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ALMOST CONFORMAL 2-COSYMPLECTIC PSEUDO-SASAKIAN MANIFOLDS

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INTRODUCTION. In the last years several papers have been concerned with *almost r-contact* or *r-paracontact* manifolds (see [6] and [14]). On the other hand, V.V. Goldberg and R. Rosca have recently studied in [12] almost 1-contact pseudo-Riemannian manifolds which are endowed with a *conformal cosymplectic pseudo-Sasakian structure*.

Since the manifolds M which we are going to discuss are connected and paracompact, we denote by $d^\omega = d + e(\omega)$ ($e(\omega)$: exterior product by the *closed* 1-form ω) the *cohomology operator* (see [13]) on M. Then any form $u \in M$ such that $d^\omega u = 0$ is said to be *d^ω-closed*.

The present paper is devoted to the study of even dimensional pseudo-Riemannian manifolds of signature $(m+2, m)$ which are endowed with an *almost conformal 2-cosymplectic pseudo-Sasakian structure*. Such a manifold is denoted by $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$, and its structure tensor fields $(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ are : the *paracomplex operator* (see [15]), an *exterior recurrent* (see [9]) 2-form of rank $2m$, two structure vector fields ξ_α; $\alpha = 2m+1$, $2m+2$, two structure 1-forms $\eta^\alpha = \flat(\xi_\alpha)$ ($\flat$: $TM \to T^*M$ is the *musical isomorphism* [6] defined by g) and the pseudo-Riemannian tensor g of M respectively.

We agree to call the 2-distribution $D_c = \{\xi_\alpha\}$ and its orthogonal complementary $D_c^\perp$, respectively the *contact* and the *neutral* distribution of M, and we assume that the connection ∇ is *symmetric*.

Next setting $\xi = \sum_\alpha f_\alpha \xi_\alpha (f_\alpha \in C^\infty M)$ and $\eta = \flat(\xi) = \sum_\alpha f_\alpha \eta^\alpha$, we call ξ (respectively η) the *bicontact vector field* (respectively the *bicontact 1-form*) of M. It is proved that both η and the simple unit 2-form φ which corresponds to D_c are *exact*. Further for the 2-form of maximal rank : $\psi = \Omega + \varphi$ one finds $d^{2\eta}\psi = 0$, that is φ is $d^{2\eta}$-exact. This proves the significant fact that φ defines on M a *globally conformal symplectic structure* $CS_p(2m+2, \mathbf{R})$ and η (respectively ξ) is the *Lee covector* (respectively the *Lee vector*) of $CS_p(2m+2, \mathbf{R})$. Next, any M is foliated by M_k and M_c where M_k is a $2m$-dimensional *para-Kählerian manifold* tangent to $D_c^\perp$ and M_c is a *flat* surfce tangent to D_c.

The proper immersion κ: $M_k \to M$ is *pseudo umbilical* [8], and the *mean curvature vector* associated with κ is the restriction $\xi|_{M_k}$.

Some properties of the Lie algebra involving ψ and η are also outlined. It is proved that for any vector field $Z_a \in D_c^\perp$, the Lie derivative $\mathcal{L}_{Z_A}\psi$ is *$d^{2\eta}$-closed* and any $(2q+1)$-form $\eta_q = \eta \wedge \psi^q$ is a *relative invariant* of Z_a.

Lee's vector field ξ enjoys the following properties:

1) ξ defines an *infinitesimal homothety* on M and is *pregeodesic*;

2) The *Ricci curvature* of ξ is expressed by $2m||\xi||^2$;

3) The *φ-dual* $\xi^\perp$ of ξ is a *Killing vector field* and *commutes* with ξ.

In Section 3 we give the following definition: A vector field $\mathcal{T}$ on an almost contact manifold whose structure is defined by the pairing (η, ξ) is called a *contact torse forming* if it satisfies the equation

$$\nabla_Z \mathcal{T} = \lambda Z + \eta(Z)\mathcal{T} - \flat(\mathrm{T})(Z)\xi.$$

It is proved that any $\mathcal{T}$ is a *conformal vector field* (i.e. $\mathcal{L}_{\mathcal{T}} g = pg$), and on $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ the existence of $\mathcal{T}$ is defined by an exterior differential system in *involution* (see [7]).

Some properties of the Lie algebra involving $\mathcal{T}$ and $U\mathcal{T}$ and the structure tensor of M are also discussed.

In the last Section 4 some improper foliations on the manifold M are considered.

Thus the following significant result emerges: any $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ may be regarded as foliated by M_a and M_{a^*}, where M_a and M_{a^*} are both $(m+1)$-dimensional *coisotropic* and of *defect* m submanifolds of M.

Finally we prove that for any $CICR$-submanifold (coisotropic CR-submanifold) M_ξ (see [12]) the *vertical distribuition* is an *isotropic foliation* (as in [5]).

1. PRELIMINARIES

Let (M, g) be an even dimensional Riemannian or pseudo-Riemannian manifold, say $\dim M = 2m + 2$ and let ∇ be the covariant differential operator defined by the metric tensor g.

We assume in the following that M is *orientable* and that the connection ∇ is *symmetric*.

Let $\Gamma(TM) = \mathcal{X}M$ (respectively $\flat\colon TM \to T^*M$) be the set of sections of the *tangent bundle* T respectively the *musical isomorphism* [18] defined by g).

Next, following [18], we set $A^q(M, TM) = \Gamma\,\mathrm{Hom}(\wedge^q TM, TM)$ and notice that elements of $A^q(M, TM)$ are vector valued q-forms, $q < \dim M$.

Denote by $d^\nabla : A^q(M,TM) \to A^{q+1}$ the *exterior covariant derivative* operator with respect to ∇ (it should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike d^2) and by $dp \in A^1(M, TM)$ the *soldering form* of M (dp is a canonical vector valued 1-form of M [10] and one has $d^\nabla(dp) = 0$).

The operator

$$d^\omega = d + e(\omega) \tag{1.1}$$

acting on $\wedge M$, where $e(\omega)$ means the exterior product by the *closed* 1-form $\omega \in \wedge^1 M$, is called the *cohomology operator* [13].

One has $(d^\omega)^2 = 0$, and any form $u \in \wedge M$ such that

$$d^\omega u = 0 \tag{1.2}$$

is said to be d^{ω}*-closed*.

Let $\mathcal{T} \in \mathcal{X}M$ be a *conformal vector field* on an n-dimensional Riemannian or pseudo-Riemannian manifold (M, g), that is such that

$$\mathcal{L}_{\mathcal{T}} g = \rho g; \quad \rho \in C^{\infty} M, \tag{1.3}$$

the if ω is any 1-form, one has (see [B 82]):

$$\mathcal{L}_{\mathcal{T}} \omega = \rho\omega + \flat[\mathcal{T}, \flat^{-1}(\omega)]; \quad [\ ,\ ] \text{is the Lie bracket.} \tag{1.4}$$

If ω is any q-form and $\star$ means the *star isomorphism*, then one has [B 82]:

$$\mathcal{L}_{\mathcal{T}} \star \omega = \star \mathcal{L}_{\mathcal{T}} \omega + \frac{n-2q}{2} \rho \star \omega. \tag{1.5}$$

Consider now a pseudo-Riemannian manifold of signature $(m+2, m)$ and with a (1,1)-tensor field U of square $+1$. Assume that there exists on M two structure fields $\xi_{\alpha} \in \mathcal{X}M$ and two structure 1-forms $\eta^{\alpha} = \flat(\xi^{\alpha})$ such that:

$$\begin{cases} \eta^{\alpha}(\xi_{\beta}) = \delta_{\alpha\beta}, \\ U^2 = \mathrm{Id} - \eta^{\alpha} \otimes \xi_{\alpha}; \quad \alpha, \beta = 2m+1, 2m+2. \end{cases} \tag{1.6}$$

Then in a manner similar to [6], we say that the triplet $(U, \xi_{\alpha}, \eta^{\alpha})$ defines an *almost 2-contact structure.*

By abusing language, the vector valued 1-form

$$l_c = \eta^{\alpha} \otimes \xi_{\alpha} \in A^1(M, TM) \tag{1.7}$$

will be called the *contact line element* and the 2-distribution $D_c = \{\xi_{\alpha}\}$ the *contact distribution* of M.

2. ALMOST CONFORMAL 2-COSYMPLECTIC PSEUDO-SASAKIAN MANIFOLDS

Let $f_{\alpha} \in C^{\infty}$, $\alpha = 1, 2$ be two nowhere vanishing scalar fields on M. Setting

$$\xi = \sum_{\alpha} f_{\alpha} \xi_{\alpha} \in D_c, \tag{2.1}$$

and

$$\eta = f_{\alpha} \eta^{\alpha} = \flat(\xi) \in D_c, \tag{2.2}$$

we agree to call ξ (respectively η) the *bicontact vector field* (respectively the *bicontact 1-form*) on M.

Using the definition of *conformal cosymplectic pseudo-Sasakian manifolds* from our paper [22], we now assume that ξ_α and η^α satisfy

$$\nabla\xi_\alpha = -f_\alpha(dp - l_c), \tag{2.3}$$

and

$$df_\alpha = c\eta^\alpha; \quad c = \text{const}. \tag{2.4}$$

First of all we notice that by (1.7) the equations (2.3) define $\{\xi_\alpha\}$ as a *quasi-concurrent pairing* (see [4]). Secondly, if we set

$$\langle\xi,\xi\rangle = \sum_\alpha f_\alpha^2 = f^2, \tag{2,5}$$

where $\langle\ ,\ \rangle$ replaces g, we get at once by (2.2) and (2.4) that

$$\eta = df^2/(2c) \tag{2.6}$$

which proves that η is an *exact form*.

Further let $\Omega \in \wedge^2 M$ be an *exterior recurrent* structure 2-form of M of rank $2m$ and having -2η as a *recurrence 1-form* (cf. [9]), that is

$$d\Omega = -2\eta\wedge\Omega. \tag{2.7}$$

If the structure tensor saatisfy the conditions

$$\begin{cases} \eta^\alpha(UZ) = 0, \quad g(Z,\xi_\alpha) = \eta^\alpha(Z), \\ g(Z,UZ') + g(Z',UZ) = 0 \\ \Omega(Z,Z') = -g(UZ,Z') \to i_Z\Omega = \flat(UZ), \\ (\nabla U)Z = \eta(Z)Udp + \flat(UZ)\otimes\xi; \quad Z,Z' \in \mathcal{X}M, \end{cases} \tag{2.8}$$

then any manifold $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ for which conditions (1.6), (2.3), (2.4), (2.7) and (2.8) are satisfied is defined as an *almost conformal 2-cosymplectic pseudo-Sasakian manifold*.

It follows from (2.7) and (1.1) that

$$d^{2\eta}\Omega = 0 \tag{2.9}$$

which proves that Ω is $d^{2\eta}$-closed.

In wiew of hereafter discussion we shall make use of the *adapted Witt* local field frames (see [15], [12] and [5]).

Denote by $W = \text{vect}.\{h_a, h_{a^*}, h_\alpha = \xi_\alpha; a = 1, \ldots, m; a^* = a + m; \alpha = m^* + 1, m^* + 2\}$ such a frame, and let $W^* = \text{covect}.\{w^a, w^{a^*}, w^\alpha = \eta^\alpha\}$ be the associated coframe.

The distribution $\{h_a, h_{a^*}\}$ defines a *real basis* and by (1.6), (2.9) one has (see [L 51]):

$$(2.10) \qquad Uh_a = h_a, \quad Uh_{a^*} = -h_{a^*}, \quad U\xi = 0,$$

and U is also called the *para complex opertor* [15].

Clearly $\{h_a, h_{a^*}\}$ defines the orthogonal complementary distribution of D_c and we agree to dentoe it by $D_c^\perp$. On the other hand, since the metric tensor associated with $D_c^\perp$ has a *neutral structure* [19], we shall call $D_c^\perp$ the *neutral distribution* of M.

Further the W-basis being normed one has

$$(2.11) \qquad \begin{cases} g(h_q, \xi_\alpha) = 0, & g(h_{a^*}, \xi_\alpha) = 0, \\ g(h_a, h_{b^*}) = \delta_{ab}, & g(\xi_\alpha, \xi_\alpha) = 1, \end{cases}$$

and one may say that ξ_α are the *anisotropic* vector fields of W.

If θ^A_B; $A, B, C \in \{a, a^*, \alpha\}$ and Θ^A_B are the local *connection forms* in the tangent bundle TM and the *curvature 2-forms* on M respectively, then the structure equations of M may be written in the indexless form as

$$(2.12) \qquad \nabla h = \theta \otimes h \in A^1(M, TM),$$

$$(2.13) \qquad d\omega = -\theta \wedge \omega,$$

$$(2.14) \qquad d\theta = -\theta \wedge \theta + \Theta.$$

Using (2.3), (2.4), (1.2), (2.9) and (2.10), one finds (see [12], [15]) that

$$(2.15) \qquad \begin{cases} \theta^a_b + \theta^{b^*}_{a^*} = 0 & \theta^{a^*}_b = 0, \quad \theta^a_{b^*} = 0, \\ \theta^\alpha_a + \theta^{a^*}_\alpha = 0, & \theta^a_\alpha + \theta^\alpha_{a^*} = 0, \end{cases}$$

and

$$(2.16) \qquad \theta^\alpha_a = f_\alpha \omega^{a^*}, \quad \theta^\alpha_{a^*} = f_\alpha \omega^\alpha.$$

Next in terms of the W^*-basis, the *soldering form* dp and the structure 2-form Ω are expressed by

$$dp = \omega^a \otimes h_a + \omega^{a^*} \otimes h_{a^*} + \eta^\alpha \otimes \xi_\alpha \Rightarrow g = \\ = 2\sum_a \omega^a \otimes \omega^{a^*} + \sum_\alpha \eta^\alpha \otimes \eta^\alpha \tag{2.17}$$

and

$$\Omega = \sum_a \omega^a \wedge \omega^{a^*}. \tag{2.18}$$

It should be noted that Ω is exchangeable with the *para-Hermitian* component $2\sum_a \omega^a \otimes \omega^{a^*}$ of g and by (2.15), (2.16) and (2.2), it is a routine matter to verify the equation (2.7).

Let us go back to the bicontact vector field ξ defined by (2.1). Using (1.7), (2.3), (2.4) and (2.8), one finds

$$\nabla\xi = -f^2\, dp + (f^2 + c)lc. \tag{2.19}$$

Taking into account the trace g expressed by (2.17), at point $p \in M$ one has

$$\operatorname{div}\xi = \operatorname{tr}(\nabla\xi) = \sum_a (\omega^a(\nabla_{h_{a^*}}\xi) + \\ + \omega^{a^*}(\nabla_{h_a}\xi) + \eta^\alpha(\nabla_{\xi_\alpha}\xi)) = 2c. \tag{2.20}$$

Hence since $c = \text{const.}$, it follows that ξ defines an *infinitesimal homothety* on M.

Further it is easily seen by means of (2.17) that

$$\nabla_\xi \xi = c\xi \tag{2.21}$$

which shows that ξ is a *pregeodesic*.

On the other hand, applying the general formula of K. Yano (see [28]), we find

$$\operatorname{div}(\nabla_Z Z) - \operatorname{div}(\operatorname{div} Z) + (\operatorname{div} Z)^2 = \operatorname{Ric}(Z) + \\ + \sum_{A,B} g(\nabla_{h_A} Z, h_B)\, g(h_A, \nabla_{h_B} Z),$$

where $Z \in xM$ and Ric is the Ricci curvature. Setting $Z = \xi$, one finds by (2.19) and (2.20) that

$$\mathrm{Ric}(\xi) = 2mf^2. \tag{2.22}$$

Denote now by

$$\varphi = \eta^{m^*+1} \wedge \eta^{m^*+2} \tag{2.23}$$

the simple unit form which corresponds to D_c, and by $\mu\colon TM \to TM^*$ the bundle isomorphism defined by

$$\mu(Z) = -i_Z\varphi. \tag{2.24}$$

One readily finds

$$(\flat^{-1} \circ \mu)\xi = f_{m^*+1}\xi_{m^*+2} - f_{m^*+2}\xi_{m^*+1} = \xi^{\perp} \in D_c = c(\xi_{m^*+2} \wedge \xi_{m^*+1}). \tag{2.25}$$

Taking the covariant derivative of $\xi^{\perp}$ and using (2.3) and (2.4), one finds

$$\nabla\xi^{\perp} = c(\eta^{m^*+1} \otimes \xi_{m^*+2} - \eta^{m^*+2} \otimes \xi_{m^*+1}). \tag{2.26}$$

We notice that $\langle l_c, \nabla\xi^{\perp}\rangle$ is a metric tensor exchangeable (up to $2c$) with φ.

Further one readily derives form (2.26) that $\langle\nabla_Z\xi^{\perp}, Z'\rangle + \langle\nabla_{Z'}\xi^{\perp}, Z\rangle = 0$ which proves that $\xi^{\perp}$ is a *Killing vector field.*

In addition, by (2.19) and (2.26) one finds: $\nabla_{\xi^{\perp}}\xi = c\xi^{\perp} = \nabla_{\xi}\xi^{\perp}$, and this moves to

$$[\xi, \xi^{\perp}] = 0, \tag{2.27}$$

that is ξ and $\xi^{\perp}$ *commute.*

In the following we agree to call $\xi^{\perp}$ the *conctact dual* (or φ-dual) vector field of th contact vector fields ξ.

In connection with $\xi^{\perp}$ it is worth to emphasize the following fact. If we set $\eta^{\perp} = \flat(\xi^{\perp})$, then by (2.4) and (2.24) one gets

$$d\eta^{\perp} = 2c\varphi, \tag{2.28}$$

and one may state that φ is an *exact 2-form.*

Denote now by $\varphi^{\perp}$ the simple unit form which corresponds to the neutral $2m$-distribution $D_c^{\perp}$. Clearly by (2.8), $\varphi^{\perp}$ is an exterior recurrent $2m$-form on M. Then, since φ is closed and $D_c^{\perp}$ (respectively D_c) is annihilated by φ (respectively by $\varphi^{\perp}$), it follows from Frobenius theorem that both D_c and $D_c^{\perp}$ are involutive. Therefore one may say that any manifold $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ is *foliate.*

On the other hand, if Z_c, $Z_c' \in D_c$ are any vector fields of D_c, one finds by (2.3) that $\nabla_{Z_c'} Z_c \in D_c$. This proves that D_c is a *totally geodesic foliation* (cf. [17]). If we denote by M_c the surface tangent to D_c, it is readily seen by (2.12) and (2.13) that M_c is *flat.*

On the other hand, let M_k be the $2m$-dimensional leaf of D_c. Then by (2.7) M_k has a *symplectic structure* and it is readily deduced from (2.17) and (2.18) that M_k is a *para-Kählerian manifold* (see [15] or [20]). Therefore we may conclude that the manifold M under discussion may be viewed as foliated by M_k and M_c.

Consider on M the *almost symplectic form*

$$\psi = \Omega + \varphi. \tag{2.29}$$

Operating on ψ by $d^{2\eta}$ and using (2.9) and (2.23), one gets

$$d^{2\eta}\psi = 0. \tag{2.30}$$

In addition, since η is an exact form, the equation (2.30) expresses that the ψ is $d^{2\eta}$-exact. This proves the significant fact that ψ defines a *globally conformal symplectic structure* $CS_p(2m+2, \mathbf{R})$ (see [13]) on M. In this case the q^{th} space of cohomology $H^q(M, \eta)$ is isomorphic to the q^{th} space of cohomology $H^q(M, \mathbf{R})$ of G. de Rham (see [13]).

It should be noticed that since the pairing $(\psi, 2\eta)$ defines a conformal symplectic structure, then it turns out that η and $\flat^{-1}(\eta) = \xi$ define respectively the *Lee covector* and the *Lee vector* field of this structure.

We shall now outline a general property of any conformal symplectic structure defined by

$$d^{2\eta}\psi = 0. \tag{2.31}$$

First of all any vector field $Z_a \in \mathcal{X}M$ such that $\eta(Z_a) = a = \text{const.}$, will be called a *constant Lee section*. Set $i_{Z_a}\psi = \alpha$ and take the Lie derivative of ψ with respect to Z_a. One has

$$\mathcal{L}_{Z_a}\psi = -2a\psi + d^{2\eta}\alpha \tag{2.23}$$

and since $(d^{2\eta})^2 = 0$, it follows by operating on (2.32) by $d^{2\eta}$ (see Section 1) and taking into account (2.31) that

$$d^{2\eta}(\mathcal{L}_{Z_A}\psi) = 0. \tag{2.33}$$

Hence all 2-forms $\mathcal{L}_{Z_a}\psi$ are $d^{2\eta}$-closed. Moreover, if L is the (1.1)-operator defined by $L : u \to u \wedge \psi;\ u \in \wedge^1 M$ (note that one has $dLu = Ldu + u \wedge Lu$), we set

$$\eta_q = L^q \eta = \eta \wedge \psi^q \in \wedge^{2q+1} M. \tag{2.34}$$

Since obviously $\mathcal{L}_{Z_a}\eta = 0$, one derives $\mathcal{L}_{Z_A}\eta_q = (q\eta \wedge \mathcal{L}_{Z_A}\psi) \wedge \psi^{q-1}$. Taking account of (2.32), one finally gets

$$d\mathcal{L}_{Z_A}\eta_q = 0. \tag{2.35}$$

Hence if η is the Lee covector of any conformal symplectic structure, then $L^q\eta$ is a *relative integral invariant* of Z_a (see [1]).

Consequentyl we may state the following theorem:

Theorem 2.1. *Let M be a Riemannian or pseudo-Riemannian manifold endowed with a conformal symplectic structure, such that $d\psi + 2\eta \wedge \psi = 0$, and let $Z_a \in \mathcal{X}M$ be a constant Lee section of the Lee covector η. Then for any Z_a, the Lie derivative $\mathcal{L}_{Z_a}\psi$ is $d^{2\eta}$-closed. Further if η_q is the $(2q+1)$-form defined by $\eta_q = \eta \wedge \psi^q$, then any η_q is a relative integral invairnat of the constant Lee section Z_a.*

Finally we shall outline some crucial properties of the proper immersion $\kappa:\ M_k \to M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$. Since the soldering form of the para-Kählerian manifold M_k is

$$dp_k = dp|_{M_k} = \omega^a \otimes h_a + \omega^{a^*} \otimes h_{a^*}, \tag{2.36}$$

the *mean curvature vector valued* $(2m-1)$*-form* $\textcircled{H} \in A^{2m-1}(M, TM)$ of M_k (see [8], [12], [19], [5], [19]) is defined by

$$\begin{aligned} \textcircled{H} = \star\, dp_k = \sum_a (-1)^{a-1}\omega^1 \wedge \ldots \wedge \widehat{\omega}^a \wedge \ldots \wedge \omega^m \wedge \\ \wedge\omega^{1^*} \wedge \ldots \wedge \omega^{m^*} \otimes h_{a^*} + \sum_a (-1)^{a^*-1}\omega^1 \wedge \ldots \wedge \omega^m \wedge \\ \wedge\omega^{1^*} \wedge \ldots \wedge \widehat{\omega}^{a^*} \wedge \ldots \wedge \omega^{m^*} \otimes h_a. \end{aligned} \tag{2.37}$$

Remind that $\star$ is the star isomorphism and that we denote the elements induced by κ by the same letters.

If σ represents the volume element of M_k, then one has $d^\nabla \textcircled{H} = 2mH \otimes \sigma$ where H denotes the *mean curvature vector fields* associated with κ. Taking into aaccount (2.13) and

(2.14) and using (2.10), one finds by operating d^∇ on Ⓗ that in the case under discussion H is defined by $\xi|_{M_k}$. In addition, since on M_k the contact line element l_c vanishes, it follows from (2.19) that the *mean quadratic form* $II = -\langle dp_k, \nabla H\rangle$ associated with κ is expressed by $II = f^2 g_k$ where $g_k = g|_{M_k}$. Hence, following a well-known definition (see [8]) the immesion $\kappa\colon M_k \to M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ is *pseudo-umbilical.*

We close this section combining the results which we have obtained in the following theorem:

Theorem 2.2. *Let $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ be a $(2m+2)$-dimensional almost conformal 2-cosymplectic pseudo-Sasakian manifold. Let $D_c = \{\xi_\alpha\}$ (respectively φ) the contact 2-distribution (respectively the simple unit form corresponding to D_c). Let $\xi \in D_c$ (respectively $\eta = \flat(\xi) \in D_c^\perp$) the bicontact vector field (respectively the bicontact 1-form) and let $d^{2\eta} = d + e(2\eta)$ be the cohomology operator on M with respect to 2η. Then one has the following properties:*

(1) Any manifold M is foliated by M_c and M_k where M_c is a totally geodesic surface tangent to D_c and M_k a $2m$-dimensional para-Kählerian manifold tangent to the complementary orthogonal distribution $D_c^\perp$ of D_c ($D_c^\perp$ is the neutral distriibution).

(2) The 2-form of maximal rank *$\psi = \Omega + \varphi$ is $d^{2\eta}$-closed, i.e., ψ defines a conformal symplectic structure $CS_p(2m+2, \mathbf{R})$ whose Lee covector (respectively Lee vector) is η (respectively ξ).*

(3) For any vector field $Z_a \in D_c^\perp$ the Lie derivative $\mathcal{L}_{Z_a}\psi$ is $d^{2\eta}$-closed, and any $(2q+1)$-form $\eta_q = \eta \wedge \psi^q$ is a realtive integral invariant of Z_a.

(4) The proper immersion $\kappa\colon M_k \to M$ is pseudo-umbilical, and the mean curvature vector field associated with κ is $\xi|_{M_k}$.

(5) The vector field ξ enjoys the following properties:

(i) ξ defines an infinitesimal homothety on M and is pregeodesic;

(ii) the Ricci curvature of ξ is expressed by $2m\|\xi\|^2$;

(iii) the φ-dual $\xi^\perp$ of ξ is a Killing vector field and commutes with ξ.

3. CONTACT TORSE FORMING ON $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$

Let $M(\nabla, g)$ be an oriented Riemannian or pseudo-Riemannian manifold with soldering form dp. Assume that M is endowed with an almost contact or almost r-contact structure having ξ (respectively $\eta = \flat(\xi)$) as an *r-contact vector field* (respectively *r-contact vector 1-form*). As an extension of a definition given in [26], we agree to call any vector field $\mathcal{T} \in \mathcal{X}M$ such that

$$(3.1) \qquad \nabla\mathcal{T} = \lambda dp + \eta \otimes \mathcal{T} - \flat(\mathcal{T}) \otimes \xi; \quad \lambda \in C^\infty M; \quad \Leftrightarrow \nabla\mathcal{T} = \lambda dp + \mathcal{T} \wedge \xi$$

a *contact torse forming* (abbreviation c.t.f.). By (3.1) one has for any vector fields Z, $Z' \in \mathcal{X}M$

$$\langle \nabla_Z \mathcal{T}, Z' \rangle + \langle \nabla_{Z'} \mathcal{T}, Z \rangle = 2\lambda. \tag{3.2}$$

The above equation proves in intrinsic manner that any $\mathcal{T}$ is a *conformal vector field* (see [18]) and is equivalent to

$$\mathcal{L}_{\mathcal{T}} g = \rho g; \quad \rho = 2 \operatorname{div} \mathcal{T} / \dim M \tag{3.3}$$

that is

$$\operatorname{div} \mathcal{T} = 2(m+1)\lambda. \tag{3.4}$$

We shall search now under what conditions the manifold M under discussion carries a c.t.f. vector field. Setting

$$\gamma = \langle \mathcal{T}, l_c \rangle = \sum_a \mathcal{T}^\alpha \eta^\alpha \in \Lambda^1 M \tag{3.5}$$

and using (2.3) and (2.12), one derives from (3.1) that

$$\begin{cases} \frac{1}{2} d\|\mathcal{T}\|^2 & = (\lambda + \eta(\mathcal{T}))\flat(\mathcal{T}) - \|\mathcal{T}\|^2 \eta, \\ d\mathcal{T}^\alpha & = \lambda \eta^\alpha - \mathcal{T}^\alpha \eta + f_\alpha(\gamma - 2\flat(\mathcal{T})). \end{cases} \tag{3.6}$$

Further by exterior differentiation one finds

$$\begin{cases} d\flat(\mathcal{T}) + 2\eta \wedge \flat(\mathcal{T}) = 0 \Leftrightarrow d^{2\eta} \flat(\mathcal{T}) = 0, \\ d\gamma + 2\eta \wedge \gamma = 0 \Leftrightarrow d^{2\eta}\gamma = 0, \\ (\lambda + \eta(\mathcal{T}))\, d\flat(\mathcal{T}) = -(2(\lambda + \eta(\mathcal{T}))\eta + \\ \qquad\qquad + d(\lambda + \eta(\mathcal{T})) \wedge \flat(\mathcal{T}), \\ (d\lambda + d\eta(\mathcal{T})) \wedge \flat(\mathcal{T}) = 0. \end{cases} \tag{3.7}$$

Hence from the equation (3.7) and by (2.6) one may say that both 1-forms $\flat(\mathcal{T})$ and γ are $d^{2\eta}$-exact.

Denote now by Σ the exterior differential system defined by the equations (3.6) and (3.7). It is easy to see that for the system Σ the *Cartan characters* are $r = 7$, $x_0 = 3$, $s_1 = 4$, and therefore, by *Cartans's test* (see [7]) the system Σ is *in involution*, and its solution depends on 4 arbitrary functions of one variable.

Next by the last equation (2.8) and by (3.1) one gets

$$\nabla U\mathcal{T} = (\lambda - \eta(\mathcal{T}))U dp - \eta \otimes U\mathcal{T} + \flat(U\mathcal{T}) \otimes \xi \tag{3.8}$$

and one quickly derives

$$[U\mathcal{T}, \mathcal{T}] = 2\eta(\mathcal{T})U\mathcal{T} \tag{3.9}$$

which shows that $\mathcal{T}$ defines an *infinitesimal confromal transformation of* $U\mathcal{T}$.

On the other hand, consider the 1-form $\flat(U\mathcal{T})$. Using (3.8) and (2.13), one gets by exterior differentiation of this form the following equation:

$$d\flat(U\mathcal{T}) = 2(\lambda - \eta(\mathcal{T}))\Omega \tag{3.10}$$

and since $i_{\mathcal{T}}\flat(U\mathcal{T}) = 0$, one has

$$\mathcal{L}_{\mathcal{T}}\flat(U\mathcal{T}) = 2(\lambda - \eta(\mathcal{T}))\flat(U\mathcal{T})). \tag{3.11}$$

But by (3.9) one may write

$$\flat[\mathcal{T}, U\mathcal{T}] = -2\eta(\mathcal{T})\flat(U\mathcal{T})$$

and so we can see from (3.4) that the equation (3.11) is coherent withe the general equation (1.4).

It is worth to emphasize that by means of the general formula (1.5) the property defined by (3.11) is invariant under the star isomorphism. Effectively since in the case under discussion $\rho = 2\lambda$, one quickly finds

$$\mathcal{L}_{\mathcal{T}} \star\flat(U\mathcal{T}) = 2(\lambda(1+m) - \eta(\mathcal{T})) \star\flat(U\mathcal{T})$$

and the above equation shows that $\mathcal{T}$ defines an infinitesimal conformal transformation of $\star\flat(U\mathcal{T})$.

We shall now discuss some additional properties of the Lie algebra involving $\mathcal{T}$, $U\mathcal{T} \in \mathcal{X}M$ and the structure tensor of the manifold under consideration.

Denote by $(\flat(\mathcal{T}), \flat(U\mathcal{T}))_P$ the *Poisson bracket* with respect to Ω of the 1-forms $\flat(\mathcal{T})$ and $\flat(U\mathcal{T})$. Recall that $(\ )_P$ is an isomorphism $Z \to -i_Z\Omega$ which moves the Lie bracket from $\mathcal{X}M$ to $\wedge^1 M$. Accordingly one has

$$(\flat(\mathcal{T}), \flat(U\mathcal{T}))_P = i_{[\mathcal{T}, U\mathcal{T}]}\Omega$$

and by (3.9) one finds

$$(\flat(\mathcal{T}),\flat(U\mathcal{T}))_P = 2\eta(\mathcal{T})(\gamma - \flat(\mathcal{T})). \tag{3.12}$$

But $\flat(\mathcal{T})$ and γ being both $d^{2\eta}$-exact one derives from (3.12) that

$$d(\flat(\mathcal{T}),\flat(U\mathcal{T}))_P = \frac{d\eta(\mathcal{T})}{\eta(\mathcal{T})} \wedge (\flat(\mathcal{T}),\flat(U\mathcal{T}))_P$$

which shows that $(\flat(\mathcal{T}),\flat(U\mathcal{T}))_P$ is $d^{-d\eta(\mathcal{T})/\eta(\mathcal{T})}$-exact.

Finally take the Lie derivative of ψ with respect to $U\mathcal{T}$. One has $i_{U\mathcal{T}}\psi = \flat(\mathcal{T}) - \gamma$. Using the first equation of (3.7), one derives

$$\mathcal{L}_{U\mathcal{T}}\psi = 2\eta \wedge (\gamma - \flat(\mathcal{T})). \tag{3.13}$$

By reference to (2.6) one finds that the exterior differentiation of (3.13) gives $d(\mathcal{L}_{U\mathcal{T}}\psi) = 0$ and this shows that ψ is a relative invariant of $U\mathcal{T}$.

Theorem 3.1. *Let $\mathcal{T}$ be a contact torse forming on the manifold M defined in Section 2 and let d^ω be the cohomology operator with respect to ω. Then any $\mathcal{T}$ is a conformal vector field and on any M the existeance of $\mathcal{T}$ is determined by an exterior differential system in involution. The c.t.f. $\mathcal{T}$ enjoys the following properties:*

(1) The dual from $\flat(\mathcal{T})$ is $d^{2\eta}$exact, and $\mathcal{T}$ defines an infitesimal conformal transformation of the 1-form $\flat(U\mathcal{T})$.

(2) The Poisson bracket $(\flat(\mathcal{T}),\flat(U\mathcal{T}))_P$ with respect to the structure 2-form Ω is $d^{-d\eta(\mathcal{T})/\eta(\mathcal{T})}$-exact.

(3) The conformal symplectic form ψ of M is a relative integral invariant of $U\mathcal{T}$, i.e. $d(\mathcal{L}_{U\mathcal{T}}\psi) = 0$.

4. IMPROPER IMMERSIONS IN $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$

We say taht an n-foliation F on m-dimensional pseudo-Riemannian manifold $M(n < m)$ is an *improper foliation* if the maximal leaf of F is an improper manifold of M.

Consider at each point p of M the two complementary distributions:

$$D_a = \text{vect}.\{h_a, \xi_{m^*+1}; a = 1, \dots, m\},$$

$$D_{a^*} = \text{vect}.\{h_{a^*}, \xi_{m^*+2}; a^* = a + m\}$$

and denote by

$$\sigma_a = \omega^1 \wedge \dots \wedge \omega^m \wedge \eta^{m^*+1}$$

and

$$\sigma_{a^*} = \omega^{1^*} \wedge \ldots \wedge \omega^{m^*} \wedge \eta^{m^*+2}$$

the simple unit forms corresponding respectively to D_a and D_{a^*}. By (2.4), (2.15), (2.16) and making use of (2.13) one finds

$$(4.1) \qquad \begin{cases} d\sigma_a = -(m\eta + \theta) \wedge \sigma_a \Leftrightarrow d^{m\eta+\theta}\sigma_a = 0, \\ d\sigma_{a^*} = -(m\eta + \theta) \wedge \sigma_{a^*} \Leftrightarrow d^{m\eta+\theta}\sigma_{a^*} = 0 \end{cases}$$

where

$$(4.2) \qquad \theta = \sum_a \theta_a^a \in \Lambda^1 M$$

is called teh *Ricci 1-form* [19] (one always has $d\theta = 0$).

Since both $(m+1)$-forms σ_α and σ_{a^*} are exterior recurrent and σ_a (respectively σ_{a^*}) annihilates D_{a^*} (respectively D_a), it follows by Frobenius theorem that both, D_a and D_{a^*}, are $(m+1)$-foliations.

Consider for instance the foliation D_a and denote by orth D_a the distribution which is orthogonal to D_a. Clearly by (2.10) and (2.11) one has orth $D_a \subset D_a$ which shows that D_a is a *coisotropic foliation*. Obviously D_{a^*} enjoys the same property.

Denote by M_a and M_{a^*} the maximal leaves of D_a and D_{a^*} respectively and by

$$dp_a = \omega^a \otimes h_a + \eta^{m^*+1} \otimes \xi_{m^*+1}$$

and

$$dp_{a^*} = \omega^{a^*} \otimes h_{a^*} + \eta^{m^*+2} \otimes \xi_{m^*+2}$$

the corresponding soldering forms.

Since $\xi_\alpha (\alpha = m^* + 1, m^* + 2)$ are the only *anisotropic vectors* of these forms, it follows at once that $g_a = (\eta^{m^*+1})^2$, $g_{a^*} = (\eta^{m^*+2})^2$ (we denote the induced elements on M_a and M_{a^*} by the same letters).

Therefore M may be also viewed as foliated by M_a and M_{a^*} where M_a and M_{a^*} are coisotropic and of *defect* $d = m$ submanifolds of $M (d = \dim M_a - \operatorname{rank} g_a)$.

It should be noted that M_a and M_{a^*} can also be regarded as *anti-invariant submanifolds* of M [27].

Effectively let us consider M_a and denote by $T_{p_a}(M_a)$ and $T_{p_a}^{\perp}(M_a)$ the tangent space and the normal space respectively at any point $p_a \in M_a$.

By reference to (2.10) one has $UT_{p_a}(M_a) = T_{p_a}^{\perp}(M_a)$ which proves the above assertion.

The equation (4.1) also shows that the manifold M is endowed with an *exterior recurrent* structure [2]. Then the recurrence form $m\eta+\theta$ (respectively $(m\eta-\theta)$ defines an element of $H^1(D_{a^*};\mathbf{R}))$ which constitutes the *first class of cohomology* of the foliation D_{a^*} (respectively of D_a (see [16]). It is easy to see that on M_a the form $-(m\eta+\theta)$ moves to $-\theta$, and on M_{a^*} the form $-(m\eta-\theta)$ moves to θ.

Using a generalization of Tachibana theorem [25] and results of [23], one may say that $\flat^{-1}(-\theta)$ (respectively $\dot{\flat}^{-1}(\theta))$ represents the *improper mean curvature vector* of M_{a^*} (respectively of M_a).

Using (2.3) and (2.4), one readily finds that on M_a and M_{a^*} we have

$$\nabla^2\xi_{m^*+1} = (f^2_{m^*+1}-c)\eta^{m^*+1}\wedge dp_a$$

and

$$\nabla^2\xi_{m^*+2} = (f^2_{m^*+2}-c)\eta^{m^*+2}\wedge dp_{a^*}.$$

This proves that on the coisotropic submanifolds M_a and M_{a^*} the anisotropic vecotr fields ξ_α are *exterior concurrent* [23]. This property allows at once to write

$$\mathrm{Ric}(\xi_\alpha) = m(f^2-c).$$

Let now $\kappa\colon\ M_I \to M(U,\Omega,\xi_\alpha,\eta^\alpha,g)$ be the improper immerison of a general coisotropic submanifold M_I in M. By definition, one has $T^\perp_{p_I}(M_I)\subset T_{p_I}(M_I)$ and, without loss of generality, we may assume that $T^\perp_{p_I}(M_I)\subset S_p=\mathrm{vect}.\{h_a\}\subset D_a$.

Following [12] we call S_p the *normal self-orthogonal* space associated with κ, and we assume that $\dim T^\perp_{p_I}(M_I)=l(l<m)$.

Consider now on M_I the two complimentary differentiable distributions:

$$D: p_I\to D_{p_I}=T_{p_I}(M_I)\backslash T^\perp_{p_I}(M_I)$$

$$D^\perp: p_I\to D^\perp_{p_I}=T^\perp_{p_I}(M_I)\subset T_{p_I}(M_I).$$

It easy to find from (2.7) that one has

$$UD_{p_I}\subset D_{p_I},\quad UD^\perp_{p_I}=T^\perp_{p_I}(M_I), \tag{4.3}$$

and therefore follwing [12] one can say that the submanifold M_I under consideration is a $CICR$submanifold (i.e. coisotropic contact CR-submanifold).

Suppose that M_I is defined by

$$(4.4) \qquad \omega^{r^*} = 0, \quad r^*, s^* = 2m+2-l, \dots, 2m.$$

Then one has $D_{p_I} = \text{vect}.\{h_i, h_{i^*}, \xi_\alpha\}$ and $D_{p_I}^\perp = \text{vect}.\{h_r;\ r = m+2-l, \dots, m\}$ and D_{p_I} (respectively $D_{p_I}^\perp$) is called the *horizontal* (respectively the *vertical*) distribution of M_I. Denote by

$$(4.5) \qquad \psi_I = \psi|_{M_I} = \sum_i \omega^i \wedge \omega^{i^*} + \varphi$$

the restriction of the conformal symplectic form ψ on M_I. Then (up to sign) the simple unito form corresponding to the horizontal distribution D_{p_I} is expressed by

$$\sigma_I = \psi_I^{m-l+1}.$$

Obviously by (2.23) σ_I is exterior recurrent and since it annihilates the vertical distribution $D_{p_I}^\perp$, it follows that the $D_{p_I}^\perp$, it follows that the $D_{p_I}^\perp$ is involutive. One refinds in this manner a general property of $CICR$-submanifolds (see [28], [24] and also [4]) and CR-submanifolds (see [2]).

Denote by $M_I^\perp$ the maximal leaf of $D_{p_I}^\perp$. By (2.12), (2.16), (5.4) and making use of (2.10), it follows that $\kappa\colon M_{p_I}^\perp \to M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ is a *totally geodesic improper immersion* (see also [12]).

Similar discussion to that of [21] can be developed.

Theorem 4.1. *Let $M(U, \Omega, \xi_\alpha, \eta^\alpha, g)$ be the manifold defined in Section 2. Any such manifold may be also regarded as foliated by M_a and M_{a^*}, where M_a and M_{a^*} are $(m+1)$-dimensional coisotropic and of defect m submanifolds of M. In addition, the anisotropic vector field on each of these submanifolds is exterior concurrent. If M_I is a general coisotropic submanifold of M, then it is $CICR$-submanifold and the corresponding vertical distribution of M_I is involutive.*

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