3) For all Banach spaces G (or only $G = C_p$ *for some p)*

$$T' \otimes \mathrm{id}_G : F' \otimes_\varepsilon G \to E' \otimes_\alpha G.$$

In this case the operators in (2) and (3) have norms $\le A(T)$ *and*

$$A(T) = ||T \otimes \mathrm{id}_{C_p} : \otimes_\varepsilon \to \otimes_{\alpha^t}|| = ||T' \otimes \mathrm{id}_{C_p} : \otimes_\varepsilon \to \otimes_\alpha||.$$

Proof . To prove (1) $\leftrightarrow$ (2) it is enough, by the theorcm and the lemma, to show that for all $M \in FIN$

$$||T' \otimes \mathrm{id}_{M'} : F' \otimes_{\alpha^*} M' \to E' \otimes_\pi M'|| \le c$$

if and only if

$$||T \otimes \mathrm{id}_M : E \otimes_\varepsilon M \to F \otimes_{\alpha^t} M|| \le c.$$

But this follows from

$$(E \otimes_\varepsilon M)' \overset{1}{=} E' \otimes_\pi M' \qquad \text{and} \qquad F \otimes_{\overleftarrow{\alpha}^t} M \overset{1}{\hookrightarrow} (F' \otimes_{\alpha^*} M')'$$

and the fact that

$$||F' \otimes_{\alpha^*} M' \to (F \otimes_{\alpha^t} M)'|| \le 1$$

As bcforc, the cquivalcncc (1) $\leftrightarrow$ (3) is a consequcnce of (1) $\leftrightarrow$ (2) by obscrving that T is α-integral if and only if T' is α^t-intcgral. ∎

If α is *not necessarily accessible* the proof showed that (1) $\curvearrowright$ (2) holds if F has the *metric approximation property* and *(1)* $\curvearrowright$ *(3)* if E' has the *metric approximation property.*

For special operator ideals it is possible to find «better» fixed spaces G (than C_p); for example: If $\mathfrak{A} = \mathcal{P}_p$ it is enough to take $G = \ell_p$; this is the tensor product formulation of the simple, but useful characterization of absolutely-p-summing operators due to Kwapien: $T \in \mathcal{L}(E, F)$ is in $\mathcal{P}_p$ iff $TS \in \mathcal{P}_p$ for all $S \in \mathcal{L}(\ell_{p'}, E)$.

5.4. To see some particular cases of these results take

$$g_p \sim \mathcal{I}_p \qquad \text{and} \qquad d'_{p'} = g^*_{p'} \sim \mathcal{P}_p.$$

Since g_p and $d'_{p'}$ are accessible (see later 9.4) it follows

Proposition. *Take* $1 \leq p \leq \infty$.

(1) *For* $T \in \mathcal{L}(E, F)$ *are equivalent:*

(a) T *is p-integral,*

(b) *for all Banach spaces G (or only* $G = F'$*)*

$$T \otimes \mathrm{id}_G : E \otimes_{g'_p} G \to F \otimes_\pi G$$

is continuous,

(c) *for all Banach spaces G*

$$T \otimes \mathrm{id}_G : E \otimes_\varepsilon G \to F \otimes_{d_p} G$$

is continuous.

(2) $T \in \mathcal{L}(E, F)$ *is integral if and only if for all Banach spaces (or only* $G = F'$*)*

$$T \otimes \mathrm{id}_G : E \otimes_\varepsilon G \to F \otimes_\pi G$$

is continuous.

(3) *For* $T \in \mathcal{L}(E, F)$ *are equivalent:*

(a) T *is absolutely-p-summing,*

(b) *for all Banach spaces G (or only* $G = F'$*)*

$$T \otimes \mathrm{id}_G : E \otimes_{d'_{p'}} G \to F \otimes_\pi G$$

is continuous,

(c) *for all Banach spaces*

$$T \otimes \mathrm{id}_G : E \otimes_\varepsilon G \to F \otimes_{g'_{p'}} G$$

is continuous.

Clearly, there are norm estimates as in 5.2, for example,

$$I(T) = \|T \otimes \mathrm{id}_{F''} : \otimes_\varepsilon \to \otimes_\pi\|$$

5.5. Another interesting and very important consequence of the theorem (and its corollary) is the

Proposition. *Let $[\mathcal{A}, A]$ be a maximal operator ideal such that the associated tensornorm α is accessible. Then*

$$\mathcal{A}^* \circ \mathcal{A} \subset \mathcal{I} \qquad \text{and} \qquad I(T \circ S) \leq A^*(T)A(S).$$

In 9.2 accessibility of α will be explained in terms of the operator ideal d . If CY is not necessarily accessible (remember that there is no example known!) it follows

$$\mathcal{A}^*(F,G) \circ \mathcal{A}(E,F) \subset \mathcal{I}(E,G)$$

with norm inequality, if F has the metric approximation property as the proof will show as well.

Proof. If $\mathcal{A} \sim \alpha$, then $\mathcal{A}^* \sim \alpha^*$. This implies that for $S \in \mathcal{A}(E,F)$ and $T \in \mathcal{A}^*(F,G)$ the map

$$(T \circ S) \otimes \mathrm{id}_{G'} : E \otimes_{\otimes_\varepsilon} G' \to F \otimes_{\alpha^t} G' \to G \otimes_\pi G'$$

has norm $\leq A^*(T)A(S)$ by 5.3 and 5.2, whence $T \circ S \in \mathcal{I}$ with the norm estimate by 5.4.∎

To see a concrete example (see also [22])

$$\mathcal{D}_{p',q'} \circ \mathcal{L}_{p,q} \subset \mathcal{I} \qquad \textit{and} \qquad I(T \circ S) \leq D_{p',q'}(T)L_{p,q}(S)$$

and even

$$\mathcal{D}_{p',q'} \circ \mathcal{L}_{p,q} \subset \mathcal{N} \qquad \textit{and} \qquad N(T \circ S) \leq D_{p',q'}(T)L_{p,q}(S)$$

if $(p,q) \notin \{(1,1),(1,\infty),(\infty,1)\}$. In the excluded cases the product is nof contained in the ideal of nuclear operators.

Proof. It will be shown in 9.4 that $\alpha_{p,q}$ is accessible, whence the first statement is clear. Coming to the second statement take $S \in \mathcal{L}_{p,q}(E,F)$ and $T \in \mathcal{D}_{p',q'}(F,G)$ and observe first that for $1 < q < \infty$

$$\mathcal{D}_{p',q'} \subset \mathcal{P}_{q'}^{\text{dual}} \subset \mathcal{W} \qquad \text{(weakly compact operators)}$$

whence the astriction $T^\pi : F'' \to G$ of T" is (by the results of 4.4) also (p', q')-dominated Since S is (p, q) -factorable 4.6 implies the factorization

$$\begin{array}{ccccccc} E & \hookrightarrow & E'' & \overset{S''}{\to} & F'' & \overset{T^\pi}{\to} & G \\ & & 1 & & \downarrow V & & \\ & & L_{q'} & \overset{J}{\to} & L_p & & \end{array}$$

whence $R := T^\pi V J$ is an integra1 operator on a reflexive space with the approximation property and therefore nuclear with I(R) = $N(R)$ (see [13], p. 248).

If $q = 1$ and $1 < p < \infty$

$$\mathcal{D}_{p',\infty} \circ \mathcal{L}_{p,1} = \mathcal{P}_{p'} \circ \mathcal{I}_p = \mathcal{W} \circ \mathcal{P}_{p'} \circ \mathcal{P}^*_{p'} \subset \mathcal{W} \circ \mathcal{I} \subset \mathcal{N}$$

(again by Radon-Nikodym arguments, see e.g. [60] 24.6.2). For (p, q) = (1,1)

$$\mathcal{D}_{\infty,\infty} \circ \mathcal{L}_{1,1} = \mathcal{L} \circ \mathcal{I} \neq \mathcal{N}.$$

For the remaining two cases $(p, q) = (1, \infty)$ or $(\infty, 1)$ take an operator $T : C[0,11 \to c_0$ which is absolutely-l-summing and not nuclear ([13], p. 175). Then T' is not nuclear as well ([13], p. 243) and

$$T \in \mathcal{P}_1 \circ \mathcal{L}_\infty = \mathcal{D}_{1,\infty} \circ \mathcal{L}_{\infty,1}$$
$$T' \in \mathcal{P}_1^{\text{dual}} \circ \mathcal{L}_1 = \mathcal{D}_{\infty,1} \circ \mathcal{L}_{1,\infty}$$

and this completes the proof.

A special case is Grothendieck's

$$\mathcal{P}_2 \circ \mathcal{P}_2 = \mathcal{P}_2 \circ \mathcal{I}_2 = \mathcal{P}_2 \circ \mathcal{P}_2^* \subset \mathcal{N}$$

$$N(TS) \leq P_2(T) P_2(S)$$

, 5.6. The rest of this paragraph will contain some more applications of this type of characterizations of α-integral operators/maximal operator ideals. First, when is the natura1 map

$$I : E\tilde{\otimes}_\alpha F \to E\tilde{\otimes}_\varepsilon F \overset{1}{\hookrightarrow} \mathcal{L}(E', F)$$

injective? If α is totally accessible the duality theorem 3.4 for tensomorms implies

$$\begin{array}{ccccccc} E\tilde{\otimes}_\alpha F & = & E\tilde{\otimes}_{\overleftarrow{\alpha}} F & \overset{1}{\hookrightarrow} & (E'\tilde{\otimes}_{\alpha'} F')' & \hookrightarrow & \mathcal{L}(E', F'') \\ & & & \searrow^{I} & & {}^{1}\nearrow & \\ & & & & E\tilde{\otimes}_\varepsilon F & & \end{array}$$

whence I is injective.

Proposition. *If α is a finitely generated tensornorm, E and F Banach spaces, one of which has the approximation property, then the natural map*

$$I : E\tilde{\otimes}_\alpha F \to E\tilde{\otimes}_\varepsilon F$$

is injective.

Proof . Assume that F has the approximation property, $z \in E\tilde{\otimes}_\alpha F$ and $I(z) = 0$. It is to show that $\langle \varphi, z\rangle = 0$ for all

$$\varphi \in (E\tilde{\otimes}_\alpha F)' \hookrightarrow \mathcal{L}(E, F').$$

By theorem 5.2 (and, clearly, the correspondence between maximal operator ideals and tensomorms)

$$L_\varphi \otimes \mathrm{id}_F : E\tilde{\otimes}_\alpha F \to F'\tilde{\otimes}_\pi F$$

is continuous. The lower map in the diagram

$$\begin{array}{ccc} E\tilde{\otimes}_\alpha F & \xrightarrow{I} & E\tilde{\otimes}_\varepsilon F \\ {\scriptstyle L_\varphi\tilde{\otimes}_{\alpha,\pi}\mathrm{id}\, F}\downarrow & \%\!. & \downarrow {\scriptstyle L_\varphi\tilde{\otimes}_\varepsilon \mathrm{id}\, F} \\ F'\tilde{\otimes}_\pi F & \rightarrow & F'\tilde{\otimes}_\varepsilon F \end{array}$$

is injective by the approximation property, whence

$$L_\varphi\tilde{\otimes}_{\alpha,\pi}\mathrm{id}_F(z) = 0 \in F'\tilde{\otimes}_\pi F$$

and formula (2) in 5.1 implies

$$\langle \varphi, z\rangle = \langle \mathrm{tr}_F, L_\varphi\tilde{\otimes}_{\alpha,\pi}\mathrm{id}_F(z)\rangle = 0.$$

Since

$$E' \otimes_\alpha F \to \mathcal{A}(E,F) = (E \otimes_{\alpha'} F')' \cap \mathcal{L}(E,F)$$

is continuous and

$$E'\tilde{\otimes}_\varepsilon F \overset{1}{\hookrightarrow} \mathcal{L}(E, F)$$

it follows: *If $[\mathcal{A}, A]$ and α are associated, then the natural map*

$$E'\tilde{\otimes}_\alpha F \to \mathrm{d}(E, F)$$

is injective if E' or F has the approximation property (or if α is totally accessible).

5.7. For the bounded approximation property of Banach spaces one obtains the

Proposition. *Let α' be totally accessible and $\alpha \sim [\mathcal{A}, A]$. Every Banach space E with* $id_E \in$ d *has the bounded approximation property with constant* $\leq A(id_E)$.

Proof. To apply the criterion 3.5 (about $\pi \leq \lambda \overleftarrow{\pi}$) for the bounded approximation property, take $z \in E \otimes E'$ and apply theorem 5.2 to $id_E \in \mathcal{A}$:

$$\pi(z; E, E') \leq A(id_E)\alpha'(z; E, E') = A(id_E)\overleftarrow{\alpha'}(z; E, E') \leq$$
$$\leq A(id_E)\overleftarrow{\pi}(z; E, E').$$

$id_E \in$ d means: $E \in$ space(d) in the terminology of Pietsch [60]; by 5.2. this is equivalent to

$$E \otimes_{\alpha'} G = E \otimes_\pi G \qquad \text{for all } G \text{ (or } G = E')$$

(isomorphically) – or, by 5.3 (if α is accessible),

$$E \otimes_\varepsilon G = E \otimes_{\alpha'} G \qquad \text{for all } G \text{ (or } G = C_p)$$

(isomorphically). The proposition has also a negative favour: If there is a Banach space $E \in$ space(d) without the bounded approximation property, then α' is *not* totally accessible. Anticipating the results of §8 take $w_p\backslash \sim \mathcal{L}_p^{inj}$ and recall that all ℓ_p (for $p \neq 2$) have subspaces without the approximation property; then the proposition says that $(w_p\backslash)' = w_p'/$ is not totally accessible $(p \neq 2)$.

5.8. For tensomorms α and β, and operators $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(E, F)$ it is not exactly known, under which circumstances the continuity of

$$S \otimes T : X \otimes_\alpha E \to Y \otimes_\beta F$$

implies the continuity of

$$S \otimes T'' : X \otimes_\alpha E'' \to Y \otimes_\beta F''$$

(see also problem 2 in 2.3). If $\alpha = \overleftarrow{\pi}$ and $\beta = \pi$

$$S := id_{E'}, \qquad T := id_E$$

the continuity of $id_{E'} \otimes id_E : \otimes_{\overleftarrow{\pi}} \to \otimes_\pi$ is, by 3.5, the bounded approximation property of E which does not imply the one of E', i.e. the continuity of $id_{E'} \otimes id_{E''} : \otimes_{\overleftarrow{\pi}} \to \otimes_\pi$. So, the answer to the above problem is negative for arbitrary α and β!

TO obtain at least some positive answers, fix $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and consider for Banach spaces V, W

$$\mathcal{A}(V,W) := \{R \in \mathcal{L}(V,W) | S \otimes R : X \otimes_{\overrightarrow{\alpha}} V \to Y \otimes_{\overleftarrow{\beta}} W \quad \text{continuous}\}$$

$$A(R) := \|S \otimes R : \otimes_{\overrightarrow{\alpha}} \to \otimes_{\overleftarrow{\beta}}\|$$

It is easily seen that [d, A] is a maximal Banach operator idea1 (for the maximality use the property stated in 4.2). The fact that $R \in$ **d** if and only if R" $\in$ **d** (by corollary 3 in 4.4) is the key for the

Proposition. *Let α and β be tensornorms, α finitely generated, X, Y, E and F Banach spaces, $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(E, F)$ such that*

$$S \otimes T : X \otimes_{\alpha} E \to Y \otimes_{\beta} F$$

is continuous. Then in each of the following five cases

$$S \otimes T'' : X \otimes_{\alpha} E'' \to Y \otimes_{\beta} F''$$

is continuous:

(1) *β is totally accessible,*

(2) β is accessible and: Y or F'' has the bounded approximation property,

(3) Y and F'' have the bounded approximation property,

(4) T is weakly compact,

(5) whenever G_1 c G_2 then $Y \otimes_{\beta} G_1$ is an isomorphic subspace of $Y \otimes_{\beta} G_2$.

Proof . To apply the construction above, observe that $\alpha = \overrightarrow{\alpha}$ and $\beta = \overleftarrow{\beta}$ in the cases (1) – (3) by the definition and the approximation lemma. Case (4) follows by using that T"(E") $\subset F$: it is not too difficult (using the extension lemma) to check that for the astricition $T^{\pi} : E'' \to F$ even

$$S \otimes T^{\pi} : X \otimes_{\alpha} E'' \to Y \otimes_{\beta} F$$

is continuous. The last case follows from a refinement of the construction of **d:** Define first a tensomorm 7 by

$$\gamma(z;\ V,W) := \sup\{\beta(\mathrm{id}_V \otimes Q_L^W(z);\ V, W/L) | L \in COFIN(W)\};$$

7 coincides with β on *NORM* x *FIN* whence, by the approximation lemma, on all spaces $Y \otimes \ell_{\infty}(\Gamma)$. Now use the maximal Banach operator ideal

$$\{R \in \mathcal{L}(V,W) \text{ IS} \otimes R : X \otimes_{\alpha} V \to Y \otimes_{\gamma} W \quad \text{continuous}\},$$

the continuous maps

$$Y \otimes_\beta F \to Y \otimes_\gamma F, \qquad Y \otimes_\gamma F'' \to Y \otimes_\gamma \ell_\infty(B_{F''}),$$

and the isomorphic embcdding

$$Y \otimes_\beta F'' \to Y \otimes_\beta \ell_\infty(B_{F''}) = Y \otimes_\gamma (B_{F''}).$$ ■

Unfortunately, this result does not cover the general case of $\beta = \pi$ – which seems to be unknown. It is clear (by 4.4) that in case (1) $\|S \otimes T : \ldots\| = \|S \otimes T'' : \ldots\|$ – and this is also true in (2) and (3) if the spaces have the metric instead of the bounded approximation property. For $\alpha = \varepsilon$, $\beta = \pi$ and Y having the metric approximation property the result was proven in [38], p. 355.

Note di Matematica Vol. VIII - n. 2, 235-239(1988)

6. $\mathcal{L}_p$-SPACES

6.1. A Banach space E is called an $\mathcal{L}^g_{p,\lambda}$*-space* (for $1 \le p \le \infty$ and $1 \le \lambda < \infty$) if for each $\varepsilon > 0$ and $N \in$ FIN(E) there exist a natura1 number n and a factorization

$$\begin{array}{ccc} N & \overset{I_N^E}{\hookrightarrow} & E \\ S \searrow & \circlearrowleft & \nearrow T \\ & \ell_p^n & \end{array}$$

such that $\|T\| \, \|S\| \le \lambda + \varepsilon$. A space is called $\mathcal{L}^g_p$ if it is an ,$,-space for some λ. Obviously, *every* $\mathcal{L}_{p,\lambda}$*-space in t he sense of Lindenstrauss and Pełczyński* ([51], for every $N \in$ FIN(E) there is an $M \in$ FIN(E) with N c M and Banach-Mazur-distante $d(M, \ell_p^{\dim M}) \le \lambda$) *is an* $\mathcal{L}^g_{p,\lambda}$*-space* and it will be seen soon (6.3) that the difference between these two classes of spaces is not very large; the great advantage of the class of $\mathcal{L}^g_{p\ \lambda}$-spaces is that the constant λ does not vary under dualization – a fact which is false for $\mathcal{L}_{p,\lambda}$-spaces and seemingly unknown if an additional ε is allowed.

Since $L_p(\mu)$-spaces are $\mathcal{L}_{p,1+\varepsilon}$-spaces for all $\varepsilon > 0$ they are $\mathcal{L}^g_{p\ 1}$-spaces and it follows the same way that the spaces C(K) are $\mathcal{L}^g_{\infty,1}$-spaces .

Following Pietsch, a Banach spaces E is in space $(\mathbf{d})$ (for an operator ideal $\mathbf{d}$) if $\mathrm{id}_E \in \mathbf{d}$. Recall that $(\mathcal{L}_p, L_p)$ is the maximal normed operator ideal of the p-factorable operators which is associated with the tensomorm w_p . Anticipating the fact that w_p is accessible (9.4) the equivalences (2) – (5) of the following proposition are immediate from the characterizations 5.2 and 5.3:

Theorem. *Let* $1 \le p \le \infty$ *and* $1 \le \lambda < \infty$*. Thenfor every Banach space E thefollowing statements are equivalent:*

(1) *E is an* $\mathcal{L}^g_{p,\lambda}$*-space*

(2) E is in space$(\mathcal{L}_p)$ *and* $L_p(\mathrm{id}_E) \le \lambda$

(3) For all Banach spaces G (or only G = E′ or G some predual of E)

$$w'_p \le \pi \le \lambda w'_p \quad \text{on} \quad E \otimes G$$

(4) For all Banach spaces G

$$\varepsilon \le w_p \le \lambda \varepsilon \quad \text{on} \quad G \otimes E$$

(5) E' is in space$(\mathcal{L}_{p'})$ *and* $L_{p'}(\mathrm{id}_{E'}) \le \lambda$

(6) For every $\varepsilon > 0$ *there is a factorization of* $\mathrm{id}_{E''}$ *through some* $L_p(\mu)$

$$\begin{array}{ccc} E'' & \overset{\mathrm{id}_{E''}}{\to} & E'' \\ S \searrow & \circlearrowleft & \nearrow T \\ & L_p(\mu) & \end{array}$$

with $\|S\|\,\|T\| \leq \lambda + \varepsilon$. *(Inparticular: E'' is isomorphic to a complemented subspace of some $L_p(\mu)$).*

It is clear form (6) that the $\mathcal{L}_2^g$ -spaces are exactly those isomorphic to Hilbert spaces. (4) implies that $\mathcal{P}_\lambda$-spaces (i.e. spaces with the X-extension property) are $\mathcal{L}_{\infty,\lambda}^g$-spaces.

Proof :

(2) $\Leftrightarrow$ (6): id $_E$ is in $\mathcal{L}_p$ iff id $_{E''}$ is in $\mathcal{L}_p$ by corollary 3 in 4.4; now the factorization theorem 4.6 for p-factorable operators shows the equivalence.

(4) $\Rightarrow$ (1): Take $N \in$ FIN(E) and

$$I_N^E \in \mathcal{F}(N,E) \overset{1}{=} N' \otimes_\varepsilon E = N' \otimes_{w_p} E,$$

then there is a reprcsentation of I_N^E by $z = \sum_{m=1}^n \varphi_m \otimes y_m$ with

$$w_p(z;\, N',E) \leq w_p(\varphi_m) w_{p'}(y_m) \leq \varepsilon(z;\, N',E)(\lambda+\delta) = \lambda+\delta$$

and whence

$$\begin{array}{ccc} N & \overset{I_N^E}{\hookrightarrow} & E \\ S \searrow & \circlearrowleft & \nearrow T \\ & \ell_p^n & \end{array} \qquad S(x) := (\langle \varphi_m, x\rangle) \qquad T(\xi_m) := \sum_{m=1}^n \xi_m y_m$$

is the desired factorization since

$$\|S\| = w_p(\varphi_m), \qquad \|T\| = w_{p'}(y_m)$$

(1) $\Rightarrow$ (4): Observe first that for all Banach spaces G

$$\varepsilon = w_p \qquad \text{on} \qquad G \otimes \ell_p^n$$

by 1.9; now the implication is immediate from the following lemma which is of its own interest. ∎

Corollary. *E is un $\mathcal{L}_{p,\lambda}^g$ -space ifand only if E' is an $\mathcal{L}_{p',\lambda}^g$-space.*

6.2. The «local techniques» for the $\mathcal{L}_p^g$-spaces are somehow concentrated in the

Local technique lemma. *Let α and β be tensornorms, $c > 0$ and G a normed space such that*

$$\alpha \leq c\beta \qquad on \qquad G \otimes \ell_p^n$$

for all n ∈ N , thcn

$$\overrightarrow{\alpha} \leq c\lambda \overrightarrow{\beta} \quad o \quad n \qquad G \otimes E$$

for every $\mathcal{L}^g_{p,\lambda}$-space G.

Proof . Take a factorization

$$\begin{array}{ccccc} N & \hookrightarrow & M & \hookrightarrow & E \\ & S\searrow & \not. & \nearrow T & \\ & & \ell_p^n & & \end{array} \qquad \|T\|\,\|S\| \leq \lambda + \varepsilon$$

then, for every z ∈ G⊗ N,

$$\begin{aligned} \alpha(z;\ G, M) &= \alpha((\mathrm{id}_G \otimes T \circ S)(z); G, M) \leq \|T\|\alpha(\mathrm{id}_G \otimes S(z); G, \ell_p^n) \leq \\ &\leq \|T\|c\beta(\mathrm{id}_G \otimes S(z); G, \ell_p^n) \leq \|T\|\,\|S\|c\beta(z;\ G, N) \leq \\ &\leq (\lambda + \varepsilon)c\beta(z; G, N). \end{aligned}$$

This implies the statement. .

(Note that the finite hull only was taken on the right side of the tcnsor product; this will be used in 8.8 and 8.9). It is obvious by the definition, that more or less the same *local techniques for operafors* apply for $\mathcal{L}^g_p$-spaces as they do for $\mathcal{L}_p$-spaces.

6.3. To obtain the precise conncction between the $\mathcal{L}_p$-spaces and the $\mathcal{L}^g_p$-spaces, observe first that for every 1 < p < ∞ the Hilbert space ℓ_2 (by using Rademacher functions) is a complemented subspace of $L_p[0, 1]$, whence an Cg-space; it follows now easily from the definition that *every Hilbert space is an Li-space for all* 1 < p < ∞ (but ℓ_2 is not an Cr-space for $p \neq 2$). Results of Lindenstrauss-Rosenthal ([52], 2.1 and 3.2) even imply (with the aid of 6.1 (6))

$1 < p < \infty$: *A Banach space is an $\mathcal{L}^g_p$-space if and only if it is an L,-space or isomorphic io a Hilbert-space.*

$p = 1$ or ∞: *The class of $\mathcal{L}^g_p$-spaces coincides with the class of $\mathcal{L}_p$-spaces.*

Note that $\mathcal{L}^g_{p,\lambda}$-spaces are exactly those which were used in the assumption of [52], theorem 4.3. Again using 6.1 (6) it follows that

A Banach space is an Li-space if and only if it is isomorphic to a complemente subspace of an $\mathcal{L}_p$-space.

This implies that tensomorm inequalities hold for $\mathcal{L}^g_p$-spaces if and only if they hold fc $\mathcal{L}_p$-spaces – but the constants may vary.

6.4. Grothendieck's inequality in tensorial form 1.11 stated that

$$\pi \leq K_G w_2 \qquad \text{on} \qquad \ell_\infty^n \otimes \ell_\infty^m$$

whence, by the local technique lemma for $\mathcal{L}_p^g$-spaces

$$\pi \leq K_G \lambda \mu w_2 \qquad \text{on} \qquad E \otimes F$$

whenever E is an $\mathcal{L}_{\infty,\lambda}^g$-space and F and $\mathcal{L}_{\infty,\mu}^g$-space. Since

$$\begin{array}{ll} \mathcal{L} \sim \varepsilon & \varepsilon' = \pi \\ \mathcal{D}_2 \sim w_2^* & w_2^{*\prime} = w_2 \\ \mathcal{P}_2 \sim g_2^* & g_2^{*\prime} = d_2 \end{array}$$

and, by 1.5,

$$w_2 = \alpha_{2,2} \leq \alpha_{1,2} = d_2$$

the «transfer argument» 4.10 implies the

Proposition. *If E is an $\mathcal{L}_{\infty,\lambda}^g$-space and F an $\mathcal{L}_{1,\mu}^g$-space, then*

$$\mathcal{L}(E,F) = \mathcal{D}_2(E,F) = \mathcal{P}_2(E,F)$$
$$P_2(T) \leq D_2(T) \leq K_G \lambda \mu \|T\|.$$

In 8.5 the result that every operator $\mathcal{L}_\infty^g \to \mathcal{L}_1^g$ is absolutely-2-summing will be improved to operators $\mathcal{L}_\infty^g \to \mathcal{L}_p^g$ for $1 \leq p \leq 2$.

Dualizing

$$\pi \leq K_G w_2 \qquad \text{on} \qquad \ell_\infty^n \otimes \ell_\infty^m$$

gives

$$w_2^* \leq K_G \varepsilon \qquad \text{on} \qquad \ell_1^n \otimes \ell_1^m$$

whence, by the local technique lemma,

$$w_2^* \leq K_G \lambda \mu \varepsilon \qquad \text{on} \qquad E \otimes F$$

if E is an $\mathcal{L}_{1,\lambda}^g$-space and F and $\mathcal{L}_{1,\mu}^g$-space. For operators this means (again by the transfer argument 4.10): Every 2-factorable $\mathcal{L}_1^g \to \mathcal{L}_\infty^g$ is integral (see also 8.13).

6.5. Another application of this simple way of arguing comes from

$$\pi \leq K_G d_\infty \qquad \text{on} \qquad \ell_1^n \otimes \ell_2^m$$

(see 1.12), and whence

$$\pi \leq K_G \lambda \mu d_\infty \qquad \text{on} \qquad \mathcal{L}_{1,\lambda}^g \otimes \mathcal{L}_{2,\mu}^g .$$

Since $\mathcal{L} \sim \varepsilon$ and $\mathcal{P}_1 \sim g_\infty^* = d_\infty'$ this implies the famous [51]

Proposition. *If E is an $\mathcal{L}_{1,\lambda}^g$-space and F an $\mathcal{L}_{2,\mu}^g$-space, rhen $\mathcal{L}(E, F) = \mathcal{P}_1(E, F)$ and* $P_1(T) \leq K_G \lambda \mu \|T\|$.

Note di Matematica Vol. VIII - n. 2, 241-245(1988)

7. MINIMAL OPERATOR IDEALS

7.1. Now another crucial link between Banach operator ideals and tensor norms will be proved *the representation theorem for minima1 ideals.*

If [d, *A]* is a quasi-Banach ideal, then its minimal kemel is defined by

$$[\mathcal{A}, A]^{\min} := [\overline{\mathcal{F}}, \|\,.\,\|] \circ [\mathcal{A}, A] \circ [\overline{\mathcal{F}}, \|\,.\,\|]$$

where $[\overline{\mathcal{F}}, \|\cdot\|]$ denotes the ideal of all approximable operators (an operator T $\in \mathcal{L}(E, F)$ is said to be approximable if it is in the operator-norm closure of all finite dimensiona1 operators). $[\mathcal{A}, A]$ is called minima1 if it coincides with its minima1 kemel (see [60], 8.6).

Let $\alpha \sim$ [d, *A]* . Then for $M \in FIN(E')$ and $N \in$ FIN(F) the diagram

$$\begin{array}{ccccc} E' \otimes_\alpha F & \overset{\psi}{\hookrightarrow} & \mathcal{A}^{\min}(E,F) & 3 & I_N^F T Q_{M^0}^E \\ \uparrow & & \uparrow & & \wr \\ M \otimes_\alpha N & \overset{1}{=} & \mathcal{A}(E/M^0, N) & \ni & T \end{array}$$

obviously commutes. Hence for $z \in E' \otimes F$ and $u \in M \otimes N$ with $I_M^{E'} \otimes I_N^F(u) = z$

$$A^{\min}(L_z) = A^{\min}(I_N^F L_u Q_{M^0}^E) \leq A(L_u) = \alpha(u; M, N),$$

which implies

$$\|\psi : E' \otimes_\alpha F \hookrightarrow \mathcal{A}^{\min}(E,F)\| \leq 1.$$

Even more holds:

Theorem. *If* $\alpha \sim [\mathcal{A}, A]$ *the canonica1 map*

$$\tilde{\Psi} : E' \tilde{\otimes}_\alpha F \overset{1}{\twoheadrightarrow} \mathcal{A}^{\min}(E,F)$$

is a metric surjection for all Banach spaces E and F.

Proof : (1) Let $S_0 \in$ d(X, Y) , T $\in \mathcal{F}(E, X)$, $R \in \mathcal{F}(Y, F)$ and consider w $\in E' \otimes F$ corresponding to $RS_0 T \in \mathcal{F}(E, F)$. Then

$$\alpha(w; E', F) \leq \|R\| A(S_0) \|T\|.$$

Indeed, if

$$\begin{array}{lll} R = I_M^F R_0 & \text{with} & M \in FIN(F),\ R_0 \in \mathcal{L}(Y, M) \\ T = T_0 Q_N^E & \text{with} & N \in COFIN(E), T_0 \in \mathcal{L}(E/N, X) \end{array}$$

then $RS_0T = I_M^F R_0 S_0 T_0 Q_N^E$, and hence

$$\begin{aligned}\alpha(w; E', F) &= \alpha((Q_N^E)' \otimes I_M^F(z_{R_0S_0T_0}); E', F) \\ &\leq \alpha(z_{R_0S_0T_0}; (E/N)', M) \\ &= A(R_0S_0T_0) \leq \|R\| A(S_0) \|T\| .\end{aligned}$$

(2) Let now $S \in \mathcal{A}^{\min}(E, F)$. Then by definition there are $S_0 \in$ d(X, Y), $T \in \overline{\mathcal{F}}(E, X)$, $R \in \overline{\mathcal{F}}(Y, F)$ such that

$$S = RS_0T \qquad \text{and} \qquad \|S\| A(S_0) \|T\| \leq (1+\varepsilon) A^{\min}(S).$$

For sequences (T_n) in $\mathcal{F}(E, X)$ and (R_n) in $\mathcal{F}(Y, F)$ with

$$\|T - T_n\| \to 0 \qquad \text{and} \qquad \|R - R_n\| \to 0$$

choose $w_{nm} \in E' \otimes F$ corresponding to $R_nS_0T_m \in \mathcal{F}(E, F)$; then, by (1),

$$\begin{aligned}\alpha(w_{nn} - w_{mm}; E', F) &\leq \\ &\leq \alpha(w_{nn} - w_{mn}; E', F) + \alpha(w_{mn} - w_{mm}; E', F) \\ &\leq \|R_n - R_m\| A(S_0) \|T_n\| + \|R_m\| A(S_0) \|T_n - T_m\| ,\end{aligned}$$

which implies that $w := \lim w_{nn} \in E'\tilde{\otimes}_\alpha F$ exists. Obviously,

$$\psi(w) = \lim \psi(w_{nn}) = RS_0T = S$$

and, again by (1),

$$\begin{aligned}\alpha(w; E', F) &= \lim \alpha(w_{nn}; E', F) \\ &\leq \lim \|R_n\| A(S_0) \|T_n\| \\ &= \|R\| A(S_0) \|T\| \leq (1+\varepsilon) A^{\min}(T).\end{aligned}$$

It is a well-known fact (see 0.7) that the extension

$$E'\tilde{\otimes}_\pi F \twoheadrightarrow \mathcal{N}(E, F)$$

of the canonical embedding is a metric surjection. Hence in the special case $\alpha = \pi \sim \mathcal{I}$ the preceding result implies that $[\mathcal{I}, I]^{\min} = [\mathcal{N}, N]$. This is the reason why operators in $\mathcal{A}^{\min}$ sometimes are called *α-nuclear* .

The following statement follows directly from 5.6:

Corollary. *Let $\alpha \sim [\mathcal{A}, A]$ and let E, F be Banach spaces. If α is totally accessible or if E' or F has the approximation property, then*

$$E'\tilde{\otimes}_\alpha F = \mathcal{A}^{\min}(E, F)$$

isometrically.

7.2. With the last theorem, the third of the three basic links between the metric theory of tensor products and the theory of Banach-operator ideals was obtained: If the maximal Banach operator ideal [A, A] and the finitely generated tensomorm α are associated, i.e.

$$M' \otimes_\alpha N = \mathcal{A}(M,N)$$

isometrically for all M, N $\in$ FIN, then for all Banach spaces E and F the following theorems hold: (4.3, 4.4, 7.1)

(1) The representation theorem for maximal operator ideals:

$$\mathcal{A}(E,F') \stackrel{1}{=} (E \otimes_{\alpha'} F)'$$

(2) The embedding theorem:

$$E'\tilde{\otimes}_{\overleftarrow{\alpha}} F \stackrel{1}{\hookrightarrow} A(E,F)$$

(3) The representation theorem for minima1 operator ideals:

$$E'\tilde{\otimes}_\alpha F \stackrel{1}{\twoheadrightarrow} \mathcal{A}^{\min}(E,F).$$

In order to illustrate the interplay of these three facts the following extension of a result of Schwarz [76] (see also [60], 10.3.5) is proved:

Proposition. *Let [A , A] be a maximal Banach ideal. If the associated tensornorm α of A is totally accessible or if E or F' has the approximation property, then*

$$A^*(E, F'') = (\mathcal{A}^{\min}(F, E))'.$$

Proof : The representation theorem for maximal ideals shows

$$\mathcal{A}^*(E,F'') \stackrel{1}{=} (E \otimes_{\alpha^t} F')' \stackrel{1}{=} (F'\tilde{\otimes}_\alpha E)'$$

(4.5 implies $\alpha^* = (\alpha^t)' \sim \mathcal{A}^*$) and corollary 7.1 of the representation theorem for minima ideals gives

$$F'\tilde{\otimes}_\alpha E = \mathcal{A}^{\min}(F,E),$$

hence

$$\mathcal{A}^*(E,F'') = (\mathcal{A}^{\min}(F,E))'.$$

The duality bracket can be calculated with the trace: Use 5.2 to see (first on elementary tensors) that for $T \in \mathcal{A}^*(E, F'')$

$$
\begin{array}{ccccccc}
\mathcal{A}^{\min}(F, E) = F'\tilde{\otimes}_\alpha E & \overset{id_{F'}\tilde{\otimes}T}{\longrightarrow} & F'\tilde{\otimes}_\pi F'' & \longrightarrow & \mathcal{N}(F', F') \\
\in & & \big\downarrow \mathrm{tr}_{F'} & & \in \\
S & \rightsquigarrow & & \rightsquigarrow & S' \circ T' \circ \kappa_{F'} \\
 & & \rightsquigarrow \langle T, S\rangle \in \mathbb{K} & &
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
\mathcal{A}^{\min}(F, E) = F'\tilde{\otimes}_\alpha E & \hookrightarrow & F'''\tilde{\otimes}_\alpha E & \overset{T'\tilde{\otimes}id_E}{\longrightarrow} & E'\tilde{\otimes}_\pi E & \longrightarrow & \mathcal{N}(E, E) \\
\in & & & & \big\downarrow \mathrm{tr}_E & & \in \\
S & \rightsquigarrow & & & & \rightsquigarrow & S^\pi \circ T \\
 & & & & \rightsquigarrow \langle T, S\rangle \in \mathbb{K} & &
\end{array}
$$

where $S^\pi : F'' \to E$ is the astriction of S''; it follows that

$$
\langle T, S\rangle = \begin{cases} \mathrm{tr}_{F'}(S' \circ T' \circ \kappa_{F'}) & \text{if } F' \text{ has a.p.} \\ \mathrm{tr}_E(S^\pi \circ T) & \text{if } E \text{ has a.p.} \end{cases}
$$

In the case of α being totally accessible, the duality bracket cannot always be calculated with the trace on *operators*: for an example, take $\alpha = \varepsilon$ whence $\mathcal{A}^* = \mathcal{I}$ and $\mathcal{A}^{\min} = \overline{\mathcal{F}}$ and E a reflexive space without the approximation property; then

$$
\mathcal{I}(E, E) = \mathcal{N}(E, E) = \mathcal{I}(c_0, E) \circ \overline{\mathcal{F}}(E, c_0) = \overline{\mathcal{F}}(\ell_1, E) \circ \mathcal{I}(E, \ell_1)
$$

so neither $S' \circ T'$ nor $T \circ S$ (for $T \in \mathcal{A}^*$ and $S \in \mathcal{A}^{\min}$) have in general a trace (see also 0.8).

7.3. The following trivial consequence of the representation theorem for minimal ideals sometimes is useful:

Take E and F Banach spaces, $\alpha \sim \mathcal{A}$ and $\beta \sim \mathcal{B}$, then

$$
\alpha \le c\beta \quad \text{on} \quad E' \otimes F
$$

implies

$$
\mathcal{B}^{\min}(E, F) \subset \mathcal{A}^{\min}(E, F), \qquad A^{\min}(T) \le cB^{\min}(T)
$$

As an application a «nuclear» version of Grothendieck's theorem 6.5 is given: Since $g_p \sim \mathcal{I}_p$ for $1 \leq p \leq \infty$, proposition 1.6 (and 1.7) and the representation theorem for minimal operator ideals imply that an operator $T \in \mathcal{L}(E, F)$ belongs to $\mathcal{I}_p^{\min}(E, F)$ if and only if it has a nuclear representation of the form

$$T = \sum_{i=1}^{\infty} x_i' \otimes y_i$$

such that $(\|x_i'\|) \in \ell_p$ (in c_0 if $p = \infty$) and $w_p(y_i) < \infty$. Moreover, in this case

$$I_p^{\min}(T) = \inf \ell_p(x_i') w_p(y_i)$$

where the infimum is taken over all possible representations. This proves that $(\mathcal{I}_p^{\min}, I_p^{\min})$ coincides isometrically with the Banach ideal $(\mathcal{N}_p, N_p)$ of all p-nuclear operators (see [60], 18.2.1).

Since $\pi \leq K_G g_\infty$ on $\ell_2^n \otimes \ell_1^m$ (see 1.12) the local technique lemma implies that for every $\mathcal{L}_{2,\lambda}^g$-space E' and $\mathcal{L}_{1,\mu}^g$-space F

$$\pi \leq K_G \lambda\mu g_\infty \leq K_G \lambda\mu g_p \qquad \text{on} \qquad E' \otimes F$$

and whence, by the above observation:

Proposition. *Let E be an $\mathcal{L}_{2,\lambda}^g$-space and F an $\mathcal{L}_{1,\mu}^g$-space, then for all* $1 \leq p \leq \infty$

$$\mathcal{N}(E, F) = \mathcal{N}_p(E, F)$$
$$N(T) \leq K_G \lambda\mu N_p(T)$$

See the results of 8.5, 10.2 and 10.3 in order to obtain other results of this type.

Note di Matematica Vol. VIII - n. 2, 247-262(1988)

8. PROJECTIVE AND INJECTIVE TENSORNORMS

8.1. A tensomorm α on *NORM* (or on *FIN)* is called *right-injective on NORM* (or *on* FIN), shorthand: (r) -injective, if for all metric injections $I : F \overset{1}{\hookrightarrow} G$

$$\mathrm{id}_E \otimes I : E \otimes_\alpha F \hookrightarrow E \otimes_\alpha G$$

is a metric injection *(E, F, G* $\in$ *NORM* or *FIN,* respectively) and *right-projective on NORM* (or *on FIN),* shorthand: (r)-projective, if for all metric surjections Q : $F \overset{1}{\twoheadrightarrow} G$

$$\mathrm{id}_E \otimes Q : E \otimes_\alpha F \to E \otimes_\alpha G$$

is a metric surjection $(E, F, G \in$ *NORM* or *FIN,* respectively). If α^t is (r)-injective (resp. (r) -projective), then α is called *left-injective* (resp. *left-projective*); if α is left- and right-injective (resp. projective) it is called *injective* (resp. *projective).* Clearly, ε is injective and π projective on *NORM* (this follows directly from the definitions, see 0.7). The duality

$$M \otimes_{\alpha'} N = (M' \otimes_\alpha N')' \qquad\qquad M, N \in FIN$$

implies: *α is (r) -injective on FIN ifand only if α' is (r)-projective on FIN.*

8.2. This result will be extended to tensomorms on *NORM* . Unfortunately, (r) -projective tensomorms are more difficult to treat for normed spaces than (r) -injective ones, so their study will be prepared by a precise investigation of their behaviour with respect to dense subspaces. For this, let β be a tensomorm on *NORM* x C , where C is either the class of all Banach - or of all normed spaces, and define for *(E, F)* $\in$ *NORM* x *NORM* and $z \in E \otimes F$ «the right-finite hull» (1)

$$\beta^{\rightarrow}(z; E, F) := \inf\{\beta(z; E, N) \mid N \in FIN(F),\ z \in E \otimes N\}.$$

Clearly, this is a tensomorm on *NORM* x *NORM* and $\beta \leq \beta^{\rightarrow}$.

Lemma.

(1) If β is (r) -projective on NORM x C, then $\beta = \beta^{\rightarrow}$ on NORM x C.

(2) If β is a tensorrwrm on NORM such that $\beta = \beta^{\rightarrow}$ on NORM x BAN, then $\beta = \beta^{\rightarrow}$ on NORM x NORM and

$$E \otimes_\beta F \overset{1}{\hookrightarrow} E \otimes_\beta \tilde{F}$$

(1) A similar «right-cofinite-hull» was used in 5.8.

for all $(E, F) \in NORM \times NORM$.

(3) *If β is a tensornorm on NORM, (τ)-projective on $NORM \times BAN$. then it is (τ)-projective on $NORM \times NORM$.*

Proof :

(1) If $G \in C$, then there is a metric surjection

$$Q : F \overset{1}{\twoheadrightarrow} G$$

such that F has the $(1+\varepsilon)$-approximation property for all $\varepsilon > 0$ (if G is complete take $F := \ell_1(B_G)$ and in the general case a dense subspace of $\ell_1(B_{\tilde{G}})$); then, for every normed space E

$$\beta(\cdot; E, F) = \beta^{\rightarrow}(\cdot; E, F)$$

by the approximation lemma. It follows that for $z \in E \otimes G$ there is an $N \in FIN(F)$ and a $\hat{z} \in E \otimes N$ with $id_E \otimes Q(\hat{z}) = z$ and

$$\beta(\hat{z}; E, N) \leq (1+\varepsilon)\beta(z; E, G)$$

and therefore

$$\beta(z; E, G) \leq \beta^{\rightarrow}(z; E, G) \leq \beta(z; E, QN) \leq \beta(\hat{z}; E, N) \leq$$
$$\leq (1+\varepsilon)\beta(z; E, G).$$

(2) Take $z \in E \otimes F$, then the metric mapping property gives

$$\beta^{\rightarrow}(z; E, \tilde{F}) \leq \beta^{\rightarrow}(z; E, F).$$

For $N \in FIN(F)$ with $z \in E \otimes N$ and

$$\beta(z; E, N) \leq (1+\varepsilon)\beta^{\rightarrow}(z; E, \tilde{F})$$

choose an operator $R : N \to F$ with $\|R\| \leq 1+\varepsilon$ and $Ry = y$ whenever $y \in N \cap F$ *(the* existence of R will be shown in a moment). Then

$$z \in (E \otimes F) \cap (E \otimes N) \subset E \otimes RN \qquad \text{and} \qquad id_E \otimes R(z) = z$$

whence

$$\beta^{\rightarrow}(z; E, F) \leq \beta(z; E, RN) = \beta(id_E \otimes R(z); E, RN) \leq$$
$$\leq \|R\|\beta(z; E, N) \leq (1+\varepsilon)^2\beta^{\rightarrow}(z; E, \tilde{F})$$

which proves (2). For the existence of R take a projection $Q : \tilde{F} \to N \cap F$; since

$$\mathcal{L}(N,F) = N' \otimes_\varepsilon F \hookrightarrow N' \otimes_\varepsilon \tilde{F} = \mathcal{L}(N,\tilde{F})$$

is dense there is an $R_0 \in \mathcal{L}(N, F)$ with

$$\|I_N^{\tilde{F}} - R_0\| \leq \varepsilon(2\|Q\|)^{-1}.$$

Now $R := R_0 + (I_N^{\tilde{F}} - R_0)Q|_N$ has the desired properties.

(3) To see this look at the following result: Let U and V be normed spaces, $P \in \mathcal{L}(U, V)$ surjective, $U_0 \subset U$ dense and

$$P_0 := P|_{U_0} : U_0 \to V_0 := P(U_0).$$

Then P_0 is a metric surjection if and only if $\ker P = \overline{\ker P_0}^U$ and P is a metric surjection.

This is perhaps not very well-known (see [78]); a proof follows from

(a) If P_0 is a metric surjection, then $P'(V') = P_0'(V_0')$ is $\sigma(U', U_0)$-closed, whence

$$\overline{\ker P_0}^{\sigma(U,U')} = ((\ker P_0)^0)^0 = (P'(V'))^0 = \ker P.$$

(b) If $x \in U$, then

$$\inf\{\|x+z\| \mid z \in \ker P_0\} = \inf\{\|x+z\| \mid z \in \overline{\ker P_0}\}$$

Coming back to statement (3) take for normed spaces F and G a metric surjection $Q : F \overset{1}{\twoheadrightarrow} G$. Then $\tilde{Q} : \tilde{F} \to \tilde{G}$ is a metric surjection, $\ker \tilde{Q} = \ker Q$ and

$$\mathrm{id}_E \otimes \tilde{Q} : E \otimes_\beta \tilde{F} \to E \otimes_\beta \tilde{G}$$

is a metric surjection as well. Since, by (1) and (2)

$$E \otimes_\beta F \overset{1}{\hookrightarrow} E \otimes_\beta \tilde{F} \qquad \text{and} \qquad E \otimes_\beta G \overset{1}{\hookrightarrow} E \otimes_\beta \tilde{G}$$

are dense subspaces, the mapping

$$\mathrm{id}_E \otimes Q : E \otimes_\beta F \to E \otimes_\beta G$$

is a metric surjection (by the above result) if

$$\ker(\mathrm{id}_E \otimes \tilde{Q}) = E \otimes \ker \tilde{Q} \overset{!}{\subset} \overline{\ker(\mathrm{id}_E \otimes Q)}^{E \otimes_\beta \tilde{F}}$$

which is obvious by $\ker \tilde{Q} = \ker Q$. ∎

This lemma allows to restrict the attention to Banach spaces when investigating projective tensornorms.

8.3. Now the announced duality between (τ) -injective and (r) -projective tensomorms can be proved. At the same time, and this is somehow natural, a first observation on accessibility of these tensomorms is made (a more careful investigation will be made in $9).

Proposition. *Let* α *be tensornorm on NORM.*

(1) *If* α *is* (τ) *-injective on FIN, then* $\overleftarrow{\alpha}$ *and* $\overrightarrow{\alpha}$ *are (r) -injective on NORM.*

(2) If α *is (r) -projective on FIN, then* $\overrightarrow{\alpha}$ *is (T) -projective on NORM.*

(3) If α *is finitely or cofinitely generated, then:* α *is (T) -injective on NORM if and only if* α' *is* (τ) *-projective* on $NORM$.

(4) If α *is (T) -injective or (T) -projective on FIN, then* α *is (T) -accessible.*

Proof

(1) and (4): If α is (r)-injective on *FIN,* then for $F \overset{1}{\hookrightarrow} G$ and $z \in E \otimes F$

$$\begin{aligned} \overrightarrow{\alpha}(z;\ E,G) &\leq \overrightarrow{\alpha}(z;\ E,F) = \\ &= \inf\{\alpha(z;\ M, N \cap F) \mid M \in FIN(E),\ N \in FIN(G),\ z \in M \otimes N\} = \\ &= \inf\{\alpha(z;\ M,N) \mid \ldots\} = \overrightarrow{\alpha}(z;\ E,G); \end{aligned}$$

so $\overrightarrow{\alpha}$ is (T) -injective. To treat the cofinite hull, first (4) will be shown: For this take (N, *F)* $\in$ *FIN* x *NORM* and z $\in$ N $\otimes$ *F* and assume α being (r)-injective on *FIN.* Then, by what was already shown and the approximation lemma, it follows

$$\begin{aligned} \overrightarrow{\alpha}(z;\ N,F) = \overrightarrow{\alpha}(z;\ N,\ell_\infty(B_{F'})) = \overleftarrow{\alpha}(z;\ N,\ell_\infty(B_{F'})) &\leq \\ \leq \overleftarrow{\alpha}(z;\ N,F) \end{aligned}$$

whence α is (τ)-accessible. Now remembcr that α is (τ)-accessible if α' is (see 3.6): Whence, if α is (τ)-projective on *FIN,* the dual α' is (τ)-injective on *FIN,* whence α' is (T) -accessible and so is α.

Now it is possible to show that $\overleftarrow{\alpha}$ is (τ)-injective on *NORM* if α is on FIN: For $F \overset{1}{\hookrightarrow} G$ and z $\in$ *E* $\otimes$ *F* the following holds by the two results which were already shown:

$$\begin{aligned} \overleftarrow{\alpha}(z;\ E,\ F) &= \sup\{\overleftarrow{\alpha}(Q_K^E \otimes \mathrm{id}_F(z);\ E/K,\ F) \mid K \in COFIN(E)\} = \\ &= \sup\{\overrightarrow{\alpha}(Q_K^E \otimes \mathrm{id}_F(z);\ E/K,\ F) \mid K \in COFIN(E)\} = \\ &= \sup\{\overrightarrow{\alpha}(Q_K^E \otimes \mathrm{id}_G(z);\ E/K,\ G) \mid K \in COFIN(E)\} = \\ &= \sup\{\overleftarrow{\alpha}(Q_K^E \otimes \mathrm{id}_G(z);\ E/K,\ G) \mid K \in COFIN(E)\} = \\ &= \overleftarrow{\alpha}(z;\ E,G). \end{aligned}$$

(2) Using lemma 8.2 (3) it is enough to consider a metric surjection Q : F $\to$ G between Banach spaces. By (4) the tensomorm α' is (r)-accessible, whence for every N $\in$ FIN the result (1) implies

$$(N \otimes_{\overrightarrow{\alpha}} G)' = N' \otimes_{\alpha'} G' \overset{1}{\hookrightarrow} N' \otimes_{\alpha'} F' = (N \otimes_{\overrightarrow{\alpha}} F)'$$

and therefore

$$N \otimes_{\overrightarrow{\alpha}} F \to N \otimes_{\overrightarrow{\alpha}} G$$

is a metric surjection. Now take E an arbitrary normed space:

$$\begin{aligned}\overrightarrow{\alpha}(z; E, G) &= \inf\{\overrightarrow{\alpha}(z; N, G) \mid N \in FIN(E), z \in N \otimes G\} = \\ &= \inf\{\overrightarrow{\alpha}(w; N, F) \mid N \in FIN(E), \mathrm{id}_N \otimes Q(w) = z\} = \\ &= \inf\{\overrightarrow{\alpha}(w; E, F) \mid \mathrm{id}_E \otimes Q(w) = z\}.\end{aligned}$$

The last statement (3) follows from (1) and (2). .

It is *not true* that the cofinite hull $\overleftarrow{\alpha}$ is right-projective on BAN if α is right-projective on FIN; to see an example take $\alpha = \pi$ and $\ell_1(B_F) \overset{1}{\twoheadrightarrow} F$ fora Banach-space F without the metric approximation property, then

$$F' \otimes_{\overleftarrow{\pi}} \ell_1(B_F) = F' \otimes_{\pi} \ell_1(B_F) \overset{1}{\twoheadrightarrow} F' \otimes_{\pi} F \neq F' \otimes_{\overleftarrow{\pi}} F.$$

Since there is no Hahn-Banach-theorem for operators, π is neither (r)- nor (ℓ)-projective; see also 8.15.

8.4. For the $\alpha_{p\,q}$-tensornorms the following result holds:

Proposition. *Let* $1 \le p \le \infty$. *Then*

(1) d_p *is* (r) *-projective und, consequently,* g_p *is* (ℓ) *-projective and* $g_p^* = d_p'$ (r) *-injective.*

(2) $\alpha_{2,p}$ *is (r) -injective,* $\alpha_{p,2}$ (ℓ) *-injective and* $\alpha'_{2,p}$ (r) *-projective. In particular:* w_2 *is injective and* $w_2^* = w_2'$ *projective.*

Proof . Since

$$d_p(z; E, F) = \inf\{w_{p'}(x_i)\ell_p(y_i) \mid z = \sum x_i \otimes y_i\}$$

the result (1) follows directly from the following observation: If Q : $F \twoheadrightarrow G$ is a metric surjection, $\varepsilon > 0$ and $y_1, \ldots, y_n \in$ G, then there are $\hat{y}_i \in F$ with Q($\hat{y}_i$) = y_i and

$$\ell_p(y_i) \le \ell_p(\hat{y}_i) \le (1 + \varepsilon)\ell_p(y_i)$$

TO see that $\alpha_{2,p}$ is (r) -injective, take an isometric injection F c-1 G, an element z $\in E \otimes F$ and $\varepsilon > 0$: Choose a representation in $E \otimes G$ of z with

$$\ell_r(\lambda_i) w_{p'}(x_i) w_2(y_i) \leq (1+\varepsilon)\alpha_{2,p}(z;\ E, G)$$

then the associated operator $T_z : E' \to F$ has an obvious factorization

$$\begin{array}{ccccccccc}
\varphi & \in & E' & \overset{T_z}{\to} & F & \hookrightarrow & G & \ni & \sum \xi_i y_i \\
\wr & & \downarrow R & & & \nearrow S & & & \uparrow \wr \\
(\langle \varphi, \ x_i \rangle) & \in & \ell_{p'}^n & \underset{D_\lambda}{\to} & \ell_n^2 & & & \ni & (\xi_i)
\end{array}$$

(D_λ the diagonal operator associated with (X,)). Then

$$\|R\| = w_{p'}(x_i) \qquad \text{and} \qquad \|S\| = w_2(y_i).$$

If P is the orthogonal projection $\ell_2^n \to H$:= S-'(F) and $S_0 : H \to F$ the astriction of SI,, then $D_\lambda R(E') \subset H$ implies $T_z = S_0 P D_\lambda R$. This means

$$z = \sum \lambda_i x_i \otimes S_0 Pe_i \in E \otimes F$$

and therefore

$$\begin{aligned}
\alpha_{2,p}(z;\ E, F) &\leq \ell_r(\lambda_i) w_{p'}(x_i) w_2(S_0 Pe_i) \leq \\
&\leq \ell_r(\lambda_i) w_{p'}(x_i) \|S_0\|\, \|P\| w_2(e_i) \leq \\
&\leq (1+\varepsilon)\alpha_{2,p}(z;\ E, G).
\end{aligned}$$

The other statements in (2) follow easily by transposition and dualization. .

8.5. There is a nice application of the fact that d_2 is (r) -projective. Grothendieck's inequality 1.11 implies (see 6.4) that

$$d_2 \leq \pi \leq K_G w_2 \leq K_G d_2 \qquad \text{on} \qquad \ell_\infty^m \otimes F'$$

whenever $F = L_1(\nu)$. An old result of Kadec (see [59], p. 272 and [60], 21.1.3) says that for every $1 \leq$ p ≤ 2 and $n \in$ N there is an isometric embedding

$$\ell_p^n \overset{1}{\hookrightarrow} L_1(\nu)$$

for some finite measure ν; dualizing this, the fact that π and d_2 are (r) -projective implies that

$$d_2 \leq \pi \leq K_G d_2 \qquad \text{on} \qquad \ell_\infty^m \otimes \ell_{p'}^n$$

and whence, by the loca1 technique lemma 6.2 for $\mathcal{L}_p^g$-spaces,

$$d_2 \leq \pi \leq \lambda\mu K_G d_2 \qquad \text{on} \qquad E \otimes F'$$

whenever *E is an* $\mathcal{L}_{\infty,\lambda}^g$*-space and F an* $\mathcal{L}_{p,\mu}^g$*-space* (with $1 \leq$ p ≤ 2). Since $\mathcal{P}_2 \sim g_2^* = d_2'$ and $\mathcal{L} \sim \varepsilon$ the transfer argument 4.10 gives Grothendieck's well-known [51]

If E is an $\mathcal{L}^g_{\infty,\lambda}$*-space and F an* $\mathcal{L}^g_{p\,\mu}$*-space (for* $1 \le p \le 2$*), then*

$$\mathcal{L}(E,F) = \mathcal{P}_2(E,F) \quad \text{and} \qquad P_2(T) \le K_G \lambda\mu ||T||.$$

Clearly this result can also easily be deduced from the case p = 1 using Kadec's result and loca1 techniques for operators.

8.6. Every tensomorm α is less than or equal to π and π is projective. Whence it is reasonable to search fora closest tensomorm $\beta \ge \alpha$ which is projective.

Theorem. *Let* α *be a tensornorm on NORM. Then there is a unique* (τ)*-projective tensornorm* $\alpha/ \ge \alpha$ *on NORM with the following property: If* $\beta \ge \alpha$ *is (r) -projective, then* $\beta \ge \alpha/$.

The *right-projective associate* $\alpha/$ of α can be calculated using the following property:

If E is normed and F a Banach space, then

$$E \otimes_\alpha \ell_1(B_F) \overset{1}{\twoheadrightarrow} E \otimes_{\alpha/} F$$

is a metric surjection. If E and F are arbitrary normed spaces and $z \in E \otimes F$*, then*

$$\alpha/(z;\ E,F) = \inf\{\alpha/(z;\ E,N) \,|\, N \in FIN(F), z \in E \otimes N\}. \qquad (\star)$$

The symbol $\alpha/$ comes from the fact that $\alpha/$ respects quotient mappings $F \overset{1}{\twoheadrightarrow} F/G$.

Proof . Uniqueness is clear if it exists. $\alpha/$ will be constructed first on *NORM* x *BAN* and then extended, using the introductory lemma 8.2.

(a) If $(E,F) \in$ *NORM* x *BAN*, define $\alpha/$ to be the quotient seminorm on $E \otimes F$ given by the mapping

$$E \otimes_\alpha \ell_1(B_F) \to E \otimes F.$$

Using the lifting property of the space $\ell_1(\Gamma)$:

$$\begin{array}{ccc} \ell_1(B_{F_1}) & \overset{\hat{T}}{\dashrightarrow} & \ell_1(B_{F_2}) \\ \downarrow & \circlearrowright & \downarrow \\ F_1 & \overset{T}{\longrightarrow} & F_2 \end{array} \qquad ||\hat{T}|| \le (1+\varepsilon)||T||$$

and the test 1.1 it is easy to see that $\alpha/$ is a tensornorm on *NORM x BAN*.

(b) If $Q : F \to G$ is a metric surjection between Banach spaces, the same lifting property gives

$$\begin{array}{ccc} \ell_1(B_F) & \overset{\hat{Q}}{\longleftarrow} & \ell_1(B_G) \\ {\scriptstyle 1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle 1} \\ F & \underset{Q}{\overset{1}{\twoheadrightarrow}} & G \end{array} \qquad \|\hat{Q}\| \leq 1 + \varepsilon$$

and this implies easily that

$$\mathrm{id}_E \otimes Q : E \otimes_{\alpha/} F \to E \otimes_{\alpha/} G$$

is a metric surjection for all normed spaces E. Lemma 8.2 now implies

$$\alpha/ = \alpha/^{\rightarrow} \quad \text{on} \quad NORM \times BAN.$$

(c) This means that

$$\alpha/ := \alpha/^{\rightarrow} \quad \text{on} \quad NORM \times NORM$$

is an extension of the tensomorm $\alpha/$ to $NORM \times NORM$. Lemma 8.2 shows that $\alpha/$ is (r)-projective and $\alpha \leq \alpha/$ since, by definition, $\alpha \leq \alpha/$ on $NORM \times FIN$.

(d) If $\alpha \leq \beta$, then, again by the very definitions, $\alpha/ \leq \beta/$. If β is (r)-projective, then $\beta = \beta^{\rightarrow}$ by lemma 8.2 and therefore $\beta = \beta/$. These two observations show that $\alpha/$ has the universal property stated in the theorem. ∎

A lifting argument as in (b) shows the

Corollary 1. *If E is a normed space, then*

$$\alpha(\cdot; E, \ell_1(\Gamma)) = \alpha/(\cdot; E, \ell_1(\Gamma))$$

for all sets Γ.

Remember that by a result of Grothendieck's [26] all spaces with the lifting property (as it was used) are isometric to some $\ell_1(\Gamma)$. Köthe [44] showed that spaces with the lifting property (without norm-restriction) are isomorphic to some $\ell_1(\Gamma)$. Clearly,

$$\backslash\alpha := ((\alpha^t)/)^t$$

is called the *left-projective associare* of α.

Corollary 2. *Let α be a tensornorm. Then*

$$\backslash(\alpha/) = (\backslash\alpha)/ =: \backslash\alpha/$$

is called the projective associate of α; it is the unique smallest projective tensornorm $\geq \alpha$, is finitely generated and

$$\ell_1(B_E) \otimes_\alpha \ell_1(B_F) \overset{1}{\twoheadrightarrow} E \otimes_{\backslash\alpha/} F$$

is a metric surjection if E and F are Banach spaces.

The proof follows easily from the «transitivity of metric surjections» and the theorem.

8.7. Fortunately, the injective case is simpler.

Theorem. *Let α be a tensornorm on NORM. Then there is a unique (r) -injective tensornorm $\alpha\backslash \leq \alpha$ on NORM such that $\beta \leq \alpha\backslash$ for all (P) -injective tensornorms $\beta \leq \alpha$. For all normed spaces E, F*

$$E \otimes_{\alpha\backslash} F \overset{1}{\hookrightarrow} E \otimes_\alpha \ell_\infty(B_{F'}) \qquad (\star)$$

is a metric injection.

$\alpha\backslash$ is called the *right-injective associate* of α.

Proof . Define $\alpha\backslash$ on $E \otimes F$ to be the subspace norm of

$$E \otimes F \hookrightarrow E \otimes_\alpha \ell_\infty(B_{F'}) .$$

Since all $\ell_\infty(\Gamma)$ have the 1-extension-property

$$\begin{array}{ccc} G & & \\ \cup & \searrow^{\hat{T}} & \\ F & \overset{T}{\to} & \ell_\infty(\Gamma) \end{array} \qquad \|\hat{T}\| \leq \|T\|$$

test 1.1 gives easily that $\alpha\backslash$ is a tensornorm on $NORM$ - as well as that $\alpha\backslash$ is (τ)-injective. The definition implies immediately that $\beta \leq \alpha\backslash$ if $\beta \leq \alpha$ is (r) -injective. .

As in the projective case:

$$/\alpha := ((\alpha^t)\backslash)^t$$

is the *left-injective associate* of α and

$$/\alpha\backslash := (/\alpha)\backslash = /(\alpha\backslash)$$

is *the injective associate* which is the unique largest injective tensomorms smaller than α. I follows:

$$E \otimes_{/\alpha\backslash} F \overset{1}{\hookrightarrow} \ell_\infty(B_{E'}) \otimes_\alpha \ell_\infty(B_{F'}) .$$

Note that injective tensomorms are clearly finitely generated.

Corollary. *If the Banach space F has the X-extension-property, then*

$$\alpha\backslash \leq \alpha \leq \lambda\alpha\backslash \qquad on \qquad E \otimes F$$

for all normed spaces E.

8.8. The following is clear by what has been already shown:

Proposition. *For every tensornorm α, normed space E and $n \in N$*

$$E \otimes_\alpha \ell_1^n = E \otimes_{\alpha/} \ell_1^n \qquad isometrically$$
$$E \otimes_\alpha \ell_\infty^n = E \otimes_{\alpha\backslash} \ell_\infty^n \qquad isometrically$$

Now the local technique lemma 6.2 for $\mathcal{L}_p^g$-spaces will be applied to give the

Corollary. *Let α be a tensornorm and E a normed space.*
(1) *If F is an $\mathcal{L}_{1,\lambda}^g$-space, then*

$$\alpha \leq \alpha/ \leq \lambda\alpha^{\rightarrow} \qquad on \qquad E \otimes F.$$

Note that $\alpha^{\rightarrow} \leq \mu\alpha$ on $E \otimes F$ if F has the μ-approximation property (by the approximation lemma) and $\alpha = \alpha^{\rightarrow}$ if α is finitely generated.

(2) F is an $\mathcal{L}_{\infty\lambda}^g$-space, then

$$\alpha\backslash \leq \alpha \leq \lambda\alpha\backslash \qquad \text{on} \qquad E \otimes F.$$

Proof . The proof of the local technique lemma actually gave $\alpha^{\rightarrow} \leq c\beta^{\rightarrow}$ instead of $\overrightarrow{\alpha} \leq c\,\overrightarrow{\beta}$ as it was stated. Now (1) is immediate and (2) follows from $\alpha\backslash = \alpha\backslash^{\rightarrow}$. .

8.9. This result helps to state a simple test for recognizing whether a tensomorm β is the projective/injective associate of α:

Proposition. *Let α and β be tensornorms.*
(1) *If β is (r) -projective, then the following are equivalent:*
(a) $\beta = \alpha/$
(b) *For all $E \in NORM$ and $n \in N$*

$$E \otimes_\beta \ell_1^n = E \otimes_\alpha \ell_1^n \qquad isometrically$$

(2) If β is (τ) -injective, then the following are equivalent:

(a) $\beta = \alpha\backslash$

(b) *For all $E \in NORM$ and $n \in N$*

$$E \otimes_\beta \ell_\infty^n = E \otimes_\alpha \ell_\infty^n \qquad \textit{isometrically}$$

(3) If α and β are finitely generated, then it is enough in both cases to test only for finite-dimensiona1 E.

Proof . Assume (1) (b), then (again by the proof of the loca1 technique lemma) $\beta^{\rightarrow} = \alpha^{\rightarrow}$ on all $E \otimes \ell_1(\Gamma)$ and whence $\beta = \alpha$ on all $E \otimes \ell_1(\Gamma)$ by the approximation lemma: the properties $(\star)$ in theorem 8.6 give (a); the reverse implication follows from the last proposition. (2) can be shown the same way and (3) is obvious. ∎

Clearly, it would be enough in (3) that α and β are finitely generated on the left side. Note that the result (together with 8.3) implies in particular that $\alpha/$ and $\alpha\backslash$ are finitely generated if CY is finitely generated.

The same arguments give:

Let α and β be finitely generated tensornorms.

(4) If β isprojective, then $\beta = \backslash\alpha/$ ifand only if for all $n \in N$

$$\ell_1^n \otimes_\beta \ell_1^n = \ell_1^n \otimes_\beta \ell_1^n \qquad \textit{isometrically}$$

(5) If β is injective, then $\beta = /\alpha\backslash$ ifand only iffor all $n \in \mathbb{N}$

$$\ell_\infty^n \otimes_\beta \ell_\infty^n = \ell_\infty^n \otimes_\alpha \ell_\infty^n \qquad \textit{isometrically}$$

8.10. The following formulas contain many of the phenoma conceming projective/injective associates and finite/cofinite hulls; they create a type of «calculus» which will be helpfull when dealing with accessibility:

Proposition. *Let α be a tensornormorm on NORM.*

(1) $(\overrightarrow{\alpha})\backslash = \overrightarrow{\alpha\backslash}$ *and* $(\overrightarrow{\alpha})/ = \overrightarrow{\alpha/}$.

(2) $(\overleftarrow{\alpha})\backslash = \overleftarrow{\alpha\backslash}$ *but in general* $(\overleftarrow{\alpha})/ \neq \overleftarrow{\alpha/}$.

(3) $(\alpha/)' = (\alpha')\backslash$ *and* $(\alpha\backslash)' = (\alpha')/$.

(4) $(\alpha/)^* = /\alpha^*$ *and* $(\alpha\backslash)^* = \backslash\alpha^*$.

Proof :

(1) By 8.8 it follows that

$$\overrightarrow{\alpha} = \alpha = \alpha\backslash = \overrightarrow{\alpha}\backslash \quad \text{on} \quad N \otimes \ell_\infty^n$$

Since $\beta := \overrightarrow{\alpha\backslash}$ is (r) -injective by proposition 8.3 the test gives

$$\overrightarrow{\alpha\backslash} = \overrightarrow{\alpha}\backslash$$

The same for the (r) -projective associate.

(3) and (4) follow again from the test, since α' and $(\alpha/)'$ are finitely generated and clearly

$$N \otimes_{\alpha'} \ell_\infty^n = (N' \otimes_\alpha \ell_1^n)' = (N' \otimes_{\alpha/} \ell_1^n)' = N \otimes_{(\alpha/)'} \ell_\infty^n$$

whence $(\alpha')\backslash = (\alpha\backslash)'$ which implies all formulas in (3) and (4).

(2) Note first that $\overleftarrow{\alpha\backslash}$ is (r)-injective by proposition 8.3. Since, by (3) and 8.8.

$$E' \otimes_{\alpha'} \ell_1(B_{F'}) = E' \otimes_{(\alpha\backslash)'} \ell_1(B_{F'})$$

and, by the duality theorem 3.4,

$$E \otimes_{(\overleftarrow{\alpha})\backslash} F \overset{1}{\hookrightarrow} E \otimes_{\overleftarrow{\alpha}} \ell_\infty(B_{F'}) \overset{1}{\hookrightarrow} (E' \otimes_{\alpha'} \ell_1(B_{F'}))'$$

$$E \otimes_{\overleftarrow{\alpha\backslash}} F \overset{1}{\hookrightarrow} E \otimes_{\overleftarrow{\alpha\backslash}} \ell_\infty(B_{F'}) \overset{1}{\hookrightarrow} (E' \otimes_{(\alpha\backslash)'} \ell_1(B_{F'}))'$$

one obtains $(\overleftarrow{\alpha}) = \overleftarrow{\alpha\backslash}$. The related formula for the (r) -projective associate is not true, since – as it was already seen in 8.3 –

$$(\overleftarrow{\pi})/ = \pi \neq \overleftarrow{\pi} = \overleftarrow{\pi/}.$$ ∎

8.11. Let α bea finitely generated tensomorm and $(\mathcal{A}, A)$ the associated maximal Banach operator ideal. Take $(\mathcal{B}, B) \sim \alpha\backslash$ and $T \in \mathcal{L}(E, F)$. Since $(\ell_\infty(B_{F'}))'$ is an $\mathcal{L}^g_{1,1}$-space corollary 8.8 implies

$$E \otimes_{\alpha'} (\ell_\infty(B_{F'}))' = E \otimes_{\alpha/} (\ell_\infty(B_{F'}))'$$

and whence, by the representation theorem for maximal operator ideals

$$\begin{array}{ccccc} (E \otimes_{(\alpha\backslash)'} F')' & = & (E \otimes_{\alpha/} F')' & \overset{1}{\underset{(\mathrm{id}_E \otimes I')'}{\hookrightarrow}} & (E \otimes_{\alpha'} (\ell_\infty(B_{F'}))')' \\ \uparrow 1 & & ∕. & & \uparrow 1 \\ \mathcal{B}(E,F) \ni T & & - - & & I \circ T \in \mathcal{A}(E, \ell_\infty(B_{F'})) \end{array}$$

whence $T \in \mathcal{B}$ iff $I \circ T \in \mathcal{A}$ (with equal norms). This shows that $(\mathcal{B}, B) = (\mathcal{A}^{inj}, A^{inj})$ is the injective hull of $\mathcal{A}$ in the sense of Pietsch (note that it was shown that $\mathcal{A}^{inj}$ is maximal, if $\mathcal{A}$ is). This was the first part of the

Proposition. *Let $\alpha \sim (\mathbf{d}, A)$ be associated.*

(1) $\alpha\backslash \sim (\mathcal{A}^{\text{inj}}, A^{\text{inj}})$. *In particular: the tensornorm α is (r)-injective if und only if the operutor ideal* $(\mathbf{d}, A)$ *is injective.*

(2) $/\alpha \sim (\mathcal{A}^{\text{surj}}, A^{\text{surj}})$. *In particular: the tensornorm α is (ℓ)-injective if und only if the operutor ideal* (d, A) *is surjective.*

Proof of (2). This is along the same lines as the (r) -injective case: Take $\mathcal{B} \sim /\alpha$, then

$$\begin{array}{ccccc} (E \otimes_{(/\alpha)'} F')' & = & (E \otimes_{\backslash(\alpha')} F')' & \overset{1}{\underset{(Q\otimes \mathrm{id}_{F'})'}{\hookrightarrow}} & (\ell^1(B_E) \otimes_{\alpha'} F')' \\ \updownarrow 1 & & \circlearrowleft & & \updownarrow 1 \\ \mathcal{B}(E, F) \ni T & & \rightsquigarrow & & T \circ Q \in \mathcal{A}(\ell_1(B_E), F) \end{array}$$

which shows that the operator idea1 $\mathcal{B}$ coincides isometrically with the ideal $(\mathcal{A}^{\text{surj}}, A^{\text{surj}})$ in the sense of Pietsch. .

TO see just one consequence of these relationships:

Corollary. If $(\mathcal{A}, A)$ *is a maximal normed operutor ideal, then*

$$(\mathcal{A}^{\text{dual}})^{\text{inj}} = (\mathcal{A}^{\text{surj}})^{\text{dual}}$$

(with equal natura1 norms).

Proof. This is just $(\alpha^t)\backslash = (/\alpha)^t$. .

8.12. The projective associates of α give factorization theorems for the operator ideals. Using Kakutani's representation theorem for abstract L- and M-spaces and, clearly as before the representation theorem of maximal operator ideals, it follows

Proposition. *Let $\alpha \sim (\mathbf{d}, A)$ be associated und denote by $(\mathcal{A}/, A/)$ und $(\backslash\mathcal{A}, \backslash A)$ the operutor ideals associated with $\alpha/$ und $\backslash\alpha$, respectively.*

(1) $T \in \mathcal{A}/(E, F)$ *if and only if there exists a strictly localizable measure μ, operators $R \in \mathcal{A}$ and $S \in \mathcal{L}$ such that*

$$\begin{array}{ccccc} E & \overset{T}{\to} & F & \overset{1}{\hookrightarrow} & F'' \\ & R\searrow & \circlearrowleft & \nearrow S & \\ & & L_1(\mu) & & \end{array}$$

In this case:

$$A/(T) = \min A(R)\|S\|$$

and the minimum is attained with a metric surjection

$$S : L_1(\mu) \overset{1}{\twoheadrightarrow} F''.$$

(2) $T \in$ \d($E, F)$ *if and only if there is a compact space* K, *operators* $R \in \mathcal{L}$ *and* $S \in$ **d** *such that*

$$\begin{array}{ccccc} E & \overset{T}{\to} & F & \hookrightarrow & F'' \\ & {}_R\searrow & \not\!\cdot & \nearrow_S & \\ & & C(K) & & \end{array}$$

In this case:

$$\backslash A(T) = \min \|R\| A(S)$$

and the minimum is attained with a metric injection R.

The details of the easy proof (which is of the same type as the one of proposition 8.11) are left to the reader.

8.13. Since w_2 is injective by 8.4 the fundamental theorem 1.11 of the metric theory:

$$w_2 \le \pi \le K_G w_2 \qquad \text{on} \qquad \ell_\infty^n \otimes \ell_\infty^n$$

is, by the finite-dimensionai test 8.9 (5), just the

Theorem:

$$w_2 \le /\pi\backslash \le K_G w_2$$
$$\backslash\varepsilon/ \le w_2' = w_2^* \le K_G \backslash\varepsilon/$$

Since $\pi \sim \mathcal{I}$ the integral operators, $w_2 \sim \mathcal{L}_2$ the operators that factor through a Hilbert space (see 4.6)

$$\begin{array}{ccc} E & \overset{T}{\to} & F \\ \searrow & & \nearrow \\ & H & \end{array},$$

$(/\pi)\backslash = /(\pi\backslash) = /\pi\backslash$ and $\mathcal{I}^{inj} = \mathcal{P}_1$ (by the factorization theorems), the results of 8.11 give the

Corollary (Grothendieck's inequality in operator form):

$$(\mathcal{P}_1)^{surj} = (\mathcal{I}^{surj})^{inj} = \mathcal{L}_2$$
$$L_2(T) \le P_1^{surj}(T) = (I^{surj})^{inj}(T) \le K_G L_2(T).$$

Clearly, this implies

$$\mathcal{P}_1(\ell_1, F) = \mathcal{L}_2(\ell_1, F)$$

for all Banach spaces, and the well-known (see 6.5)

$$\mathcal{P}_1(\ell_1, \ell_2) = \mathcal{L}(\ell_1, \ell_2).$$

This latter formula (nowadays called: Grothendieck's theorem) implies (by simple factorization arguments) the corollary, which is nothing else than the theorem, i.e. the fundamental theorem of the metric theory/Grothendieck's inequality.

8.14. The following result about associates of $\alpha_{p\,q}$ will be very useful.

Proposition. *Let* $1 \leq p \leq \infty$, *then*

(1) $g_p\backslash = g_{p'}^* = d_{p'}'$

(2) $\backslash g_p^* = g_{p'}$ *and* $d_p^*/ = d_{p'}$

(3) $\backslash(g_p\backslash) = g_p$ *and* $(/d_p)/ = d_p$

(4) $\pi\backslash = g_\infty^* = w_\infty^* = w_1' = d_\infty'$ *and* $\varepsilon/ = d_\infty = w_1$

(5) $g_2^* = g_2$ *and* $d_2^* = d_2$.

Proof . (2) – (4) follow from (1) just by calculating with proposition 8.10. The fact that $g_2 = \alpha_{2\,,1}$ is (r)-injective (see 8.4) shows that (1) also implies (5).

To see (1) take first $p = \infty$, then, by 1.9,

$$g_\infty = w_\infty = \varepsilon \qquad \text{on} \qquad N \otimes \ell_\infty^n$$

therefore the test 8.9 implies $g_\infty\backslash = \varepsilon = \pi^* = g_1^*$.

The cases $1 \leq p < \infty$ follow from the fact that by the factorization theorems 4.6 and 4.8 for the p-integral ($\sim$ gr,) and absolutely-p-summing ($\sim g_{p'}^*$) operators

$$\mathcal{I}_p^{\text{inj}} = \mathcal{P}_p \qquad \text{isometrically}$$

and whence $g_p\backslash = g_{p'}^*$, since $\mathcal{I}_p^{\text{inj}} \sim g_p\backslash$ by 8. 11.

These formulas contain information about the structure of Banach-spaces. Take, for example, $\pi\backslash = w_1'$: The characterization of the $\mathcal{L}_1^g$ -spaces (these are the $\mathcal{L}_1$ -spaces, 6.3) in 6.1 and the description $(\star)$ of $\pi\backslash$ in 8.7 give the

Corollary 1. *A Banach space E is an $\mathcal{L}_1$-space if and only if $E \otimes_\pi \cdot$ respects subspaces isomorphically.*

This is a result of Stegall-Retherford [77] (see also [15]; the corresponding isometric result was mentioned in 1.1). The Hahn-Banach-theorem applied to

$$\mathcal{L}(\cdot; E') = (\cdot \otimes_\pi E)'$$

shows, that *dual $\mathcal{L}_\infty$-spaces (= dual $\mathcal{L}_\infty^g$-spaces) have the extension property.*

The formula $\varepsilon/ = w_1$ implies in rathcr the same way

Corollary 2. *A Banach space E is an $\mathcal{L}_\infty$-space if and only if $E \otimes_\varepsilon$ respects quotients isomorphically.*

This contains Kaballo's characterization [41] of (EL) -spaces, i.e. those Banach spaces E such that $E\tilde{\otimes}_\varepsilon \cdot$ respects quotients isomorphically: To see this, note first that $E \otimes_\varepsilon \cdot$ respecting quotients implies that $E\tilde{\otimes}_\varepsilon \cdot$ does; if, corfversely, E is an (EL) -space, a simple argument by contradiction shows, that there is a $\lambda \geq 1$ such that for all $Q : M \overset{1}{\twoheadrightarrow} N$ between finite-dimensional spaces and for every $z \in E \otimes_\varepsilon N$ there is an $u \in E \otimes_\varepsilon M$ with

$$\mathrm{id}_E \otimes Q(u) = z \qquad \text{and} \qquad \varepsilon(u; E, M) \leq \lambda\varepsilon(z; E, N)$$

and whence, by $(E \otimes_\varepsilon N')' = E' \otimes_\pi N$, that $E' \otimes_\pi \cdot$ respects finite-dimensional injections with a universal constant: Corollary 1 implies that E' is an $\mathcal{L}_1$ -space.

8.15. Is there a tensomorm α which is projective *und* injective? Existence would imply, by the reformulation 8.13 of Grothendieck's inequality, that ($\sim$ for equivalent norms)

$$g_2^* \leq w_2^* \sim \backslash\varepsilon/ \leq \alpha \leq /\pi\backslash \sim w_2,$$

whence (by $\mathcal{L}_2 \sim w_2$, $\mathcal{D}_2 \sim w_2^*$, $\mathcal{P}_2 \sim g_2^*$)

$$\mathcal{L}_2 \subset \mathcal{D}_2 \subset \mathcal{P}_2,$$

but the identity map of ℓ_2 is not in $\mathcal{P}_2$. More general (and much deeper)

Proposition. *There is no tensornorm which is (r) -injective and (T) -projective.*

Proof . This would imply, as before (using 8.14)

$$w_1 = \varepsilon/ \leq \pi\backslash = w_1' = g_\infty^*$$

and whence $\mathcal{P}_1 \subset \mathcal{L}_1$. But this is not true as Gordon and Lewis showed in [21] solving an old problem of Grothendieck's ([27] p. 72, question 2). .

Note di Matematica Vol. VIII - n. 2, 263-267(1988)

9. ACCESSIBLE TENSORNORMS AND OPERATOR IDEALS

9.1. As defined in 3.6 a tensornorm α is said to be right-accessible if

$$\overleftarrow{\alpha}(\cdot;\ M,F) = \overrightarrow{\alpha}(\cdot;\ M,F)$$

for all $(M,F) \in$ FIN x *NORM* , left-accessible if its transposed tensomorm α^t is right-accessible and accessible if it is both: right- and left-accessible. Moreover, α is totally accessible if α is finitely and cofinitely genemted, i.e. $\overleftarrow{\alpha} = \overrightarrow{\alpha}$. The preceding sections show that these notions are very useful for the full understanding of the duality theory of tensornorms.

Proposition. *Let α be a tensornorm.*

(1) *$\alpha\backslash$ and $\alpha/$ are right-accessible.*

(2) If α is left-accessible, then $\alpha\backslash$ is totally accessible.

(3) $(\backslash\alpha)\backslash$ and $/\alpha\backslash$ are totally accessible. In particular: Every injective tensornorm is totally accessible.

Proof :

(1) follows directly from 8.3 (4). For the proof of (2) let E, $F \in BAN$. Since α is left-accessible

$$\overleftarrow{\alpha}(\cdot;\ E,\ell_\infty(B_{F'})) = \overrightarrow{\alpha}(\cdot;\ E,\ell_\infty(B_{f'})),$$

by the approxrmation-lemma (see also 3.7); now the formulas 8.10 give for $z \in E \otimes F$

$$\begin{aligned}\overleftarrow{\alpha\backslash}(z;\ E,F) &= \overleftarrow{\alpha}\backslash(z;\ E,F)\\ &= \overleftarrow{\alpha}(z;\ E,\ell_\infty(B_{F'}))\\ &= \overrightarrow{\alpha}(z;\ E,\ell_\infty(B_{F'}))\\ &= \overrightarrow{\alpha}\backslash(z;\ E,\ F) = \overrightarrow{\alpha\backslash}(z;\ E,\ F).\end{aligned}$$

(3) is a simple consequence of (1) and (2).

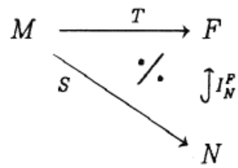
To see an example: Since gr, is (ℓ) -projective, formula 8.14 (1) implies that

$$g_p^* = g_{p'}\backslash = (\backslash g_{p'})\backslash$$

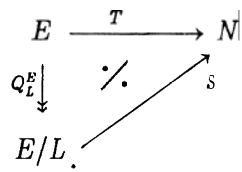
is totally accessible. But note the following: By 9.4 the tensomorm w_p' is totally accessible but $w_p'/$ is not totally accessible for $p \neq 2$ by 5.7.

9.2. It tums out that it is sometimes easier to check the accessibility of a given finitely generated tensomorm through its associated maximal Banach operator ideal.

A quasi-Banach ideal $[d, \mathcal{A}]$ is called *right-accessible* if for all $(M, F) \in FIN \times BAN$, $T \in \mathcal{L}(M,F)$ and $\varepsilon > 0$ there are $N \in FIN(F)$ and $S \in \mathcal{L}(M,N)$ suchthat



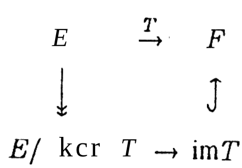
commutes and $A(S) \leq (1+\varepsilon)A(T)$. It is said to be *left-accessible* if for all $(E, N) \in BAN \times FIN$, $T \in \mathcal{L}(E,N)$ and $\varepsilon > 0$ there are $L \in COFIN(E)$ and $S \in \mathcal{L}(E/L,N)$ such that



and $A(S) \leq (1+\varepsilon)A(T)$. A left- and right-accessible idea1 is briefly called *accessible*. Moreover, $[d, A]$ is *totally accessible* if for every finite rank operator $T \in \mathcal{F}(E,F)$ between Banach spaces and $\varepsilon > 0$ there are $L \in COFIN(E)$, $N \in FIN(F)$ and $S \in \mathcal{L}(E/L, N)$ such that

$$T = I_N^T S Q_L^E \qquad \text{and} \qquad A(S) \leq (1+\varepsilon)A(T).$$

Obviously,every injective quasi-Banach ideal is right-accessible and every surjective idea1 is left-accessible. The canonica1 factorization



$T \in \mathcal{F}(E,F)$

gives that a surjective and injective quasi Banach idea1 is even totally accessible.

The key for the following result is the embedding theorem 4.4, namely

$$E' \otimes_{\overleftarrow{\alpha}} F \overset{1}{\hookrightarrow} \mathcal{A}(E,F)$$

if α and $(\mathcal{A}, A)$ are associated.

Proposition. *A finitely generated tensornorm α is right-accessible (resp. left-accessible, accessible, totally accessible) if and only if its associated maximal Banach ideal is.*

Proof. It will be shown that α is totally accessible iff $[\mathcal{A}, A]$ has this property; all other proofs are similar. Assume that α is totally accessible and let $T \in \mathcal{F}(E, F)$. Then

$$\overrightarrow{\alpha}(z_T;\ E',F) = \overleftarrow{\alpha}(z_T;\ E',F) = A(T)$$

which implies that there are $(M, N) \in FIN(E') \times FIN(F)$ and $u \in M \otimes N$ with

$$\alpha(u;\ M, N) \leq (1+\varepsilon)A(T) \qquad \text{and} \qquad I_M^{E'} \otimes I_N^F u = z_T.$$

Hence $T_u \in \mathcal{L}(E/M^0, N)$ satisfies

$$A(T_u) \leq (1+\varepsilon)A(T) \qquad \text{and} \qquad I_N^F T_u Q_{M^0}^E = T.$$

Conversely, let $[\mathcal{A}, A]$ be totally accessible. By the embedding lemma 2.4 it suffices to check that

$$\alpha(\cdot;\ E',F) = \overleftarrow{\alpha}(\cdot;\ E',F)$$

for all $E, F \in BAN$. Let $z \in E' \otimes F$. Then there are $L \in COFIN(E)$, $N \in FIN(F)$ and $S \in \mathcal{L}(E/L, N)$ such that

$$A(S) \leq (1+\varepsilon)A(T_z) \qquad \text{and} \qquad I_N^F S Q_L^E = T_z.$$

It follows, by what was said before, for $z_S \in L^0 \otimes N$

$$\alpha(z_S;\ L^0, N) \leq (1+\varepsilon)\overleftarrow{\alpha}(z;\ E',F) \qquad \text{and} \qquad I_{L^0}^{E'} \otimes I_N^F(z_s) = z,$$

which completes the proof.

Since $\mathcal{A}^{\text{dual}} \sim \alpha^t$ and $\mathcal{A}^* \sim \alpha^*$ (by proposition 4.5) it follows from 3.6 the

Corollary. *Let $[\mathcal{A}, A]$ be a maximal operator ideal.*

(1) *$[\mathcal{A}^{\text{dual}}, A^{\text{dual}}]$ is right-accessible (resp. left-accessible, totally accessible) if and only if $[\mathcal{A}, A]$ is left-accessible (resp. right-accessible, totally accessible).*

(2) *$[\mathcal{A}^*, A^*]$ is right-accessible (resp. left-accessible) if and only if $[\mathcal{A}, A]$ is left-accessible (resp. right-accessible).*

9.3. The following result will be quite useful:

Proposition. *Let [d, A] and [$\mathcal{B}$, B] be quasi-Banach ideals, [d, A] injective and left-accessible, [$\mathcal{B}$, B] totally accessible. Then [$\mathcal{B}$ o* **d,** *B o A] is totally accessible.*

It is easy to see that injective and left-accessible ideals are totally accessible.

Proof . Take T $\in \mathcal{F}(E, F)$ and $\varepsilon > 0$. Then there are $R \in$ d(E, G) and S $\in \mathcal{B}$(G, F) such that

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ & R\searrow \quad \nearrow S & \\ & G & \end{array} \qquad B(S)A(R) \leq (1+\varepsilon)(B \circ A)(T)$$

Since d is injective one can choose this factorization with $\overline{R(E)} = G$ whence S(G) c T(E) and S is finite-dimensional. Since $\mathcal{B}$ is totally accessible and **d** is left-accessible, the following factorization holds:

$$\begin{array}{ccccc} E & \xrightarrow{T} & F & & \\ & R\searrow & \nearrow S & \nwarrow & \\ \downarrow & G & & & N \\ & & \searrow & \nearrow S_0 & \\ E/K & \underset{R_0}{\rightarrow} & G/L & & \end{array} \qquad \begin{array}{l} A(R_0) \leq (1+\varepsilon)A(R) \\ B(S_0) \leq (1+\varepsilon)B(S). \end{array}$$

Consequently,

$$B(S_0)A(R_0) \leq (1+\varepsilon)A(R)(1+\varepsilon)B(S) \leq (1+\varepsilon)^3 B \circ A(T)$$

which proves the result. .

Similarly, it can be shown that if [$\mathcal{A}$, A] and [$\mathcal{B}$, B] are both right-accessible or left-accessible, then their product [$\mathcal{B}$ o d, B o A] again has this property.

9.4. Now everything is prepared to give an easy proof of the following fundamental

Theorem. *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} \geq 1$.*

(1) *$\alpha_{p,q}$ and [$\mathcal{L}_{p,q}$, $L_{p,q}$] are accessible.*

(2) *$\alpha^*_{p,q}$ and [$\mathcal{D}_{p',q'}$, $D_{p',q'}$] are totally accessible.*

Proof . Since the tensomorms and operator ideals in question are associated (4.9) and α is accessible if α^* is (3.6) it sufficcs, by 9.2, to show that $\mathcal{D}_{p',q'}$ is totally acccssiblc. Kwapien's Factorization Theorem 4.8 states that

$$\mathcal{D}_{p',q'} = \mathcal{P}^{\text{dual}}_{q'} \circ \mathcal{P}_{p'}.$$

Now, applying the preceding proposition, $\mathcal{P}_{p'}$ is injective and

$$\mathcal{P}_{p'} \sim g_p^*, \qquad \mathcal{P}_{q'}^{\text{dual}} \sim g_q^{*t}$$

are, by 9.1, both totally accessible.

For another proof of this result see [20].

Corollary. *If p or g = 2 , then* $\alpha_{p,q}$ *is totally accessible.*

Proof . This follows with 9.1 (2) from the facts that $\alpha_{2,p}$ is right-injective (8.4) and left accessible. .

9.5. The tensomorm $g_2 = g_2^*$ is totally accessible. But Reinow [65], cor. 1.2, showed the existence of a reflexive Banach space Z such that for all p $\in [1, \infty[$ with $p \neq 2$ the natura1 map

$$Z' \tilde{\otimes}_{g_p} Z \to \mathcal{L}(Z, Z)$$

is not injective (i.e. Z does not have *the p-approximation properfy).* Since

$$Z' \tilde{\otimes}_{\overleftarrow{g_p}} Z \overset{1}{\hookrightarrow} (Z \otimes_{g_p'} Z')' \hookrightarrow \mathcal{L}(Z, Z)$$

is injective, Reinow's result implies that:

For $1 \leq p < \infty$ *and* $p \neq 2$ *the tensornorm* g_p *is not totally accessible.*

Note di Matematica Vol. VIII - n. 2, 269-275(1988)

10. MORE ABOUT $\alpha_{p\ q}$

10.1. The present paragraph gives some examples for the interplay between maximal operator ideals and their associated (finitely generated) tensomorms. The transfer argument 4.10, remark 2 will be crucial: the reader should have it always in mind! Many of the results will be about the spaces ℓ_p : By 1-complementation, they always imply results on ℓ_p^n (with constants independent from n) and therefore, by the local technique-lemma for $\mathcal{L}_p^g$-spaces (6.2 for tensomorms and, the same way for operator ideals), also results for general $\mathcal{L}_p^g$-spaces (with additional constants) instead of ℓ_p are valid. The obvious consequences for minima1 operator ideals (via the representation theorem 7.1) will not be stated.

10.2. The first result contains as a particular case that all tensomorms $\alpha_{p,q}$ (for p, g $\in$ $]1,\infty[$) are equivalent on Hilbert spaces; remember $\alpha_{p,q} \leq c_{p,q} w_2$ from 1.8.

Proposition. *Let* p, g $\in]1,\infty[$ *with* $\frac{1}{p}+\frac{1}{q}\geq 1$ and $r,s\in[1,2]$.

Then

$$\varepsilon \leq \alpha_{p,q} \leq K_G c_{p,q}\varepsilon \qquad on \qquad \ell_r \otimes \ell_s$$

and

$$\alpha'_{p,q} \leq \pi \leq K_G c_{p,q}\alpha'_{p,q} \qquad on \qquad \ell_{r'} \otimes \ell_{s'}$$

Proof . By 4.10 and Grothendieck's inequality 1.11

$$w_2 \leq w_2^* \leq K_G \varepsilon \qquad \text{on} \qquad \ell_1^n \otimes \ell_1^m.$$

Since w_2 and ε are injective and

$$\ell_r^n \overset{1}{\hookrightarrow} L_1(\mu)$$

(see 8.5) the local technique lemma for $\mathcal{L}_p^g$-spaces implies

$$w_2 \leq K_G \varepsilon \qquad \text{on} \qquad \ell_r \otimes \ell_s$$

which gives the announced result on $\ell_r \otimes \ell_s$. The second one follows by dualization (remember this aspect of the transfer argument).

In terms of operators (this is a result of Lindenstrauss-Pełczyński [51] which was generalized by Kwapien [48]).

Corollary. *If* p, g $\in]1,\infty[$ und $r,s\in[1,2]$, *then*

$$\mathcal{L}_{p,q}(\ell_{r'},\ell_s) = \mathcal{L}_p(\ell_{r'},\ell_s) = \mathcal{L}(\ell_{r'},\ell_s)$$
$$\mathcal{D}_{p,q}(\ell_r,\ell_{s'}) = \mathcal{D}_p(\ell_r,\ell_{s'}) = \mathcal{I}(\ell_r,\ell_{s'})$$

10.3. To investigate the tensomorms $g_p = \alpha_{p,1}$ it is reasonable to study first the associated operator ideals of summing opcrators.

Proposition. *Take* $s, p \in [1,2]$ *and* $q \in [2, \infty[$ *, then for every Banach space* F

(1) $$\mathcal{P}_p(\ell_s, F) = \mathcal{P}_1(\ell_s, F)$$
$$P_1(T) \le K_G P_p(T) \qquad \text{for} \qquad T \in \mathcal{P}_p(\ell_s, F)$$

and

(2) $$\mathcal{P}_q(F, \ell_s) = \mathcal{P}_2(F, \ell_s)$$
$$P_2(T) \le a_s b_q P_p(T) \qquad \text{for} \qquad T \in \mathcal{P}_q(F, \ell_s).$$

(The constants a_s and b_q from Khintchine's incquality). This result is due to Kwapien as well [46]. Clearly, a special case is Pelczynski's theorem, that all $\mathcal{P}_p$ coincide on Hilbert spaces. We present a proof since it fits nicely into our setting.

Proof: (1) It is enough to take $p = 2$; for $T \in \mathcal{P}_2(\ell_s, F)$ fix $x_1, \ldots, x_n \in \ell_s$ anddctine

$$S : \ell_\infty^n \to \ell_s \qquad Se_i := x_i$$

whence $\|S\| = w_1(x_i; \ell_s)$. Since $\mathcal{P}_2 \sim g_2 \sim \mathcal{P}_2^*$ (by 8.14) the relations

$$\mathcal{P}_2 \circ \mathcal{P}_2 = \mathcal{P}_2 \circ \mathcal{P}_2^* \subset \mathcal{I} \subset \mathcal{P}_1$$

(5.5) give

$$P_1(TS) \le P_2(T) P_2(S) \le P_2(T) K_G \|S\|$$

when using $\mathcal{L}(\ell_\infty^n, \ell_s) = \mathcal{P}_2(\ell_\infty^n, \ell_s)$ (see 8.5). Therefore

$$\sum_i \|Tx_i\| = \sum_i \|TSe_i\| \le P_2(T) K_G \|S\| w_1(e_i; \ell_\infty^n)$$
$$= P_2(T) K_G w_1(x_i; \ell_s)$$

which is $P_1(T) \le K_G P_2(T)$.

(2) For $T \in \mathcal{L}(F, \ell_s)$ take $x_1, \ldots, x_n \in F$ and use Khintchine's inequality 1.8 in order

to obtain:

$$\left(\sum_{i=1}^{n} \|Tx_i\|^2\right)^{1/2} = \left(\sum_{i=1}^{n}\left(\sum_{k=1}^{\infty}|Tx_i(k)|^s\right)^{2/s}\right)^{1/2} \leq$$

$$\leq \left(\sum_{k=1}^{\infty}\left(\sum_{i+1}^{n}|Tx_i(k)|^2\right)^{s/2}\right)^{1/s} \leq$$

$$\leq a_s\left(\sum_{k=1}^{\infty}\int_{D_n}\left|\sum_{i=1}^{n}\varepsilon_i(t)Tx_i(k)\right|^s \mu_n(dt)\right)^{1/s} =$$

$$= a_s\left(\int_{D_n}\left\|T\left(\sum_{i=1}^{n}\varepsilon_i(t)x_i\right)\right\|_{\ell_s}^s \mu_n(dt)\right)^{1/s} \leq$$

$$\leq a_s\left(\int_{D_n}\left\|T\left(\sum_{i=1}^{n}\varepsilon_i(t)x_i\right)\right\|_{\ell_s}^q \mu_n(dt)\right)^{1/q}$$

Now, if T is even absolutely-q-summing , the Grothendieck-Pietsch-domination theorem gives

$$\leq a_s P_q(T)\left(\int_{D_n}\int_{B_{F'}}\left|\left\langle x', \sum_{i=1}^{n}\varepsilon_i(t)x_i\right\rangle\right|^q \nu(dx')\mu_n(dt)\right)^{1/q} \leq$$

$$\leq a_s P_q(T)\sup_{x'\in B_{F'}}\left(\int_{D_n}\left|\sum_{i=1}^{n}\varepsilon_i(t)\langle x', x_i\rangle\right|^q \mu_n(dt)\right)^{1/q} \leq$$

$$\leq a_s P_q(T) b_q w_2(x_i)$$

and this is $P_2(T) \leq a_s b_q P_q(T)$.

In terms of tensomorms (by the transfer argument and the embedding lemma)

Corollary 1. *For every Banach space F the following holds:*

(1) *If* $r, q \in [2, \infty]$, *then*

$$g_q^* \leq g_\infty^* \leq K_G g_q^* \qquad on \qquad \ell_r \otimes F$$

and if $s \in [1,2]$, $q \in [2, \infty]$, *then*

$$d_\infty \leq d_q \leq K_G d_\infty \qquad on \qquad \ell_s \otimes F.$$

(2) *If* $s \in [1,2]$ und $p \in]1,2]$, *then*

$$g_p^* \leq g_2^* \leq a_s b_{p'} g_p^* \qquad on \qquad F \otimes \ell_s$$

and if $r \in [2,\infty]$ *and* $p \in]1,2]$, *then*

$$d_2 \leq d_p \leq a_{r'} b_{p'} d_2 \qquad on \qquad F \otimes \ell_r .$$

By the transfer argument it is possible to go back to operator ideals in order to obtain the dual results for operator ideals (note that all these tensornorms are accessible and ℓ_p has the metric approximation property). The transposed of the second statement in (1) and (2) give therefore immediately

Corollary. *Let F be a Banach space, then*

$$\mathcal{I}_q(F,\ell_s) = \mathcal{L}_\infty(F,\ell_s) = \mathcal{I}_2(F,\ell_s)$$

for $q \in [2,\infty]$, $s \in [1,2]$ *and*

$$\mathcal{I}_2(\ell_s, F) = \mathcal{I}_p(\ell_s, F)$$

for $p \in]1,2]$ *and* $s \in [1,2]$.

10.4. To see what this means for Hilbert spaces H and K, observe first, that $\mathcal{P}_2(H, K) = \mathcal{HS}(H, K)$ (Hilbert-Schmidt operators) holds isometrically, whence

$$H \otimes_{g_2^*} K \overset{1}{\hookrightarrow} \mathcal{P}_2(H,K) = \mathcal{HS}(H,K)$$

and therefore – for finite orthonormal systems –

$$g_2^*\left(\sum_{i,j} \alpha_{ij} e_i \otimes f_j\right) = \left(\sum_{i,j} |\alpha_{ij}|^2\right)^{1/2}$$

which implies $g_2^* = d_2^*$. Whecce $g_2 = g_2^* = d_2^* = d_2$ is the Hilbert-Schmidt norm on $H \otimes K$. Now the preceding results imply the

Proposition. *On the tensor product* $H \otimes K$ *of two Hilbert spaces the following holds:*

$$\begin{array}{ll}
\varepsilon \leq \alpha_{p,q} \leq K_G c_{p,q} \varepsilon & p,q \in]1,\infty[\\
\alpha'_{p,q} \leq \pi \leq K_G c_{p,q} \alpha'_{p,q} & p,q \in]1,\infty[\\
g_2 \leq g_q^* \leq K_G g_2 & q \in [2,\infty] \\
g_p^* \leq g_2 \leq b_{p'} g_p^* & p \in]1,2] \\
g_q \leq g_2 \leq K_G g_q & q \in [2,\infty] \\
g_2 \leq g_p \leq b_{p'} g_2 & p \in]1,2]
\end{array}$$

So there are, up to equivalence, only three tensomorms under the $\alpha_{p\,q}$ and $\alpha'_{p\,q}$ on Hilbert spaces: ε, π and the Hilbert-Schmidt norm g_2 . In terms of operators:

$p, q \in]1, \infty[$:	$\mathcal{L}_{p,q} = \mathcal{L}_p = \mathcal{L}$	all operators
$p, q \in]1, \infty[$:	$\mathcal{D}_{p,q} = \mathcal{D}_p = \mathcal{I} = \mathcal{N}$	nuclear operators
$p \in [1, \infty[$:	$\mathcal{P}_p = \mathcal{P}_p^{\text{dual}} = \mathcal{L}_1 = \mathcal{L}_\infty =$	Hilbert-Schmidt
$q \in]1, \infty]$	$= \mathcal{I}_q = \mathcal{I}_q^{\text{dual}} = \mathcal{HS}$	operators

10.5. Some of the preceding results have remarkable extensions to Banach spaces with type and cotype. For $q \in [2, \infty[$ an operator $T \in \mathcal{L}(E, F)$ is called of *cotype q* if there is a $\rho \geq 0$ such that for all $x_1, \ldots, x_n \in E$

$$\left(\sum_{i=1}^n \|Tx_i\|^q\right)^{1/q} \leq \rho \left(\int_{D_n} \left\|\sum_{i=1}^n \varepsilon_i(t) x_i\right\|^2 \mu_n(dt)\right)^{1/2}$$

(see 1.8 for the notation); $C_q(T) := \inf \rho$. The Kahane inequality (see e.g. [53], p. 74) implies that using on the right side of the definition the L_p-norm ($1 \leq p < \infty$) instead of the L_2-norm gives an equivalent norm. It is straightforward to see that the operator ideal $(\mathcal{C}_q, C_q)$ of all cotype-q-operators is a maximal, injective Banach operator ideal, whence associated with a certain finitely generated tensomorm.

A Banach-space has cotype q if $id_E \in \mathcal{C}_q$. Following the arguments in the first part of the proof of 10.3 (2) with Khintchine's inequality it is clear that ℓ_p for $1 \leq p < \infty$ has cotype $q := \max\{p, 2\}$ and this implies, by the usual local techniques, that all $\mathcal{L}_p^g$-spaces (for $1 \leq p < \infty$) have cotype $q = \max\{p, 2\}$. A direct application of corollary 3 in 4.4 gives that E has cotype q if and only if E" has cotype q.

By the way, since there are cotype-q-spaces without the approximation property (subspaces of ℓ_1) it follows from proposition 5.7 that the dual tensomorm γ'_q of the tensornorm γ_q associated with the cotype-q-operators is not totally accessible.

10.6. Pisier's factorization theorem ([64], chap. 4) states that if E' and F have cotype 2, then each operator $T : E \to F$ which can be approximated by finite-rank operators uniformly on compact sets factors through a Hilbert space; in particular

$$E' \tilde{\otimes}_\varepsilon F =: \overline{\mathcal{F}}(E, F) \subset \mathcal{L}_2(E, F).$$

Since $w_2 \sim \mathcal{L}_2$ and ε and w_2 are totally accessible this implies

$$E' \otimes_\varepsilon F = E' \otimes_{w_2} F \qquad \text{isomorphically}$$

whence, by the embedding lemma and 1 .8:

If E and F have cotype 2 and $p,q \in]1,\infty[$ with $\frac{1}{p}+\frac{1}{q} \geq 1$, then

$$E \otimes_{\varepsilon} F = E \otimes_{\alpha_{p,q}} F \qquad \textit{isomorphically}$$

However, the dual result *fails* to be true: If E' and F' have cotype 2, then π and w_2' are in general not equivalent on $E \otimes F$. Pisier constructed a Banach space P not isomorphic to a Hilbert space, but such that P and P' have cotype 2 ([64], chap. 10). If $P \otimes_{\pi} P'$ and $P \otimes_{w_2'} P'$ were isomorphic, the representation theorem for maximal ideals would imply that every operator $P \to P$ factors through a Hilbert space which is a contradiction.

The transfer argument (4.10 remark 2(1)) is not applicable to ($\star$) by the following reason: if E' and F have cotype 2 it follows only

$$\overline{\mathcal{F}}(E,F) \subset \mathcal{L}_2(E,F)$$

but in general not

$$\mathcal{L}(E,F) = \mathcal{L}_2(E,F)$$

by Pisier's example. On the other hand if E (or F) in addition has the approximation property, then Pisier's factorization theorem implies $\mathcal{L}(E,F) = \mathcal{L}_2(E,F)$.

Now the transfer argument applied to $\mathcal{L}(E,F')$ and the symmetry of w_2' and π give

If E' and F' have cotype 2 and: E or F has the approximation property, then

$$E \otimes_{\pi} F = E \otimes_{w_2'} F \qquad \textit{isomorphically}$$

and whence also for all $\alpha'_{p,q}$ (for $p, q \neq 1$).

10.7. Analyzing the proof of 10.3 (2) it is clear that the result extends to cotype 2 spaces instead of ℓ_s: The second of the following two statements holds.

(1) $\mathcal{P}_p(E,F) = \mathcal{P}_1(E,F)$ *if* $p \in [1,2]$ *and* E *has cotype 2.*

(2) $\mathcal{P}_q(E,F) = \mathcal{P}_2(E,F)$ *if* $q \in [2,\infty[$ *and* F *has cotype 2.*

Both results are due to Maurey; for a proof of (1) see [64], chap. 5.

Using the transfer argument, the fact that all g_p^* are totally accessible and the embedding lemma, (1) and (2) imply the following generalizations of corollary 1 in 10.3.

Let $p \in]1,2]$, $q \in [2, \infty]$ *and E, F Banach spaces. Then*

$g_\infty^* \sim g_q^*$ *on* $E \otimes F$ *if* E' *has cotype 2*
$d_\infty \sim d_q$ *on* $E \otimes F$ *if E has cotype 2*
$g_2^* \sim g_p^*$ *on* $E \otimes F$ *if F has cotype 2*
$d_2 \sim d_p$ *on* $E \otimes F$ *if* F' *has cotype 2.*

Since $g_\infty^* = \pi\backslash$ and $g_2^* = g_2$ (by 8.14) the first norm equivalence gives

$$g_2^* \sim g_\infty^* = \pi\backslash = \pi \text{ on } E \otimes \ell_\infty$$

and whence

$$g_2^{*\prime} = g_2' = g_2^{*t} \sim \pi \quad \text{on} \quad \ell_\infty \otimes E$$

if E' has cotype 2. This clearly implies another result of Maurey's

(3) $\mathcal{L}(\ell_\infty, F) = \mathcal{P}_2(\ell_\infty, F)$ *if F has cotype 2*

which generalizes Grothendieck's result for $\mathcal{L}_p^g$-spaces F (with $1 \leq p \leq 2$, see 8.5).

Note di Matematica Vol. VIII - n. 2, 277-278(1988)

11. FINAL REMARKS

11.1. There are various aspects of the metric theory of tensor products which we did not treat: We want to mention at least some of them which are closely connected with what we presented.

11.2. Probably the most important is the treatment of the «semi» tensomorms Δ_p

$$L_p(\mu) \otimes_{\Delta_p} E \overset{1}{\hookrightarrow} L_p(\mu; E)$$

for which

$$d_p \leq \Delta_p \leq g^*_{p'}$$
$$\Delta_\infty = \varepsilon, \ \Delta_1 = \pi$$

holds. In general there is no tensomorm which induces Δ_p; this causes from the fact that for $T \in \mathcal{L}(L_p, L_p)$ the operator

$$T \otimes \mathrm{id}_E : L_p(\mu) \otimes_{\Delta_p} E \to L_p(\mu) \otimes_{\Delta_p} E$$

is in general not continuous: take, for example, for T the Fourier-transform on $L_2(\mathbb{R})$. There are two directions of research: First, look for spaces or, more generally, for operators $S \in \mathcal{L}(E, F)$ such that $T \otimes S$ .is Δ_p-continuous for all $T \in \mathcal{L}(L_p, L_p)$ (here are some crucial results due to Kwapien [48], see also [23], and 11.3) or, secondly, fix $T \in \mathcal{L}(L_p, L_p)$ and look for all $S \in \mathcal{L}(E, F)$ such that $T \otimes S$ is Ar-continuous; for example, take T the Fourier transform on $L_2(\mathbb{R})$ (see Kwapien [47]) or T *the* Hilbert transform on $L_p(\mathbb{R})$ (see Burkholder [3]; Bourgain [2], M. Defant [11]) or T the projection of $L_2(\{-1, 1\}^{\mathbb{N}})$ onto the space of the Rademacher functions (see Pisier [62]).

11.3. In [9] products $\rho := \alpha \otimes_G \beta$ for tensomorms were defined via the trace mapping

$$(E \otimes_\alpha G') \otimes_\pi (G \otimes_\beta F) \overset{1}{\twoheadrightarrow} E \otimes_\rho F$$

which mimics the composition of operators. Among other things, this was used to prove that $S \in \mathcal{L}(E, F)$ has the property that

$$T \otimes S : L_p \otimes_{\Delta_p} E \to L_p \otimes_{\Delta_p} F$$

is continuous for all $T \in \mathcal{L}(L_p, L_p)$ if and only if

$$T \in (\mathcal{L}_p^{\mathrm{surj}})^{\mathrm{inj}},$$

i.e. factors through a subspace of a quotient of some L_p which is the operator version of a result of Kwapien.

11.4. As a generalization of the Radon-Nikodym properry Lewis [50] studied the question of when

$$E'\tilde{\otimes}_{\alpha}F' = (E \otimes_{\alpha'} F)'$$

which for the associated maximal operator ideal means

$$\mathcal{A}^{\min}(E, F') = d(E, F')$$

by the representation theorems for minimal and maximal operator ideals. Clearly, this study allows in particular to investigate under which circumstances the space d(E, F) is rellexive (see [50], [22]).

11.5. A crucial tool in the theory of the distribution of eigenvalues of operators is the tensor stability of operator ideals d: If T, S ∈ d, then $T \otimes_{\alpha} S$ ∈ d. For example, $\mathcal{P}_p$ is ε-stable [36] and this is the key for Pietsch's trick to prove the Johnson-König-Maurey-Retherford theorem: If T ∈ $\mathcal{P}_p$ the sequence of eigenvalues of T is in ℓ_p (for $2 \leq p < \infty$, see [43], [61]). Tensor stability has various other promissing applications (see [42], [4], [5]).

11.6. The metric theory of tensomorms has an extension to locally convex spaces, due to Harksen [29], [30]: If E and F are separated locally convex spaces with defining systems P_E and P_F of seminorms, the α-tensornorm topology on $E \otimes F$ is defined by

$$E\tilde{\otimes}_{\alpha}F := \operatorname*{proj}_{p\in P_E \ q\in P_F} E_p\tilde{\otimes}_{\alpha}F_q$$

where E_p is the canonica1 normed space associated with the seminorms p. Projectivity and injcctivity properties of α for normed spaces hold also for the a-tensomorm topology. There are many applications to the theory of vector-valued continuous, differentiableor holomorphic functions, to lifting and extension properties, and to the study of the topologica1 and geometrical structure of spaces of such functions; for rcferences see [9], [10], [16], Kaballo [41] and Hollstein [31]-[35].

Note di Matemanca Vol. VIII n. 2, 279-281(1988)

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Andreas Defant
Klaus Floret
Fachbereich Mathematik
der Universität
D-2900 Oldenburg
Fed. Rep. Germany