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GEOMETRY OF EXCEPTIONAL WEBS EW(4, 2, 2) OF MAXIMUM 2-RANK

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1. PRELIMINARIES

1.1. Definitions

A d-web $W(d, n, r)$ of codimension r is given in an open domain D of a differentiable manifold M^{nr} of dimension nr by d foliations X_a, $a = 1, \dots, d$, of codimension r if leaves (web surfaces) of X_a through a point $w \in D$ are in general position.

Two webs $W(d, n, r)$ and $\widetilde{W}(d, n, r)$ with domains $D \subset M^{nr}$ and $\widetilde{D} \subset \widetilde{M}^{nr}$ are *equivalent* if there exists a local diffeomorphism ϕ: $D \to W(d, n, r)$ D such that $\phi(X_a) = W(d, n, r)X_a$, $a = 1, \dots, d$.

Suppose that a $d$web $W(d, n, r)$ is defined in some open domain $D \subset X^{nr}$ by the Pfaffian equations

$$\underset{a}{\omega}^1 = 0, \dots, \underset{a}{\omega}^r = 0, \quad a = 1, \dots, d.$$

An *obelian* q*-equation* is a relation of the form

$$\sum_{a-1}^{d} f_{ai_1 \dots i_q} \underset{a}{\omega}^{i_1} \wedge \dots \wedge \underset{a}{\omega}^{i_q} = 0, \quad q = 1, 2, \dots, r,$$

where

$$df_{ai_1 \dots i_q} \equiv 0 \bmod \{\underset{a}{\omega}^1, \dots, \underset{a}{\omega}^r\}.$$

The last condition means that the functions $f_{ai_1 \dots i_q}$ are constant on leaves $V_a \subset X_a$.

The q-rank R_q of a $W(d, n, r)$ is the maximum number of linearly independent abelian q-equations admitted by the $W(d, n, r)$ (see [1]).

The important problems are:

1) To determine an upper bound on R_q.

2) To describe webs $W(d, n, r)$ of maximum q-rank .

Let P^{r+n-1} be a projective space of dimension $r + n - 1$, $G(n - 1, r + n - 1)$ be the Grassmann manifold of its $(n - 1)$-dimensional subspaces P^{n-1}, and let $\Sigma(x)$ be the Schubert variety of its $(n - 1)$-dimensional subspaces that pass through a point $x \in P^{r+n-1}$. A smooth manifold $V \subset P^{r+n-1}$ of dimension r determines in some domain $\widetilde{U} \subset G(n - 1, r + n - 1)$ a foliation of codimension r whose leaves are $\Sigma(x)$, $x \in V$. If V_a, $a = 1, \dots, d$, $d \geq n+1$, are d given smooth manifolds of P^{r+n-1}, then in some domain $\widetilde{D} \subset P^{r+n-1}$ they determine a d-web of codimension r. Such a web $W(d, n, r)$ is said to be a *Grassmann* web. We denote it by $GW(d, n, r)$.

A Grassmann d-web is said to be *algebraic* if the varieties V_a generating it belong to an algebraic variety V_d^r of dimension r and degree d. Such a web will be denoted by $AW(d, n, r)$.

A web $W(d, n, r)$ which is equivalent to $GW(d, n, r)$ (resp. $AW(d, n, r)$) is called *Grassmannizable* (resp. *algebraizable*).

Let us consider now an $(n+1)$-web $W(d, n, r)$ on a manifold M^{nr}. Let $T_p(M^{nr})$ be the tangent space of M^{nr} at p. Its subspaces $T_p(V_u)$, $u = 1, \ldots, n+1$, tangent to leaves V_u through a point $p \in M^{nr}$ define in $T_p(M^{nr})$ a Segre cone $C_p(r, n)$ with the vertex p. Note that the projectivization of $C_p(r, n)$ with the vertex p. Note that the projectivization of $C_p(r, n)$ is a Segre manifold $S(r-1, n-1) = P^{r-1} \times P^{n-1}$. A Segre cone $C_p(r, n)$ carries two families of flat generators $\xi_p(r)$ and $\zeta_p(n)$ of dimension r and n respectively. The field of Segre cones defines on M^{nr} an *almost Grassmann structure* $AG(n-1, r+n-1)$ associated with $W(d, n, r)$. Its structure group is the group $G = GL(r) \times SL(n)$ of transformations of $T_p(M^{nr})$. The cone $C_p(r, n)$ is invariant under transformations of G [2].

An almost Grassmann structure $AFG(n-1, r+n-1)$, is *r-semi- integrable* (resp. *n-semi-integrable*) if there exists a family of r-dimensional surface W^r (resp. W^n) on M^{nr} such that $T_p(W^r) = \xi_p(r)$ (resp. $T_p(W^n)$ W^r (resp. W^n) [2].

If the structure $AG(n-1, r+n-1)$ associated with $W(n-1, n, r)$ is r-semi- integrable (resp. n-semi-integrable), the web $W(n-1, n, r)$ is *isocline* (resp. *transversally geodesic*) (see [2], [3]).

If $AG(n-1, r+n-1)$ is both $r-$ and n-semi-integrable, and only in this case, the structure $AG(n-1, r+n-1)$ is locally *Grassmann* and the corresponding web $W(n-1, n, r)$ is *Grassmannizable* [2, 3].

Thus we see that any $(n+1)$-web $W(n-1, r+n-1)$ defines an almost Grassmann strucutre $AG(n-1, r+n-1)$.

A web $W(d, n, r)$, $d > n+1$, is *almost Grassmannizable* if all almost Grassmann structures defined by its $(n+1)$-subwebs coincide. We will denote sucha web by $AGW(d, n, r)$.

If $n = 2$, i.e. for $W(d, 2, r)$, the condition of almost Grassmannizability is that all its basis affinors $\underset{\alpha}{\lambda}{}^i_j$, $\alpha = 4, \ldots, d$; $i, l, \ldots, r$, are scalar: $\underset{\alpha}{\lambda}{}^i_j = \delta^i_j \underset{\alpha}{\lambda}$ (see [4]).

For almost Grassmannizable webs $AGW(d, 2, r)$ we can define the notions of isoclinity and transversal geodesicity as we did above for $W(n-1, r+n-1)$ because we have only one $AG(1, r+1)$ connected with $AGW(d, 2, r)$ (see [5], [6]).

If a web $AGW(d, 2, r)$ is isocline (but not transversally geodesic and therefore not Grassmannizable) and satisfies the condition $\sum\limits_{a=1}^{d} \underset{a}{k}_{ij} = 0$ (which is the condition of algebraizability for a Grassmannizable web), we will call such a web *almost algebraizable* and will denote it by $AAW(d, 2, r)$ (see [5], [6]).

1.2. Developments in the rank problems

During the intensive development of web geometry in the 30's these problems were considered for webs $W(d, n, 1)$ int he plane $(n = 2)$, 3-space $(n = 3)$, and n-space, sometimes for any $d > 3$, sometimes for some particular values of d (see [7]). SS. Chern in 1936 found the upper bound $\pi(d, n, 1)$ for the rank R_1 of $W(d, n, 1)$ for any n (see [8]). At that time he did not find webs $W(d, n, 1)$ of maximum rank.

Recently when S.S. Chern and P.A. Griffiths recognized that the same number $\pi(d, n, 1)$ is the maximum genus of a non-degenerate algebraic curve of degree d in a complex projective space P^n of dimension n, they used algebraic geometry results and proved (see [9], [10]) that so called «normal» webs $W(d, n, 1)$ of maximum rank $\pi(d, n, 1)$ are linearizable (equivalent to a web $W(d, n, 1)$ whose all leaves ar hyperplanes) and algebraizable (hyperplanes mentioned belong to an algebraic curve of degree d in the dual space).

In all mentioned papers concerning with the rank of webs the rank was considered with respect to abelian 1-equations. It was the only possibility because webs studied in the papers were webs of codimension one.

As it was mentioned, in the case of $W(d, n, 1)$, where $r > 1$, the rank can be defined with respect to abelian q-equations where $q = 1, 2, \ldots, r$.

Int he paper [11], S.S. Chern and P.A. Griffiths found the upper bound $\pi(d, 2, r)$ on the rank R_r and showed that webs of maximum r-rank problems for webs $W(d, 2, r)$:

i) *If* $r > 2$ *and* $d \leq r + 1$, *the* r-rank R_r *of a web* $AGW(d, 2, r)$ *is equal to 0.*

ii) *The maximum* 2-rank *of* $AGW(d, 2, r)$ *is equal to* $\pi(d, 2, 2) = \frac{1}{6}(d-1)(d-2)(d-3)$ (it matches the Chern and Griffiths result obtained in [11] for general $W(d, n, r)$)).

iii) *A web* $AGW(d, 2, 2)$, $d > 4$, *is of maximum* 2-rank *if and only if it is algebraizable.*

iv) *A web* $W(4, 2, 2)$ *is of maximum* 2-rank *one if and only if*

a) it is an almost algebraizable web $AAW(4, 2, 2)$ *or*

b) it is a non-isocline almost Grassmannizable web $AGW(d, 2, r)$ *for which the middle affine connection of the canonical affine connections induced by alla 3-subwebs of* $AGW(d, 2, r)$ *is equiaffine.*

These results show that the P.A. Griffith's conjecture that webs $W(d, 2, r)$ of maximum r-rank are algebraizable is true for webs $AGW(D, 2, 2)$, $d > 4$, and is not true for webs $W(4, 2, 2)$.

Because of this, it is natural to call webs $W(4, 2, 2)$ of maximum 2-rank *exceptional webs*. We will denote them by $EW(4, 2, 2)$.

1.3. Double fibrations and webs

A *double fibration* (abr. DF) is a diagram

$$\begin{array}{ccc} & Z & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ X_1 & & X_2 \end{array}$$

where Z, X_1 and X_2 are smooth manifolds and

a) Z is a smooth fibration with respect to π_1 and π_2,

b) $\pi_1 \times \pi_2 : Z \to X_1 \times X_2$ is a non-degenerate injective diffeomorphism,

c) for any $x_1, x_2 \in X_1, x_1 \neq x_2$ and $\xi_1, x_2 \in X_2, \xi_1 \neq \xi_2$ we have correspondingly $\pi_2 \cdot \pi_1^{-1} x_1 \neq \pi_2 \cdot \pi_1^{-1} x_2, \pi_1 \cdot \pi_2^{-1} \xi_1 \neq \pi_1 \cdot \pi_2^{-1} \xi_2$.

The ideal of DF can be found in the paper [12] of S.S. Chern. It was extensively used by I.M. Gelfand and his collabrorators in their works on integral geometry (see for example the paper [13]).

In [13] the authors write that «π_1 and π_2 enable us to carry the various analytical objects (functions, forms etc.) from X_1 to X_2 by first lifting them from X_1 to Z and subsequently descending them to X_2».

Any 3-web $W(3,2,r)$ defines an r-parameter family of DF. To get a DF, one has to take the first two foliation of $W(3,2,r)$ as X_1 and X_2, fix a leaf V_3 of 3rd foliation and consider Z as the manifold of pairs $(V_1, V_2), V_1 \subset X_1, V_2 \subset X_2, V_1 \cap V_2 = A \in V_3$. It is easy to see that $Z \to X_1 \times X_2$ is a smooth embedding. Note also that one gets more DF taking any pair of foliations of $W(d, 2, r)$ as X_1 and X_2.

For a d-web $W(d, 2, r), d > 3$, one can construct d-2 r-parameter families of DF.

In general for $W(d, n, r), d \geq n+1$, one can get d-n r-parameter families of n-fold fibrations taking the first foliations as $X_1, \ldots, X_n$.

In the present paper we will study geometry of exceptional webs $EW(4,2,2)$. In particular, some analytical objects (function, vector fields, forms) given on different foliations will be introduced. The program outlined in [13] can be applied to these objects. Note that all our considerations and results are of local nature.

2. EXCEPTIONAL WEBS EW(4, 2, 2) OF MAXIMUM 2-RANK

Let $W(4,2,2)$ be a four-web of codimension two given on a four-dimensional differentiable manifold M^4 by four foliations $X_a, a = 1,2,3,4$, of codimension two.

Suppose that foliations $X_a, a = 1,2,3,4$, are given by the following systems of Pfaffian equations:

$$\underset{a}{\omega}^i = 0, \quad a = 1,2,3,4; \quad i = 1,2, \tag{2.1}$$

where $\underset{1}{\omega}^i$ and $\underset{2}{\omega}^i$ are basis forms of the cotangent bundle $T_p^*(M^4)$ at point $p \in M^4$. Then as is shown by the first author (see [14] or [15]) the forms $\underset{3}{\omega}^i$ and $\underset{4}{\omega}^i$ are expressed in terms of $\underset{1}{\omega}^i$ and $\underset{2}{\omega}^i$ as follows:

$$-\underset{3}{\omega}^i = \underset{1}{\omega}^i + \underset{2}{\omega}^i, \quad -\underset{4}{\omega}^i = \lambda^i_j \underset{1}{\omega}^j + \underset{2}{\omega}^i, \tag{2.2}$$

$$\det(\lambda^i_j) \neq 0, \quad \det \delta^i_j - \lambda^i_j) \neq 0 \tag{2.3}$$

where λ^i_j is the *basis affinor* of $W(4,2,2)$.

We intend to consider webs $W(4,2,2)$ of maximum 2-rank. As we mentioned in Section 1, such webs are always almost Grassmannizable and this implies [4]

$$\lambda^i_j = \delta^i_j \lambda. \tag{2.4}$$

Therefore

$$-\underset{4}{\omega}^i = \lambda \underset{1}{\omega}^i + \underset{2}{\omega}^i. \tag{2.5}$$

For a point $p \in D \subset M^4$ we have

$$dp = \underset{1}{\omega}^i e^1_i + \underset{2}{\omega}^i e^2_i \tag{2.6}$$

where $\{e^1_i, e^2_i\}$ is a basis in the tangent bundle $T_p(M^4)$ at p.

It follows from (2.6), (2.1) and (2.5) that at p the vectors e^1_i and e^2_i are tangent to the leaves $V_1 \subset X_1$ and $V_2 \subset X_2$ and the vectors $e^1_i - e^2_i$ and $e^1_i - \lambda e^2_i$ are tangent to the leaves $V_3 \subset X_3$ and $V_4 \subset X_4$.

It was shown in [5], [6] that for an isocline almost Grassmannizable web $AGW(4,2,2)$ we have the following equations:

$$\begin{cases} d\underset{1}{\omega}^i = \underset{1}{\omega}^j \wedge \omega^i_j + a_j \underset{1}{\omega}^j \wedge \underset{1}{\omega}^i, \\ d\underset{2}{\omega}^i = \underset{2}{\omega}^j \wedge \omega^i_j + a_j \underset{2}{\omega}^j \wedge \underset{2}{\omega}^i, \\ d\omega^i_j - \omega^k_j \wedge \omega^i_k = b^i_{jk\ell} d\underset{1}{\omega}^k \wedge \underset{2}{\omega}^\ell, \\ d\lambda = \lambda(b_i - a_i)\underset{1}{\omega}^i + (b_i - \lambda a_i)\underset{2}{\omega}^i, \end{cases} \tag{2.7}$$

$$(2.8)\qquad \begin{cases} da_i - a_j\omega^i_j = (\underset{1}{k}_{ij} - \underset{3}{k}_{ij})\underset{1}{\omega}^j + (\underset{2}{k}_{ij} - \underset{3}{k}_{ij})\underset{2}{\omega}^j, \\ db_i - b_j\omega^i_j = [b_i(b_j - a_j) + \lambda(\underset{1}{k}_{ij} - \underset{4}{k}_{ij})]\underset{1}{\omega}^j + (\underset{2}{k}_{ij} - \underset{4}{k}_{ij})\underset{2}{\omega}^j \end{cases}$$

where $\underset{a}{k}_{ij}$, $a = \{,2,3,4$, are symmetric in i and j and the quantities

$$(2.9)\qquad a^i_{jk} = a\;\delta^i_{[jk]},$$

$$(2.10)\qquad b^i_{jk\ell} = a^i_{jk\ell} + \underset{1}{k}_{jk}\delta^i_\ell + \underset{2}{k}_{\ell j}\delta^i_k + \underset{3}{k}_{k\ell}\delta^i_j$$

are the *torsion* and *curvature tensor* of $AGW(4, 2, 2)$. The quantities $a^i_{jk\ell}$ in (2.10) are symmetric in j, k, ℓ and satisfy the relation $a^i_{ik\ell} = 0$.

For a non-isocline almost Grassmannizable web $AGW(4, 2, 2)$ we have equations (2.7) and

$$(2.11)\qquad \begin{cases} da_i - a_j\omega^j_i = p_{ij}\underset{1}{\omega}^i + q_{ij}\underset{2}{\omega}^i, \\ db_i - b_j\omega^i_j = [b_i(b_j - a_j) + \lambda(b_{ij} + p_{ij} - q_{ij})]\underset{1}{\omega}^j + b_{ij}\underset{2}{\omega}^j \\ b_{[ij]} = p_{[ij]} = \lambda q_{[ij]} \neq 0. \end{cases}$$

Note that the last condition uniquely determines λ i.e. the location of leaves of X_4 with respect to leaves of X_1, X_2, and X_3. This location has been described geometrically in [6].

For webs $W(4, 2, 2)$ of maximum 2-rank, this 2-rank is equal to one and the only abelian equation for them has the form (see [5], [6]):

$$(2.12)\qquad \sum_{a=1}^{4} \Omega_a = 0$$

where

$$(2.13)\qquad \begin{cases} \Omega_1 + (\lambda - \lambda^2)\sigma\underset{1}{\omega}^1 \wedge \underset{1}{\omega}^2, \\ \Omega_2 = (\lambda - 1)\sigma\underset{2}{\omega}^1 \wedge \underset{2}{\omega}^2, \\ \Omega_3 = -\lambda\sigma(\underset{1}{\omega}^1 + \underset{2}{\omega}^1) \wedge (\underset{1}{\omega}^2 + \underset{2}{\omega}^2), \\ \Omega_4 = \sigma(\lambda\underset{1}{\omega}^1 + \underset{2}{\omega}^1) \wedge (\lambda\underset{1}{\omega}^2 + \underset{2}{\omega}^2) \end{cases}$$

are closed forms, σ satisfies the equation

$$(2.14)\qquad d\ell n[\sigma(\lambda - 1)] = \omega^k_k + \left(\frac{a_i - b_i}{\lambda}\right)\underset{2}{\omega}^i,$$

(2.15)
$$\sum_{a=1}^{4} \underset{a}{k}_{ij} = 0$$

in the case of isocline webs and

(2.16)
$$b^k_{kij} = b_{ij} - q_{ij}$$

in the case of non-isocline webs of maximum 2 - rank .

Note that $(\wedge\Omega_a)^2 = 0$ and since forms Ω_a are closed, they are *presemplectic* [16].

3. GEOMETRY OF EXCEPTIONAL WEBS EW(4, 2, 2) OF MAXIMUM 2-RANK

3.1. Interior products associated with exceptional webs

For a web $W(4, 2, 2)$ a 2-dimensional flat generator $\zeta_p(2)$ of the Segre cone $C_p(2, 2)$ is determined by a transversally geodesic bivector $E^1 \wedge E^2$ where $E^1 = \xi^i e^1_i$, $E^2 = \xi^i e^2_i$ and ξ^i satisfies the equation

(3.1)
$$d\xi^i + \xi^i \omega^i_j = 0, \quad i, j = 1, 2.$$

This bivector intersects the tangent 2-planes $T_p(V_a)$ to leaves V_a along the directions parallel to the vectors

(3.2)
$$W_1 = -\xi^i e^2_i, \quad W_3 = \xi^i (e^2_i - e^1_i),$$
$$W_2 = \xi^i e^1_i, \quad W_4 = \xi^i (e^1_i - \lambda e^2_i).$$

Proposition 3.1. *For an exceptional web* $W(4, 2, 2)$ *of maximum* 2 - rank *there are the following relations among the interior products of forms* Ω_a *with respect to the vector fields* W_b*:*

(3.3)
$$i_{W_a}\Omega_a = 0,$$

(3.4)
$$\sum_b i_{W_a}\Omega_b = 0.$$

Proof . The proof is straightforward. According to the well-known definition, for a 2 -form Ω and vector fields ξ and V we have

(3.5)
$$(i_\xi \Omega)_p(V) = \Omega_p(\xi(p), V)$$

and for two 1-forms ω and ω'

$$i_\xi(\omega \wedge \omega') = i_\xi \omega \wedge \omega' - \omega \wedge i_\xi \omega'. \tag{3.6}$$

According (3.5) and (3.6) to the vector field W_4 defined by (3.2) and forms Ω_a defined by (2.13), one easily obtains that

$$\begin{cases} \alpha_1 = i_{W_4}\Omega_1 = (\lambda - \lambda^2)\sigma(\xi^1 \underset{1}{\omega}^2 - \xi^2 \underset{1}{\omega}^1), \\ \alpha_2 = i_{W_4}\Omega_2 = (\lambda^2 - \lambda)\sigma(-\xi^1 \underset{2}{\omega}^2 - \xi^2 \underset{2}{\omega}^1), \\ \alpha_3 = i_{W_4}\Omega_3 = (\lambda^2 - \lambda)\sigma[\xi^1(\underset{1}{\omega}^2 + \underset{2}{\omega}^2) - \xi^2(\underset{1}{\omega}^1 + \underset{2}{\omega}^1)], \\ \alpha_4 = i_{W_4}\Omega_4 = 0. \end{cases} \tag{3.7}$$

The realtions (3.3) and (3.4) for W_4 follows from (3.7). The proof for W_1, W_2, and W_3 is similar.

Remark. If one changes W_a for $\widetilde{W}_a = kW_a$, then $i_{\widetilde{W}_a}\Omega_b$ will be factored by k and (3.3) and (3.4) still will be held for $\widetilde{W}_a$.

3.2. Exterior 3-forms associated with an exceptional web

Let us consider now two pairings (α_1, Ω_2) and (α_2, Ω_1) where α_1, α_2 and Ω_1, Ω_2 are defined by (3.7) and (2.13) correspondingly. Two exterior cubic forms are associated with these pairings:

$$\begin{cases} \psi_1 = \alpha_1 \wedge \Omega_2 = \sigma^2\lambda(1-\lambda)^2(\xi^2 \underset{1}{\omega}^1 - \xi^1 \underset{1}{\omega}^2) \wedge \underset{2}{\omega}^1 \wedge \underset{2}{\omega}^2, \\ \psi_2 = \alpha_2 \wedge \Omega_1 = \sigma^2(\lambda - \lambda^2)^2(\xi^2 \underset{2}{\omega}^1 - \xi^1 \underset{2}{\omega}^2) \wedge \underset{1}{\omega}^1 \wedge \underset{1}{\omega}^2. \end{cases} \tag{3.8}$$

It is known that a vector field ξ is an *infinitesimal conformal transformation* (abr. i.c.t.) of an exterior form ω if the Lie derivative $\mathcal{L}_\xi \omega$ is proportional to ω:

$$\mathcal{L}_\xi \omega = a\omega, \quad a \in C^\infty(M^4) \tag{3.9}$$

and ξ is an *infinitesimal automorphism* (abr. i.a.) of ω if

$$\mathcal{L}_\xi \omega = 0. \tag{3.10}$$

The next definition is a generalization of these two notions.

Definition 3.2. *We will say that two exterior s-forms* ω_1 *and* ω_2 *define a quasi-recurrent Lie derivative pairing generated by a vector field* ξ *if*

$$\mathcal{L}_\xi \omega_i = a_i^j \omega_j, \quad i,j = 1,2, \quad a_i^j \in C^\infty(M^4). \tag{3.11}$$

Note that in the definition we used the term that has been used by R. Rosca [17] in a similar situation for a pair of vector fields.

Theorem 3.3. *The exterior cubic forms* ψ_1 *and* ψ_2 *determined by (3.8) are exterior recurrent and define a quasi-recurrent Lie derivtive pairing generated by the vector field* W_4. *In addition, the vector fields* W_1, W_2 *,a nd* W_3 *are i.c.t. of both forms* ψ_1 *and* ψ_2 *and the vector field* W_4 *is an i.c.t. of the form* ψ_2.

Proof. First of all, it follows form (3.8), (2.14), and (3.1) that

$$d\psi_1 = \delta_1 \wedge \psi_1, \quad d\psi_2 = \delta_2 \wedge \psi_2 \tag{3.12}$$

where

$$\delta_1 = b_i \underset{1}{\omega}^i, \quad \delta_2 = [(b_i - a_i)\underset{1}{\omega}^i - a_i \underset{2}{\omega}^i]. \tag{3.13}$$

Equations (3.12) show that the forms ψ_1 and ψ_2 are exterior recurrent [18] with δ_1 and δ_2 defined by (3.13) as recurrency forms respectively.

To calculate the Lie derivatives $\mathcal{L}_{W_4}\psi_1$ and $\mathcal{L}_{W_4}\psi_2$, we need the following formulas:

$$\begin{cases} \mathcal{L}_{W_4}\psi_i = (i_{W_4}\sigma_i)\psi_i - \sigma_i \wedge i_{W_4}\psi_i + d(i_{W_4}\psi_i), \quad i = 1,2, \\ i_{W_4}\sigma_1 = \xi^i b_i, \quad i_{W_4}\sigma_2 = \xi^i[b_i + (\lambda - 1)a_i], \\ i_{W_4}\psi_1 = -i_{W_4}\psi_2 = -\delta_1 \wedge \delta_2, \\ d(-_{W_4}\psi_1) = -d(i_{W_4}\psi_2) = [(a_i - 2b_i)\underset{1}{\omega}^i + a_i\underset{2}{\omega}^i] \wedge \delta_1 \wedge \delta_2. \end{cases} \tag{3.14}$$

Using (3.14), we find that

$$\begin{cases} \mathcal{L}_{W_4}\psi_1 = (b_i + \lambda a_i)\xi^i\psi_1 + (a_i - b_i)\xi^i\psi_2, \\ \mathcal{L}_{W_4}\psi_2 = [2b_i + (\lambda - 1)a_i]\xi^i\psi_2. \end{cases} \tag{3.15}$$

Using a similar way, one can prove that

$$\begin{cases} \mathcal{L}_{W_1}\psi_1 = \left(\frac{b_i\xi^i}{\lambda}\right), & \mathcal{L}_{W_1}\psi_2 = a_i\xi^i\psi_2, \\ \mathcal{L}_{W_2}\psi_1 = b_i\xi^i\psi_1, & \mathcal{L}_{W_2}\psi_2 = (2b_i - a_i)\xi^i\psi_2, \\ \mathcal{L}_{W_3}\psi_1 = -\left(\frac{1+1}{\lambda}\right)b_i\xi^i\psi_1, & \mathcal{L}_{W_2}\psi_2 = -2b_i\xi^2\psi_2. \end{cases} \tag{3.16}$$

Equalities (3.15) and (3.16) prove the statements of Theorem 3.3.

3.3. Infinitesimal automorphisms of exterior cubic forms associated with an exceptional web

Let us find under what conditions the vector fields W_a defined by (3.2) are i.a. of a cubic form

$$\psi = f\psi_1 + g\psi_2, \quad f, g \in C^\infty(M^4). \tag{3.17}$$

The differentials of f and g have the form

$$df = f_i \underset{1}{\omega}{}^i + \tilde{f}_i \underset{2}{\omega}{}^i, \quad dg = g_i \underset{2}{\omega}{}^i + \tilde{g}_i \underset{2}{\omega}{}^i. \tag{3.18}$$

For the vector field W_1 we have

$$i_{W_1} f = -\xi^i \tilde{f}_i, \quad i_{W_1} g = -\xi^i \tilde{g}_i. \tag{3.19}$$

Using (3.19), we can calculate the Lie derivative $\mathcal{L}_{W_1}\psi$:

$$\begin{aligned} \mathcal{L}_{W_1}\psi &= (i_{W_1} f)\psi_1 + f\mathcal{L}_{W_1}\psi_1 + (i_{W_1} g)\psi_2 + g\mathcal{L}_{W_1}\psi_2 \\ &= \left(-\tilde{f}_i + b_i \frac{f}{\lambda}\right)\xi^i\psi_1 + (-\tilde{g}_i + a_i g)\xi^i\psi_2. \end{aligned} \tag{3.20}$$

The vector field W_1 is an i.a. of ψ for any ξ^i if and only if

$$\tilde{f}_i = b_i \frac{f}{\lambda}, \quad \tilde{g}_i = a_i g. \tag{3.21}$$

Because of (3.21), equations (3.17) can be written in the form

$$df = f_i \underset{2}{\omega}{}^i + f b_i \underset{2}{\omega}{}^i/\lambda, \quad dg = g_i \underset{1}{\omega}{}^i + g a_i \underset{2}{\omega}{}^i. \tag{3.22}$$

Using the same way, one can find that the vector fields W_2, W_3, and W_4 are i.a. of ψ for any ξ^i if and only if the functions f and g satisfy respectively the following equations:

$$df = -f b_i (\underset{1}{\omega}{}^i + \tilde{f}_i \underset{2}{\omega}{}^1, \quad dg = g(a_i - 2b_i)\underset{2}{\omega}{}^i + \tilde{g}_i \underset{2}{\omega}{}^i, \tag{3.23}$$

$$df = f_i(\underset{1}{\omega}{}^i + \underset{2}{\omega}{}^i) + \left(1 + \frac{1}{\lambda}\right) f b_i \underset{2}{\omega}{}^i, \quad dg = g_i(\underset{1}{\omega}{}^i + \underset{2}{\omega}{}^i) + 2g b_i \underset{2}{\omega}{}^i, \tag{3.24}$$

$$\left\{ \begin{aligned} df &= \tilde{f}_i(\lambda \underset{1}{\omega}{}^i + \underset{2}{\omega}{}^i) - (\lambda a_i + b_i) f \underset{2}{\omega}{}^i, \\ dg &= g_i(\lambda \underset{1}{\omega}{}^i + [f(b_i - a_i) - g(2b_i + (1-\lambda)a_i)]\underset{1}{\omega}{}^i. \end{aligned} \right. \tag{3.25}$$

We proved the following proposition:

Proposition 3.4. *The vector fields W_a, $a = 1, 2, 3, 4$, defined by (3.2) are i.a. of the cubic form ψ defined by (3.17) for any ξ^i if and only if the functions f and g in (3.17) satisfy respectively the equations (3.22), (3.23), (3.24), and (3.25).*

We have to study now the compability of equaitons (3.22)-(3.25) with the equations (2.7), (2.8), (2.15) for isocline exceptional webs and (2.7), (2.11), (2.16) for non-isocline exceptional webs.

Theorem 3.5. *The vector fields W_1 and W_2 (for any ξ^i) can not be i.a. of the cubic form ψ defined by (3.17) if an exceptional web is non-isocline. In all other cases the set of forms ψ, for which the vector fields W_a, $a = 1, 2, 3, 4$, are i.a. of ψ for any ξ^i, depends on two arbitrary functions of two independent variables.*

Proof. In the case of W_1 we have (3.22). If an exceptional web id non-isocline, the exterior differentiation of (3.22) by means of (2.7) and (2.11) leads to two exterior quadratic equations. The first of them is:

$$\left\{df_i - f_j\omega_i^j + f a_{[ij]}\underset{1}{\omega^j} - \left[b_j\frac{f_i}{\lambda} + f(b_{ij} + p_{ij} - q_{ij})\right]\underset{2}{\omega^i}\right\} \wedge \underset{1}{\omega^i} = f q_{ij}\underset{2}{\omega^i} \wedge \underset{2}{\omega^i} = 0. \tag{3.26}$$

It follows from (3.26) that $q_{[ij]} = 0$ and the last one contradicts to a non-isoclinity of an exceptional web (see (2.11)).

If an exceptional web is isocline, the exterior differentiation of (3.22) by means of (2.7) and (2.8) gives the following exterior quadratic equations:

$$\Delta f_i \wedge \underset{1}{\omega^i} = 0, \quad \Delta g_i \wedge \underset{1}{\omega^i} = 0 \tag{3.27}$$

where

$$\Delta f_i = df_i - f_j\omega_i^j + f \underset{[ij]}{a}\underset{1}{\omega^j} - \left[b_j + f(\underset{1}{k_{ij}} - \underset{4}{k_{ij}})\right]\underset{2}{\omega^i},$$

$$\Delta g_i = dg_i - g_j\omega_i^j + g \underset{[ij]}{a}\underset{1}{\omega^j} - \left[a_j + g(\underset{1}{k_{ij}} - \underset{3}{k_{ij}})\right]\underset{2}{\omega^i}, .$$

The number of unknown functions (Δf_i and Δg_i) is equal to $q = 4$. The consecutive Cartan's characters [19] are $s_1 = 2$, $s_2 = 2$, $s_3 = 0$, and the Cartan's number $Q = s_1 + 2s_2 = 6$. It follows from (3.27) that the general two-dimensional integral element depends on $N + 6$ parameters. Because of $Q = N$, the system (3.22), (3.27) is involution and its solution depends on two functions of two variables [19].

The proof for W_2, W_3 and W_4 is similar.

3.4. Infinitesimal conformal transformations of exterior cubic forms associated with an exceptional web

According to Theorem 3.3, the vector fields W_1, W_2, and W_3 ar i.c.t. of ψ_1 and ψ_2 and W_4 is that of ψ_2. We will find under what conditions vectors fields W_a, $a = 1, 2, 3, 4$, are i.c.t. of the cubic form ψ defined by (3.17).

We will suppose that $f \neq 0$ and $g \neq 0$ and consider for example the vector field W_1.

One can find from (3.9) and (3.20) that W_1 for any ξ^i is an i.c.t. of ψ if and only if

$$\frac{\widetilde{f}_i}{f} - \frac{\widetilde{g}_i}{g} = \frac{1}{\lambda}(b_i - \lambda a_i). \tag{3.28}$$

For the system (3.18), (3.28) for both isocline and non-isocline exceptional webs one gets: $q = 6$, $s_1 = s_2 = s_3 = 2$, $s_4 = 0$, $Q = s_1 + 2s_2 + 3s_3 = 12$, $N = 12$.

We proved the theorem.

Theorem 3.6. *The vector fields W_a, $a = 1, 2, 3, 4$, for any ξ^i are i.c.t. of the exterior cubic form ψ defined by (3.17) if and only if the functions f and g in (3.17) satisfy (3.18) and respectively (3.28) and equations similar to (3.28). The set of forms ψ depends on two functions of three independent variables.*

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