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SEMIGROUP IN WHICH S^{n+1} IS A SEMILATTICE OF RIGHT GROUPS (INFLATIONS OF A SEMILATTICE OF RIGHT GROUPS)

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ABSTRACT. In this paper we consider semigroups in which S^{n+1} is a semilattice of right groups. Also n-inflation of a semilattice of right groups is treated.

1. INTRODUCTION AND PRELIMINARIES

A semigroup S in a *n-inflation* of a semigroup T if T is an ideal of S, $S^{n+1} \subset T$ and there exists a homomorphism φ of S onto T such that $\varphi(t) = t$ for all $t \in T$, [1]. The notion of inflation (1-inflation, is introduced by A.H. Clifford, [7] and the strong inflation (2-inflation) is introduced by M. Petrich, [12]. A construction of an n-inflation is given in [1]. In [1] we have, also, some characterizations for n-inflation of a union of groups. In this paper we consider semigroups in which S^{n+1} is a semilattice of right groups. We also characterise n-inflation of a semilattice of right groups. For the related results see [4] and [2].

E.G. Shutov, [14] and N. Kimura, T. Tamura and R. Merkel, [8] considered λ-semigroups, i.e. semigroups in which every subsemigroup is a left ideal. In this paper we introduce the concept of λ_n-semigroups. We prove that S is a λ-semigroup if and only if S is a λ_1-semigroup. One simple construction for λ-semigroups is given in [9].

Here a construcion for λ_n-semigroups is given.

T. Tamura, [15] studied semigroups with the following identity $xy = f(x, y)$. In the present paper we consider semigroups in which the following identity holds:

$$\prod_{i=1}^{n+1} x_i = \prod_{j=1}^{h} \left(\prod_{i=1}^{n+1} x_i^{n_{ij}} \right).$$

A classification of these semigroups is given. Some special cases are treated in [6] and [13].

Throughout this paper, Z^+ will denote the set of all positive integers. By $\mathrm{Reg}(S)$ $(\mathrm{Gr}(S), E(S))$ we denote the set of all regular (completely regular, idempotent) elements of a semigroup S.

For undefined notions and notations we refer to [5] and [7].

Lemma 1.1. *S is a right group if and only if*

$$(1.1) \qquad (\forall x, a \in S)\, x \in aSx.$$

Proof. Let S be a right group. Then for every $a \in S$ there exists $b \in S$ such that $a = aba$. In a right simple semigroup S every idempotent is a left identity (Lemma VI. 3.2, [5]), so for any $x \in S$ we have that $x = abx \in aSx$.

Conversely, if (1.1) holds then $x \in x^2Sx$ for every $x \in S$. Hence, S is a union of groups. For every $e, f \in E(S)$ we have that $e \in fSe$, whence $e = fe$. Thus S is a right group (see [7], p. 63). □

Corollary 1.1. *S is a periodic right group if and only if*

$$(\forall x, a \in S)(\exists k \in Z^+)x = a^k x.$$

Proof. Let S be a periodic right group. Then for every $a \in S$ there exists $k \in Z^+$ such that $a^k \in E(S)$. Now, by Lemma 1.1. we have the assertion.

The converse follows by Lemma 1.1. □

2. n-INFLATION OF A SEMILATTICE OF RIGHT GROUPS

Theorem 2.1. *The following conditions are equivalent on a semigroup S:*

(i) S^{n+1} is a semilattice of right groups,

(ii) S is a semilattice Y of semigroups S_α, $\alpha \in Y$, where S_α^{n+1}, $\alpha \in Y$ is a right group and for every $x_i \in S_{\alpha_i}$, $\alpha_i \in Y$,

$$x_1 x_2 \dots x_{n+1} \in S^{n+1}_{\alpha_1 \alpha_2 \dots \alpha_{n+1}},$$

(iii) $x_1 x_2 \dots x_{n+1} \in x_{n+1} S x_1 x_2 \dots x_{n+1}$ for every $x_1, x_2, \dots, x_{n+1} \in S$.

Proof. (i) $\Rightarrow$ (ii). Let S^{n+1} be a semilattice of right groups. Then $\mathrm{Reg}(S) = \mathrm{Gr}(S)$. Thus S is a GV-semigroup. By Theorem 3, [4] we have that S is a semilattice Y of nil-extensions S_α, $\alpha \in Y$ of right groups K_α. Let $x_i \in S_{\alpha_i}$.

Then

$$x_1 x_2 \dots x_{n+1} \in K_{\alpha_1 \dots \alpha_{n+1}} = S^{n+1}_{\alpha_1 \dots \alpha_{n+1}}$$

where $K_{\alpha_1 \dots \alpha_{n+1}}$ is a right group.

(ii) $\Rightarrow$ (iii). By Lemma 1.1. we have that for every $x_i \in S_{\alpha_i}$,

$$x_1 x_2 \dots x_{n+1} \in x_{n+1} \dots x_1 S^{n+1}_{\alpha_1 \dots \alpha_{n+1}} x_1 \dots x_{n+1} \subseteq x_{n+1} S x_1 \dots x_{n+1}.$$

(iii) $\Rightarrow$ (i). Using, more a time, the hypothesis we obtain that for every $x_1, x_2, \dots, x_{n+1} \in S$ there exists $u \in S$ such that

$$x_1 x_2 \dots x_{n+1} = (x_1 x_2 \dots x_{n+1})^2 u \quad x_1 x_2 \dots x_{n+1}.$$

So S^{n+1} is a union of groups. For every $e, f \in E(S)$ there exists $u \in S$ such that $ef \dots f = fuef \dots f$, whence $ef = fef$. By theorem 2, [10] we have that S^{n+1} is a semilattice of right groups. □

Corollary 2.1. *S^{n+1} is a right group if and only if*

$$(2.1)\qquad (\forall x_1, x_2, \ldots, x_{n+1}, a \in S)\, x_1x_2\ldots x_{n+1} \in aSx_1x_2\ldots x_{n+1}.$$

Proof. Let S^{n+1} be a right group. Then by Lemma 1.1. we have that for every $x_1, x_2, \ldots, x_{n+1} \in S$,

$$x_1x_2\ldots x_{n+1} \in ax_1x_2\ldots x_{n+1}S^{n+1}x_1x_2\ldots x_{n+1} \subseteq aSx_1x_2\ldots x_{n+1}.$$

Conversely, from (2.1) we have that $x_1x_2\ldots x_{n+1} \in x_{n+1}Sx_1x_2\ldots x_{n+1}$ for every $x_1, x_2, \ldots, x_{n+1} \in S$. By Theorem 2.1. S^{n+1} is a semilattice of right groups. By (2.1) S^{n+1} is a right simple semigroup. Therefore, S^{n+1} is a right group. □

Theorem 2.2. *S^{n+1} is a semilattice of periodic right groups if and only if*

$$(2.2)\qquad (\forall x_1, x_2, \ldots, x_{n+1} \in S)(\exists m \in Z^+)\, x_1x_2\ldots x_{n+1} =$$

$$= (x_{n+1}x_1\ldots x_n)^m x_1\ldots x_{n+1}.$$

Proof. Let S^{n+1} be a semilattice of periodic right groups. Then by Theorme 2.1. S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and $x_1x_2\ldots x_{n+1}$, $x_{n+1}x_1\ldots x_n \in S^{n+1}_{\alpha_1\ldots\alpha_{n+1}} = K$ for all $x_i \in S_{\alpha_i}$, where K is a periodic right group. Now, there exists $e \in E(S)$ and $m \in Z^+$ such that $(x_{n+1}\ x_1\ \ldots\ x_n)^m = e$, and by Corollary 1.1. we have (2.2).

Conversely, let (2.2) holds. Then for every $x_1, \ldots, x_{n+1} \in S$, there exists $m \in Z^+$ such that

$$x_1\ldots x_{n+1} = (x_{n+1}x_1\ldots x_n)^m x_1\ldots x_{n+1} \in x_{n+1}Sx_1\ldots x_{n+1}$$

and by Theorem 2.1. we have that S^{n+1} is a semilattice of periodic right groups. □

Corollary 2.2. *S^{n+1} is a periodic right group if and only if*

$$(2.3)\qquad (\forall x_1, \ldots x_{n+1}, a \in S)(\exists k \in Z^+)\, x_1\ldots x_{n+1} = a^k x_1\ldots x_{n+1}.$$

Proof. Let S^{n+1} be a periodic right group. Then for every $a \in S$ there exists $k \in Z^+$ such that $a^k \in E(S)$. Now, by Corollary 1.1. we have (2.3).

The converse follows by Theorem 2.2. and from the fact that S^{n+1} is right simple (by (2.3)). □

Corollary 2.3. *S^{n+1} is a right zero band if and only if*

(2.4) $$x_1 x_2 \dots x_{n+1} = a x_1 x_2 \dots x_{n+1}.$$

Proof. Let S^{n+1} be a right zero band. Then $a x_1 x_2 \dots x_{n+1} \in E(S) = S^{n+1}$ for every a, $x_1, \dots, x_{n+1} \in S$. So

$$x_1 \dots x_{n+1} = a x_1 \dots x_{n+1} x_1 \dots x_{n+1} = a x_1 \dots x_{n+1}.$$

Conversely, it follows from (2.4) that $x_1 \dots x_{n+1} = (x_1 \dots x_{n+1})^2$ and by Corollary 2.2. we have that S^{n+1} is a right zero band. □

Theoreme 2.3. *S is an n-inflation of a semilattice of right groups if and only if*

(2.5) $$(\forall x_1 \dots x_{n+1} \in S) x_1 \dots x_{n+1} \in x_{n+1} S x_1 \dots x_{n+1} S x_{n+1}^2.$$

Proof. Let S be an n-inflation of a semilattice of right groups T. Then $S^{n+1} = T$. So, by Theorem 2.1. for every $x_1, \dots, x_{n+1} \in S$ there exists $u \in S$ such that

$$x_1 \dots x_{n+1} = x_{n+1} u x_1 \dots x_{n+1}.$$

By Theorem 3.1. [1] we have that

$$x_{n+1} u x_1 \dots x_{n+1} = x_{n+1} u x_1 \dots x_{n+1} e_{n+1}, \quad \text{where } x_{n+1}^{n+1} \in G_{e_{n+1}}$$

$$= x_{n+1} u x_1 \dots x_{n+1} \left(x_{n+1}^{n+1}\right)^{-1} x_{n+1}^{n+1} \in x_{n+1} S x_1 \dots x_{n+1} S x_{n+1}^2.$$

Thus (2.5) holds.

Conversely, from (2.5) we have that $x_1 \dots x_{n+1} \in x_1^2 S^n x_{n+1}^2$, whence by Theorem 3.1. [1] we have that S is an n-inflation of a union of groups. For everey $e, f \in E(S)$ there exists $s \in S$ such that $ee \dots ef = fsf$, whence $ef = fef$. Therefore, by Theorem 2. [10] we have that S is an n-inflation of a semilattice of right groups. □

Corollary 2.4. *S is an n-inflation of a semilattice of periodic right groups if and only if*

$$(\forall x_1, \dots, x_{n+1} \in S)(\exists m, k \in Z^+) x_1 \dots x_{n+1} =$$

$$= (x_{n+1} x_1 \dots x_n)^m x_1 \dots x_n x_{n+1}^{k+1}.$$

Proof. Follows by Theorem 3.1.[1] and by Theorem 2.3. □

Corollary 2.5. *S is an n-inflation of a right zero band if and only if in S the following identity holds:*

$$x_1 \dots x_{n+1} = x_{n+1}^{n+1} \tag{2.6}$$

Proof . Let S be an n-inflation of a right zero band E and let φ be a retraction of S onto E. Then $S^{n+1} = E$ and for every $x_1, \dots, x_{n+1} \in S$ we have that

$$\begin{aligned} x_1 \dots x_{n+1} &= \varphi(x_1 \dots x_{n+1}) = \varphi(x_1) \dots \varphi(x_{n+1}) \\ &= \varphi(x_{n+1}) = [\varphi(x_{n+1})]^{n+1} = \varphi\left(x_{n+1}^{n+1}\right) = x_{n+1}^{n+1}. \end{aligned}$$

Conversely, it follows from (2.6) that $x^{n+1} = x^2 x \dots x = x^{n+1} \in E(S)$ for all $x \in S$. So $S^{n+1} = E(S)$. For every $e, f \in E(S)$ we have that $ee \dots ef = f^{n+1} = f$, i.e. $ef = f$. Thus $E(S)$ is a right zero band. It remains to prove that $\varphi(x) = x^{n+1}$ is a homomorphism from S onto $E(S)$. Indeed, for any $x, y \in S$ we have that

$$\varphi(xy) = (xy)^{n+1} = y^{n+1} = x^{n+1} y^{n+1} = \varphi(x)\varphi(y)$$

and since $\varphi^2(x) = \varphi(x)$ for all $x \in S$ we have that φ is a retraction of S onto $E(S)$. Therefore, S is an n-inflation of a right zero band. □

Corollary 2.6. *S is an n-inflation of a right group if and only if*

$$(\forall x_1, \dots, x_{n+1}, a \in S)\, x_1 \dots x_{n+1} \in aSx_1 \dots x_{n+1} S x_{n+1}^2 . \tag{2.7}$$

Proof . Let S be an n-inflation of a right group. Then by Corollary 2.1. we have that for every $x_1, \dots, x_{n+1}, a \in S$ there exists $u \in S$ such that

$$\begin{aligned} x_1 \dots x_{n+1} &= aux_1 \dots x_{n+1} \\ &= aux_1 \dots x_{n+1} e_{n+1}, \quad \text{where } x_{n+1}^{n+1} \in G_{e_{n+1}} && \text{(Th. 3.1. [1])} \\ &= aux_1 \dots x_{n+1} \left(x_{n+1}^{n+1}\right)^{-1} x_{n+1}^{n+1} \in aSx_1 \dots x_{n+1} S x_{n+1}^2 . \end{aligned}$$

Conversely, by (2.7) and by Theorem 2.3. we have that S is an n-inflation of a semilattice of right groups. Since S^{n+1} is a right simple semigroup we have that S is an n-inflation of a right group. □

Corollary 2.7. *S is n-inflation of a periodic right group if and only if*

$$(\forall x_1, \dots, x_{n+1}, a \in S)(\exists k, m \in Z^+)\, x_1 \dots x_{n+1} = a^k x_1 \dots x_n x_{n+1}^{m+1}.$$

Proof. Let S be an n-inflation of a periodic right group. Then by Corollary 2.2. for every $x_1, \dots, x_{n+1}, a \in S$ there exists $k \in Z^+$ such that

$$\begin{aligned} x_1 \dots x_{n+1} &= a^k x_1 \dots x_{n+1} \\ &= a^k x_1 \dots x_{n+1} e_{n+1}, \quad \text{where } x_{n+1}^{n+1} \in G_{e_{n+1}} && \text{(Th. 3.1. [1])} \\ &= a^k x_1 \dots x_{n+1} x_{n+1}^m, \quad \text{where } x_{n+1}^m = e_{n+1}. \end{aligned}$$

The converse follows by Corollary 2.5. □

3. λ_n-SEMIGROUPS

S is a *λ-semigroup* if every subsemigorup of S is a left ideal of S, [8], [14]. A simple construction of λ-semigroup is given in [9].

Definition 3.1. *S is a λ_n-semigroup if for every subsemigroup A of S the following condition holds:*

$$S^n A = A^{n+1}.$$

Lemma 3.1. *S is a λ_n-semigroup if and only if*

$$(\forall a \in S)\, S^n a = \langle a \rangle^{n+1}. \tag{3.1}$$

Proof. Let S be a λ_n-semigroup. Then for every $a \in S$,

$$S^n a \subseteq S^n \langle a \rangle = \langle a \rangle^{n+1} \subseteq S^n a$$

i.e. (3.1) holds.

Conversely, let (3.1) holds and let A be a subsemigroup of S. Then for any $a \in A$ we have that $S^n a = \langle a \rangle^{n+1} \subseteq A^{n+1}$. So $S^n A \subseteq A^{n+1}$ and since $A^{n+1} \subseteq S^n A$ we have that S is a λ_n-semigroup. □

Lemma 3.2. *S is a λ_1-semigroup if and only if S is a λ-semigroup.*

Proof. Let S be a λ_1-semigroup and let A be a subsemigroup of S. Then $SA = A^2 \subseteq A$. So S is a λ-semigroup.

Conversely, let S be a λ-semigroup and let A be a subsemigroup of S. Then by Theorem 3. [14] (see also lemmas 3. and 6. [8]) for every $x \in S$ and $y \in A$, $xy \in \{y^2, y^3\}$. So $xy \in A^2$, i.e. $SA \subseteq A^2 \subseteq SA$. Thus $SA = A^2$. Therefore, S is a λ_1-semigroup. □

Example 1. The following semigroup $\langle x \rangle = \{x, x^2, x^3, x^4 = x^5\}$ is a λ_2-semigroup. But $\langle x \rangle$ is not λ_1-semigroup, since $\langle x^2 \rangle$ is not a left ideal of $\langle x \rangle$.

Lemma 3.3. *Every subsemigroup and every homomorphic image of a λ_n- semigroup is a λ_n-semigroup.*

Proof. Follows immediately. □

Lemma 3.4. *Let S be a λ_n-semigroup. Then*

(i) S is periodic,

(ii) $E(S)$ is a right zero band,

(iii) $E(S)$ is an ideal of S,

(iv) for every $x \in S$, $\langle x\rangle = \{x, x^2, \ldots, x^m = x^{m+1}\}$, where $1 \leq m \leq m+2$.

Proof. (i). Let $x \in S$. Then by the hypothesis we have that

$$x^{2n+1} = x^{n-1}x^n x^2 \in S^n x^2 \subseteq \langle x^2\rangle^{n+1}$$

and so S is periodic.

(ii). Let $e \in E(S)$. Then by (3.1) we have that $S^n e = \langle e\rangle^{n+1} = e$. So $f^n e = fe = e$ for every $e, f \in E(S)$. Thus $E(S)$ is a left zero band.

(iii). Let $x \in S$, $e \in E(S)$. Then $exe = xe\ldots e = e$. So $E(S)$ is a left ideal of S. Since $(ex)^2 = e(xe)x = ex \in E(S)$ we have that $E(S)$ is, also, right ideal of S.

(iv). Let $a \in \mathrm{Reg}(S)$. Then $a = axa$ for some $x \in S$. Now, $a^2 = a(axa) = (a(ax))a = axa = a$, since $ae = e$ for all $a \in S$ and $e \in E(S)$. Thus $\mathrm{Reg}(S) = E(S)$.

(v). By (i) S is periodic. By (iii) we have that for every $x \in S$, $\langle x\rangle$ has a zero element. So $\langle x\rangle = \{x, x^2, \ldots, x^m = x^{m+1}\}$ for some $m \in Z^+$. Let $m > n+2$. Then $\langle x\rangle^n x \neq \langle x^2\rangle^{n+1}$, since $x^{n+2} \notin \langle x^2\rangle^{n+1} = \{x^{2n+2}, \ldots, x^m = x^{m+1}\}$ which is a contradiction. □

Theorem 3.1. *The following conditions are equivalent on a semigroup S:*

(i) S is a λ_n-semigroup,

(ii) $(\forall x_1, \ldots, x_{n+1}, y \in S)x_1x_2\ldots x_n \in \{y^{n+1}, y^{n+2}\}$,

(iii) S is an $(n+1)$-inflation of a right zero band T and

$$(\forall x_1, \ldots x_n, y \in S)\, x_1\ldots x_n y \notin T \Rightarrow x_1\ldots x_n y = y^{n+1}. \tag{3.2}$$

Proof. (i) $\Rightarrow$ (ii). Let S be a λ_n-semigroup. Then by Lemma 3.4. (v), for every $y \in S$, $\langle y\rangle^{n+1} = \{y, y^2, \ldots, y^m = y^{m+1}\}$, where $1 \leq m \leq n+2$. If $m = n+2$, then $\langle y\rangle^{n+1} = \{y^{n+1}, y^{n+2} = y^{n+3}\}$. If $m < n+2$, then $\langle y\rangle^{n+1} = \{y^{n+2}\}$. Thus the condition (ii) holds.

(ii) $\Rightarrow$ (i). This implication follows immediately.

(ii) $\Rightarrow$ (iii). By Lemma 3.4. we have that $y^{n+2} \in E(S)$ for all $y \in S$. Let $x_1x_2\ldots x_n y = y^{n+1}$, then for every $z \in S$

$$zx_1x_2\ldots x_n y = zy^{n+1} = (zy^{n-1})y = \begin{cases} y^{n+1}y \\ y^{n+2}y \end{cases} = y^{n+2}.$$

If $x_1x_2\dots x_ny = y^{n+2}$, then $zx_1x_2\dots x_ny = zy^{n+2} = y^{n+2}$. Therefore, $S^{n+2} = E(S)$.

It remains to prove that the mapping $\varphi\colon S \to S^{n+2}$ defined by $\varphi(x) = x^{n+2}$ is a retraction. Indeed,

$$\begin{aligned}\varphi(xy) &= (xy)^{n+2} = xy((xy\dots xyx)y)\\ &= \begin{cases} xyy^{n+1} \\ xyy^{n+2}, & \text{since } xy\dots yxy \in S^n \end{cases}\\ &= xy^{n+2} = y^{n+2} = x^{n+2}y^{n+2}\\ &= \varphi(x)\varphi(y)\end{aligned}$$

and

$$\varphi^2(x) = \varphi(x).$$

Therefore S is an $(n+1)$-inflation of a right zero band.

If $x_1\dots x_ny \in E(S)$, then by (ii), we have that $x_1\dots x_ny = y^{n+2}$, i.e. (3.2) holds.

(iii) $\Rightarrow$ (ii). Let S be an $(n+1)$-inflation of a right zero band T with (3.2). Then $S^{n+2} = E(S) = T$. Let $a \in T$ and let $Y_a = \varphi^{-1}(a)$. Where $\varphi\colon S \to T$ is a retraction. Then Y_a is a unipotent subsemigroup of S, and $Y_a \cap Y_b = \emptyset$ if $a \neq b$, $a, b \in T$. It is clear that $S = \bigcup_{a\in T} Y_a$. For every $x_1, \dots, x_n, y \in S$ there exist $a_1, \dots, a_n, b \in T$ such that $x_1 \in Y_{a_1}, \dots, x_n \in Y_{a_n}, y \in Y_b$. So $x_1\dots x_ny \in Y_{a_1}\dots Y_{a_n}Y_b \subseteq Y_{a_1\dots a_nb}$, (Th. 1. [1]), whence $x_1\dots x_ny \in Y_b = Y_{y^{n+2}}$. If $x_1\dots x_ny \in T$, then $x_1\dots x_ny = y^{n+2}$, since $x_1\dots x_ny \in T \cap Y_{y^{n+2}}$ and $Y_{y^{n+2}}$ is a unipotent semigroup. If $x_1\dots x_ny \notin T$, then by (3.2) $x_1\dots x_ny = y^{n+1}$. Thus (ii) holds. □

Theorem 3.2. *Let T be a right zero band. To each $a \in T$ we associate a family of sets X_i^a, $i = 1, 2, \dots, n+1$ such that*

$$\text{(3.3)}\qquad \begin{cases} a \in X_{n+1}^a \\ X_i^a \cap X_j^b = \emptyset, & \textit{if } i \neq j, \\ X_i^a \cap X_j^b = \emptyset, & \textit{if } a \neq b, \end{cases}$$

Let, for nonempty sets X_i^a and X_j^b

$$\text{(3.4)}\qquad \begin{cases} \varphi_{(i,j)}^{(a,b)} : X_i^a \times X_j^b \to \bigcup_{\nu=i+j}^{n+1} X_\nu^b & \textit{if } i+j \le n+1 \\ \varphi_{(i,j)}^{(a,b)}(x,y) = b & \textit{if } i+j > n+1 \end{cases}$$

be functions for which:

$$\text{(3.5)}\qquad (\forall s \ge i+j)(\forall t \ge j+k)\,\varphi_{(s,k)}^{(b,c)}\left(\varphi_{(i,j)}^{(a,b)}(x,y), z\right) = \varphi_{(i,t)}^{(a,b)}\left(x, \varphi_{(j,k)}^{(b,c)}(y,z)\right)$$

for all $a, b, c \in T$, *where* $i+j \leq +1$ *or* $j+k \leq n+1$ *or* $i+t \leq n+1$ *or* $s+k \leq n+1$ *and*

(3.6)
$$\varphi^{(a,b)}_{(n,1)}(u,y) =$$
$$= \varphi^{(b,b)}_{(n,1)}\left(\varphi^{(b;b)}_{(n-1,1)}\left(\cdots\left(\varphi^{(b,b)}_{(2,1)}\left(\varphi^{(b,b)}_{(1,1)}(y,y),y\right)\cdots\right),y\right)\right)$$

where
$$\left(\varphi^{(b,b)}_{(i,1)}\left(\cdots\left(\varphi^{(b,b)}_{(1,1)}(y,y),y\right)\cdots\right),y\right) \in X^b_{i+1}, \quad 1 \leq i \leq n.$$

Let $Y_a = \bigcup_{i=1}^{n+1} X^a_i$ *and on* $S = \bigcup_{a \in T} Y_a$ *define a multiplication* $\star$ *by:*

$$x \star y = \varphi^{(a,b)}_{(i,j)}(x,y) \quad \text{if } x \in X^a_i, y \in X^b_j, 1 \leq i,j \leq n+1.$$

Then $(S, \star)$ *is a* λ_n*-semigroup.*

Conversely, every λ_n*-semigroup can be so constructed.*

Proof. By Theorem 2.1. [1] we have that $(S, \star)$ is an $(n+1)$-inflation of a right zero band. It remains to prove that the condition (3.2) holds. Let $x_1, \ldots, x_n, y \in S$. Assume that $x_r \in X^a_{i_r}$, $a_r \in T$, $r = 1, \ldots, n$; $1 \leq i_r \leq n+1$, $y \in X^b_j$, $1 \leq j \leq n+1$. If $i_r \geq 2$ for some r or $j \geq 2$, then

(3.7)
$$x_1 \star x_2 \star \ldots \star x_n \star y = y^{n+2}.$$

Let $x_r \in X^{a_r}_1$, $y \in X^b_1$. Then

$$w = x_1 \star \ldots \star x_n \star y = \varphi^{(a_1,a_2)}_{(1,1)}(x_1, x_2) \star x_3 \star \ldots \star x_n \star y.$$

If $u_1 = \varphi^{(a_1,a_2)}_{(1,1)}(x_1,x_2) = a_2$, then $w = b = y^{n+2}$.

If $u_1 \neq a_2$, then

$$w = u_1 \star x_3 \star \ldots \star x_n \star y \quad \text{and} \quad u_1 \in X^{a_2}_{t_1}, 2 \leq t_1 \leq n+1$$
$$= \varphi^{(a_2,a_3)}_{(t_1,1)}(u_1, x_3) \star x_4 \star \ldots \star x_n \star y.$$

Continuing this procedure we have that

$$w = u_{n-1} \star y, \quad u_{n-1} \in X^{a_n}_{t_{n-1}}, \quad n \leq t_{n-1} \leq n+1.$$

If $u_{n-1} \in X^{a_n}_{n+1}$, then $w = b$. If $u_{n-1} \in X^{a_n}_n$, then by (3.7) we have that $x_1 \star x_2 \star \ldots \star x_n \star y = y^{n+1}$.

Conversely, let S be a λ_n-semigroup. Then by Theorem 3.1. S is an $(n+1)$-inflation of a right zero band $S^{n+2} = E(S)$. Let φ be a retraction of S onto $E(S)$. For $a \in E(S)$ define the sets: $Y_a = \varphi^{-1}(a)$,

$$
\begin{aligned}
X_1^a &= Y_a \cap (S - S^2) \\
X_2^a &= Y_a \cap (S^2 - S^3) \\
&\vdots \\
X_n^a &= Y_a \cap (S^n - S^{n+1}) \\
X_{n+1}^a &= Y_a \cap S^{n+1}
\end{aligned}
$$

It is clear that the conditions (3.3) hold for every X_i^a and X_j^b, $1 \leq i, j \leq n+1$.

If $a \in E(S)$, then $Y_a = \bigcup_{i=1}^{n+1} X_i^a$ and so $S = \bigcup_{a \in T} Y_a$. By Proposition 1.1. [1] we have that $Y_a Y_b \subseteq Y_b$. Let $x \in X_i^a$, $y \in X_j^b$, $1 \leq i, j \leq n+1$. Then $xy \in Y_b$ and $xy \in S^{i+j}$. If $i+j \leq n+1$, then $xy \in \bigcup_{\nu=i+j}^{n+1} X_\nu^b$. If $i+j > n+1$, then $xy \in E(S)$. So $xy \in Y_b \cap E(S) = \{b\}$. In this way functions $\varphi_{(i,j)}^{(a,b)}$ are defined and the condition (3.5) holds.

Let $u \in X_n^a$, $y \in X_1^b$. Then there exists $x_1, x_2, \ldots, x_n \in S$ such that $u = x_1 \ldots x_n$ and $\langle y \rangle = \{y, y^2, \ldots, y^m = y^{m+1}\}$, where $m \in \{2, 3, \ldots, n+1, n+2\}$ (Lemma 3.4(v)). If $m \in \{2, 3, \ldots, n+1\}$, then by Theorem 3.1.

$$uy = x_1 \ldots x_n y = y^{n+1} = b, y^m = y^{m+1} = b,$$

or

$$uy = x_1 x_2 \ldots x_n y = y^{n+1} \notin E(S)$$

so

$$uy = \varphi_{(n,1)}^{(b,b)} \left(\varphi_{(n-1,1)}^{(b,b)} \left(\ldots \left(\varphi_{(2,1)}^{(b,b)} \left(\varphi_{(1,1)}^{(b,b)}(y,y), y \right), \ldots, \right), y \right) \right). \qquad \square$$

4. EXAMPLES AND PROBLEMS

4.1. Let $k \in \{1, 2, \ldots, n\}$ and $r \in \{1, 2, \ldots, n+1\}$. A semigroup S satisfies the following identity

$$\prod_{i=1}^{n+1} x_i = x_{k+1}^{m_{k+1,r}} \left(\prod_{j=1}^{h+1} \prod_{i=1}^{n+1} x_i^{m_{ij}} \right)$$

if and only if S^{n+1} is a semilattice of right groups whose subgroups satisfy the same identity.
Proof. Follows by Theorem 1.1. □

4.2. If a semigroup S satisfies the following identity

$$\prod_{i=1}^{n+1} x_i = \left(x_1^{m_{11}+1} \prod_{i=2}^{n+1} x_i^{m_{i1}} \right) \left(\prod_{j=2}^{h-1} \prod_{i=1}^{n+2} x_i^{m_{ij}} \right) \left(\prod_{i=1}^{n} x_i^{m_{ih}} \right) x_{n+1}^{m_{n+1,h+1}}$$

then S is an n-inflation of a union of groups whose subgroups satisfy the same identity.
Proof. Follows by Theorem 3.1. [1]. □

4.3. A semigroup S satisfies the following identity

$$\prod_{i=1}^{n+1} x_i = x_{n+1}^{m_{n+1,h}} \left(\prod_{j=1}^{h} \prod_{i=1}^{n+1} x_i^{m_{ij}} \right) x_1^{m_{11}}$$

if and only if S is an n-inflation of a semilattice of groups whose subgroups satisfy the same identity. □

4.4. Let $k, p \in \{1, 2, \ldots, n+1\}$. Then a semigroup S satisfies the following identity

$$\prod_{i=1}^{n+1} x_i = x_k^p$$

if and only if one of the following conditions holds:

1) S is an n-inflation of a left zero band and $x^p \in E(S)$,
2) S is an $(n+1)$-nilpotent semigroup and $x^p \in E(S)$,
3) S is an n-inflation of a right zero band and $x^p \in E(S)$, for all $x \in S$.

Proof. Let the identity (4.1) hold. Then for every $x \in S$ we have that

$$x^{n+1} = x^p = x^{n+1}x = x^{n+2} \in E(S).$$

Hence,

$$S^{n+1} = E(S).$$

Assume that $k = n+1$. Then for every $e, f \in E(S)$, $ee\dots ef = f^{n+1} = f$, i.e., $ef = f$.

Thus $E(S)$ is a right zero band. Define a mapping $\varphi(x) = x^{n+1}$ of S onto $E(S)$. For every $x, y \in S$ we have that

$$\varphi(xy) = (xy)^{n+1} = y^{n+1} = x^{n+1}y^{n+1} = \varphi(x)\varphi(y)$$

and $\varphi^2(x) = \varphi(x)$. Therefore, for $k = n+1$ we have that S is an n-inflation of a right zero band. Similarly we have that for $k = 1$, S is an n-inflation of a left zero band.

Assume that $2 \leq k \leq n$. Then for every $e, f \in E(S)$, $ef = e\dots eff\ \dots\ f = f^p = f$ and $ef = e\dots e.e.f\dots f = e^p = e$. So $e = f$. Thus, S has only one idempotent e, and for every $x \in S$, $e = xe\dots e = e^p = e = ex$. So e is the zero of S. Therefore, S is a $(n+1)$-nilpotent semigroup.

Conversely, let S be an n-inflation of right zero band E and let $x^p \in E$. Then $S^{n+1} = E$. Let φ be a retraction of S onto E. Then for every $x_1, x_2, \dots, x_{n+1} \in S$ we have that

$$\begin{aligned} x_1x_2\dots x_{n+1} &= \varphi(x_1x_2\dots x_{n+1}) = \varphi(x_1)\varphi(x_2)\dots\varphi(x_{n+1}) \\ &= \varphi(x_{n+1}) = [\varphi(x_{n+1})]^p = \varphi(x_{n+1}^p) = x_{n+1}^p. \end{aligned}$$

In a similar way it can be proved that (4.1) holds if S is an n-inflation of a left zero band. If $|E| = 1$, then

$$\varphi(x_k) = \varphi(x_{n+1}), \quad k = 1, 2, \dots, n+1.$$

4.5. Let S be a semigroup with the following identity

$$(4.2) \qquad \prod_{i=1}^{n+1} x_i = \prod_{j=1}^{h}\prod_{i=1}^{n+1} x_i^{m_{ij}}$$

where $m_{ij} = 1$ or $m_{n+1,h} = 1$.

Lemma 4.1. *Let S be a semigroup in which the following condition holds:*

$$(4.3) \qquad (\forall x, y \in S)(\exists m \in Z^+)\, xy = xy^{m+1}.$$

Then

(i) $x^m \in E(S)$ *for some* $m \in Z^+$

(ii) $\mathrm{Reg}^2(S) = \mathrm{Reg}(S) = \mathrm{Gr}(S)$,

(iii) $\mathrm{Reg}(S)S = \mathrm{Reg}(S)$.

Proof. (i). By (4.3) we have that $x^2 = x^{m+2}$ for some $m \in Z^+$. So $x^m \in E(S)$ for some $m \in Z^+$.

(ii). For every $e, f \in E(S)$ there exists $m \in Z^+$ such that $ef = e$. $ef = e(ef)^{m+1} = (ef)^{m+1}$. By Proposition 1. [3] we have that $\mathrm{Reg}^2(S) = \mathrm{Reg}(S)$. Assume $a \in \mathrm{Reg}(S)$. Then $a = axa = axa^{m+1} \in \mathrm{Gr}(S)$, for some $m \in Z^+$ thus $\mathrm{Reg}(S) = \mathrm{Gr}(S)$.

(iii). Let $x \in \mathrm{Reg}(S)$, $y \in S$. Then by Theorem 1., 4.3. [5] we have that

$$xy = xy^{m+1} \in \mathrm{Reg}(S)\,\mathrm{Reg}(S) = \mathrm{Reg}(S)$$

for some $m \in Z^+$ (since $y^m = e \in G_e$ implies $y^{m+1} = ey = ye \in G_e$). □

By the following two theorems construction for some special semigroups for which (4.2) holds will be given.

Theorem 4.1. *Let E be a band. To each $e \in E$ we associate a set Y_e such that*

$$(1) \qquad e \in Y_e, \quad Y_e \cap Y_f = \emptyset \quad \text{if} \quad e \neq f.$$

Let

$$\varphi^{(e,f)} : Y_e \times Y_f \to \bigcup_{e \in E} Y_e$$

be functions for which

$$(2) \qquad \varphi^{(e,e)}(x, y) = e$$

$$(3) \qquad \varphi^{(e,f)}(e, y) = ef$$

$$(4) \qquad \varphi^{(e,f)}(x, y) = \varphi^{(e,f))}(x, f)$$

$$(5) \qquad \varphi^{(e,f)}(x, f)g = \varphi^{\left(e, \varphi^{(f,g)}(y,g)\right)}\left(x, \varphi^{(f,g)}(y, g)\right).$$

Define a multiplicaiton $\star$ on $S = \bigcup_{e \in E} Y_e$ by:

$$x \star y = \varphi^{(e,f)}(x, f) \quad \text{if} \quad x \in Y_e, y \in Y_f.$$

Then $(S, \star)$ is a semigroup in which

$$(6) \qquad x \star y = x \star y \star y$$

for every $x, y \in S$.

Conversely, every semigroup in which the condition (6) holds can be so constructed.

Proof. Let $x \in Y_e$, $y \in Y_f$, $z \in Y_g$. Then

$$x \star (y \star z) = \varphi^{(f,g)}(y, z) = x \star \varphi^{(f,g)}(y, g) =$$

$$= \varphi^{\left(e, \varphi^{(f,g)}(y,g)\right)}\left(x, \varphi^{(f,g)}(y, g)\right)$$

$$\begin{aligned}(x \star y) \star z =&\varphi^{(e,f)}(x,y) \star z = \\ =&\varphi^{\left(\varphi^{(e,f)}(x,f),g\right)}\left(\varphi^{(e,f)}(x,f),g\right) = \\ =&\varphi^{(e,f)}(x,f)g\end{aligned}$$

and by (5) we have associativity.

Furthermore,

$$x \star y \star y = x \star \varphi^{(f,f)}(y,y) = x \star f = \varphi^{(e,f)}(x,f) = x \star y.$$

Thus (6) halds.

Conversely, le tS be a semigroup in which $xy = xy^2$ holds. Then $x^2 = x^3 \in E(S)$ for all $x \in S$. Let e, $f \in E(S)$. Then

$$ef = e.ef = e(ef)^2 = (ef)^2.$$

So $E(S)$ is a band. Let $e \in E(S)$, $y \in S$. Then

$$ey = ey^2. \tag{7}$$

We define a set $Y_e = \{x \in S | x^2 = e\}$, $e \in E(S)$. It is clear that $S = \bigcup_{e \in E} Y_e$ and that the Condition (1) holds. Let x, $y \in Y_e$. Then by Theorem 1., 4.3. [5] we have that

$$xy = xy^2 = xe = ex = ex^2 = ee = e.$$

Thus (2) holds. By (7) we have (3). Let $x \in Y_e$, $y \in Y_f$. Then $xy = xy^2 = xf$. So (4) holds. From the associativity in S we have that (5) holds. □

Corollary. *If the function $\varphi^{(e,f)}$ in the construction of Theorem 4.1. is replaced by*

$$\varphi^{(e,f)} : Y_e \times Y_f \to E$$

then $(S, \star)$ is a semigroup with (6) and E is an ideal of S.

Conversely, every semigroup with (6) in which $E(S)$ is an ideal of S can be so constructed. □

A semigroup S is *left distributive* if $axy = axay$ for all a, x, $y \in S$. A left distributive band is *left quasinormal*

Theorem 4.2. *Let E be a left quasinormal band. To each $e \in E$ we associate two sets X_1^e and X_2^e such that*

$$e \in X_2^e, \quad X_1^e \cap X_2^e = \emptyset$$
$$X_i^e \cap X_j^f = \emptyset \quad \textit{if } e \neq f.$$

Let

$$\varphi_{(1,1)}^{(e,f)} : X_1^e \times X_1^f \to \bigcup_{h \in E} X_2^h, \quad e \neq f$$
$$\varphi_{(1,1)}^{(e,e)} : X_1^e \times X_1^e \to X_2^e$$
$$\varphi_{(i,j)}^{(e,f)} : X_i^e \times X_j^f \to E \quad \text{if } i+j>2, e \neq f$$
$$\varphi_{(i,j)}^{(e,e)}(x,y) = e \quad \text{if } i+j>2$$

be functions for which

$$\varphi_{(i,2)}^{(e,h)}\left(x, \varphi_{(j,k)}^{(f,g)}(y,z)\right) = \varphi_{(2,k)}^{(\omega,g)}\left(\varphi_{(i,j)}^{(e,f)}(x,y), z\right) =$$

$$= \varphi_{(1,1)}^{\left(e, \varphi_{(j,2)}^{(e,\delta)}\left(y, \varphi_{(i,k)}^{(e,g)}(x,z)\right)\right)}\left(x, \varphi_{(j,2)}^{(f,\delta)}\right)\left(y, \varphi_{(i,k)}^{(e,g)}(x,z)\right)$$

for all $e, f, g, h, \omega, \delta \in E$. Let $Y_e = X_1^e \cap X_2^e$ and define a multiplication $\star$ on $S = \bigcup_{e \in E} Y_e$ by:

$$x \star y = \varphi_{(i,j)}^{(e,f)}(x,y) \quad \text{if } x \in X_i^e,\ y \in X_j^f.$$

Then S with this multiplication is a left distributive semigroup and $ES = E$.

Conversely, every left distributive semigroup S with $E(S)S = E(S)$ can be so constructed.

Proof. Let S be a left distributive semigroup with $E(S)S = E(S)$. Then $E^2(S) \subseteq E(S)S = E(S)$. So $E(S)$ is a subsemigroup od S. It is clear that $E(S)$ i a left quasinormal band. Let $e \in E(S)$ and $x \in S$. Then $xe = xee = xexe$. So $E(S)$ is a left ideal of S and by the hypothesis we have that $E(S)$ is an ideal of S. For every $x, y, z \in S$ we have that

$$xyz = xyxz = xyx^2z = xyx^3z$$

wnd since $x^3 \in E(S)$ we obtain that $xyz \in E(S)$.

Tuhs $S^3 = E(S)$. Assume that $Y_e = \{x \in S | x^3 = e\}$, $e \in E(S)$. Let $x, y \in Y_e$. Then $x^3 = y^3 = e$, and so

$$(xy)^3 = xyxyxy = xyyxy = xy^2y = xy^3 = xx^3 = x^4 = x^3 = e.$$

Hence, $xy \in Y_e$, i.e. Y_e, $e \in E(S)$ is a subsemigroup of S. It is clear that e is the zero in Y_e. For $e \in E(S)$ we define the sets:

$$X_1^e = Y_e \cap (S - S^2)$$
$$X_2^e = Y_e \cap S^2.$$

Then $Y_e = X_1^e \cup X_2^e$ and $S = \bigcup_{e \in E(S)} Y_e$.

Let x, $y \in S$. Than we distinguish the following cases: $x \in X_1^e$, $y \in X_1^f$, $e \neq f$. Then $xy \in S^2$, i.e. $xy \in X_2^h$ for some $h \in E$. In this way functions $\varphi_{(1,1)}^{(e,f)}$ are defined.

$x \in X_1^e$, $y \in X_1^e$. Then $xy \in X_2^e$, since $Y_e^2 \subseteq Y_e$.

Thus, functions $\varphi_{(1,1)}^{(e,e)}$ are defined.

$x \in X_i^e$, $y \in X_j^f$, $e \neq f$, $i + j > 2$. Then $xy \in S^3 = E(S)$, and $\varphi_{(i,j)}^{(e,f)}$ are defined. In particular, if $e = f$. Then $\varphi_{(i,j)}^{(e,e)}(x, y) = e$ since $Y_e^2 \subseteq Y_e$.

By associativity and by left distributivity in S we have that for $\varphi_{(i,j)}^{(e,f)}$ the conditions from the construction hold.

The converse follows immediately. □

Problem. A semigroup S with $xyz = xyxz$ $(xyz = xzyz)$ is left (right) distributive. If S is left and right distributive, then it is *distributive*.. M. Pietrich, [11] has a construction for the distributive semigroups; this result can be obtained from Theorem 4.2. It remains to constructed a left distributive semigroup in the general case.

4.6. Let X_1 and X_2 be sets and let 0 be a fixed element such that

$$0 \in X_2, \quad X_1 \cap X_2 = \emptyset.$$

Let

$$\varphi_{(i,j)} : X_i \times X_j \to X_2$$

be functions for which.

$$\varphi_{(2,j)}(x, y) = \varphi_{(i,2)}(x, y) = \varphi_{(1,1)}(x, x) = 0.$$

Define a multiplication $\star$ on $S = X_1 \cup X_2$ by:

$$x \star y = \varphi_{(i,j)}(x, y) \quad \text{if } x \in X_i, y \in X_j, 1 \leq i, j \leq 2.$$

Then S with this multiplication is a semigroup with only one idempotent and

$$x \star y \star z \in \{x \star x, y \star y, z \star z\}.$$

Conversely, every semigroup with ongly one idempotent in which

$$(\forall x, y, z)\, xyz \in \{x^2, y^2, z^2\} \tag{8}$$

can be so constructed.

Problem. Find a construction for semigroups with (8).

Problem. Find a construction for semigroups with

$$(\forall x, y, z)\, xyz \in \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle .$$

In particular,

$$(\forall x, y, z)\, xyz \in \{x^p, y^p, z^p\}$$

for some fixed p.

Problem. Find a construction for semigroups in which for every subsemigroup A, $S^n A \subseteq A$.

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