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# GENERALIZED JACOBI IDENTITIES

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## 1. INTRODUCTION

In any Lie ring $(L, +, \circ)$, the following equations hold for arbitrary elements $x_j \in L$:

$$x_1 \circ x_2 + x_2 \circ x_1 = 0$$

$$(x_1 \circ x_2) \circ x_3 + (x_3 \circ x_1) \circ x_2 + (x_2 \circ x_3) \circ x_1 = 0 \qquad \text{(Jacobi identity).}$$

In [4, (9a)], Wever already notes that one also has

$$((x_1 \circ x_2) \circ x_3) \circ x_4 + ((x_2 \circ x_1) \circ x_4) \circ x_3 + ((x_3 \circ x_4) \circ x_1) \circ x_2 + ((x_4 \circ x_3) \circ x_2) \circ x_1 = 0,$$

and proves some further similar assertions in the same paper [4, Hilfssatz 1]. The aim of this note is, roughly spoken, to determine *all* relations of this kind.

As the relations in question shall hold in *any* Lie ring, we consider them as relations of free generators $x_j$ of a free Lie algebra over an arbitrary commutative ring $R$ with identity 1 [(1)]. Then it turns out that in order to determine all relations of the type mentioned above one has to describe a certain right ideal of the group ring of some symmetric group over $R$. In fact, our main result will solve the problem by exhibiting an $R$-basis of this «right ideal of generalized Jacobi identities».

Our approach makes use of Witt's theorem [1, 2.3.3] on the embedding of a free Lie algebra $L$ in a free associative algebra $A$. We begin by defining group actions on $A$ and then exploit these for the study of $L$. Not all details of our introductory analysis of the module structures of $A$ and $L$ will be used in full for our description of generalized Jacobi identities. We have, however, attempted to present this analysis in its natural general setting, to avoid an unsatisfactory ad hoc collection of merely pragmatic statements.

## 2. GROUP ACTIONS ON FREE ALGEBRAS

Let $R$ be a commutative ring with $1, n \in \mathbb{N}$, and $A$ be the free associative $R$-algebra with $n$ free generators $x_1, \ldots, x_n$. The set of all monomials $x_{i_1} \ldots x_{i_m}$, where $m \in \mathbb{N}_0$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$, are an $R$-basis of $A$. The number $m$ is called the degree of $x_{i_1} \ldots x_{i_m}$.

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(1) To our knowledge, only recently rigorous and conscious attempts have been made to develop substantial parts of Lie theory over more general scalar domains than fields (compare, e.g., the expositions in [1] and [2]).

For all $m \in \mathbb{N}_0$, let $A_m$ be the $R$-span of all monomials of degree $m$. Then (writing 1 for the empty monomial, the identity element of $A$),

$$A_0 = R1,$$
$$A_1 = Rx_1 + Rx_2 + \ldots + Rx_n,$$
$$A_2 = Rx_1^2 + Rx_1x_2 + \ldots + Rx_1x_n + Rx_2x_1 + \ldots + Rx_2x_n + \ldots + Rx_n^2,$$

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For all $l \in \{1, \ldots, n\}$ we define the $x_l$-degree of the monomial $x_{i_1} \ldots x_{i_m}$ to be the number of all $j \in \{1, \ldots, m\}$ such that $x_{i_j} = x_l$. If $k_1, \ldots, k_n \in \mathbb{N}_0$, we write $A(k_1, \ldots, k_n)$ for the $R$-span of all monomials whose $x_j$-degree is $k_j$ for $1 \leq j \leq n$. Then, for example,

$$A(1,0,\ldots,0) = Rx_1,$$
$$A(1,0,\ldots,0,1) = Rx_1x_n + Rx_nx_1.$$

Obviously, for all $m \in \mathbb{N}_0$,

$$A_m = \sum_{\substack{(k_1,\ldots,k_n) \\ k_1+\ldots+k_n=m}}^{\oplus} A(k_1, \ldots, k_n).$$

Let $G := GL_R(A_1) \cong GL(n, R)$. We define an action of $G$ on $A$ by

$$(1) \qquad (x_{i_1} \ldots x_{i_m})\gamma := (x_{i_1}\gamma) \ldots (x_{i_m}\gamma) \quad \text{for all } \gamma \in G$$

and $R$-linear extension. This yields a homomorphism of $G$ into $\mathrm{End}_R\, A$, as one easily verifies $a(\gamma\delta) = (a\gamma)\delta$ for all $a \in A$, $\gamma$, $\delta \in G$. Thus $A$ is a $G$-right module and may therefore likewise be viewed as an $RG$-right module. Each element of $G$ maps any monomial of degree $m$ onto an $R$-linear combination of monomials of degree $m$. Hence $A_m$ is an $RG$-submodule of $A$, for all $m \in \mathbb{N}_0$.

The $R$-space $A_m$ admits an action of the symmetric group $S_m$, given by

$$(2) \qquad \sigma(x_{i_1} \ldots x_{i_m}) := x_{i_{1\sigma}} \ldots x_{i_{m\sigma}} \quad \text{for all } \sigma \in S_m$$

and $R$-linear extension. For $\sigma, \tau \in S_m$, we have

$$\begin{aligned}(\sigma\tau)(x_{i_1}\dots x_{i_m}) &= x_{i_{(1\sigma)\tau}}\dots x_{i_{(m\sigma)\tau}}\\ &= \sigma(x_{i_{1\tau}}\dots x_{i_{m\tau}})\\ &= \sigma(\tau(x_{i_1}\dots x_{i_m})).\end{aligned}$$

Hence we have an antihomomorphism of $S_m$ into $\mathrm{End}_R A_m$, making $A_m$ into an $RS_m$-left module. Under the action of an element $\sigma \in S_m$, the factors $x_{i_1}, \dots, x_{i_m}$ of a monomial $x_{i_1}\dots x_{i_m}$ are permuted. Hence, for $1 \leq l \leq n$, the $x_l$-degrees of $x_{i_1}\dots x_{i_m}$ and of $\sigma(x_{i_1}\dots x_{i_m})$ are the same. This yields

(3) *The $R$-spaces $A(k_1, \dots, k_n)$ where $k_1 + \dots + k_n = m$ are $RS_m$-submodules of $A_m$.*

A general version of the following statement is well known (and due to Schur to our knowledge) in the context of tensor algebras (see, e.g., [3, 4.(2.31)]):

(4) *The actions of $RG$ and $RS_m$ on $A_m$ commute.*

Let $\sigma \in S_m$, $\gamma \in G$. For $1 \leq j \leq n$ let $c_{jl}$ $(1 \leq l \leq n)$ be the elements of $R$ such that $x_j\gamma = \sum_{l=1}^{n} c_{jl}x_l$. Then

$$\begin{aligned}\sigma((x_{i_1}\dots x_{i_m})\gamma) &= \sigma\left(\left(\sum_{1\leq l\leq n} c_{i_1 l}x_l\right)\cdot\ldots\cdot\left(\sum_{1\leq l\leq n} c_{i_m l}x_l\right)\right)\\ &= \sigma\left(\sum_{(l_1,\dots,l_m)}\left(\prod_{j=1}^{m} c_{i_j l_j}\right) x_{l_1}\dots x_{l_m}\right)\\ &= \sum_{(l_1,\dots,l_m)}\left(\prod_{j=1}^{m} c_{i_j l_j}\right) x_{l_{1\sigma}}\dots x_{l_{m\sigma}}\\ &= \sum_{(l_{1\sigma},\dots,l_{m\sigma})}\left(\prod_{j=1}^{m} c_{i_j l_{(j\sigma)\sigma^{-1}}}\right) x_{l_{1\sigma}}\dots x_{l_{m\sigma}}\\ &= \sum_{(l_1,\dots,l_m)}\left(\prod_{j=1}^{m} c_{i_j l_{j\sigma^{-1}}}\right) x_{l_1}\dots x_{l_m}\\ &= \sum_{(l_1,\dots,l_m)}\left(\prod_{j=1}^{m} c_{i_{j\sigma} l_j}\right) x_{l_1}\dots x_{l_m}\end{aligned}$$

$$= \left(\sum_{1\le l\le n} c_{i_{1\sigma}l}x_l\right)\cdot\ldots\cdot\left(\sum_{1\le l\le n} c_{i_{m\sigma}l}x_l\right)$$
$$= (x_{i_{1\sigma}}\gamma)\cdot\ldots\cdot(x_{i_{m\sigma}}\gamma)$$
$$= (\sigma(x_{i_1}\ldots x_{i_m}))\gamma.$$

Obviously, this implies (4).

The natural action of $S_n$ on $A_1$, given by

$$(5) \qquad x_i\sigma := x_{i\sigma} \quad \text{for} \quad 1\le i\le n, \quad \sigma\in S_n,$$

and linear extension, yields an embedding of $S_n$ into G. Via (1), this action extends therefore to $A$ where one has, by (5),

$$(6) \qquad (x_{i_1}\ldots x_{i_m})\sigma = x_{i_1\sigma}\ldots x_{i_m\sigma} \quad \text{for all } \sigma\in S_n.$$

In particular, any $A_m$ is an $RS_n$-right module under this action.
Now we set $m = n$. The foregoing discussion yields

(7) *Via (2) and (6), $A_n$ is an $RS_n$-left and right module. By (4), the left and right actions commute.*

The $R$-subspace $A(1,\ldots,1)$ of $A_n$ is, by (3), invariant under the $RS_n$-left action, and, by (6), invariant under the $RS_n$-right action. Obviously, $\{x_{1\sigma}\ldots x_{n\sigma}\,|\,\sigma\in S_n\}$ is an $R$-basis of $A(1,\ldots,1)$. We have

$$(8) \qquad \tau(x_{1\sigma}\ldots x_{n\sigma}) = x_{1\tau\sigma}\ldots x_{n\tau\sigma} \quad \text{for all } \sigma,\tau\in S_n.$$

Hence, under the left action, $A(1,\ldots,1)$ is isomorphic to the regular $RS_n$-left module. Moreover, we have

$$(9) \qquad (x_{1\sigma}\ldots x_{n\sigma})\tau = x_{1\sigma\tau}\ldots x_{n\sigma\tau} \quad \text{for all } \sigma,\tau\in S_n,$$

whence $A(1,\ldots,1)$ is likewise isomorphic to the regular $RS_n$-right module under the right action. In particular,

$$(10) \qquad \alpha(x_1\ldots x_n) = (x_1\ldots x_n)\alpha \quad \text{for all } \alpha\in RS_n.$$

We now turn to the Lie multiplication in $A$ which is defined by

$$a\circ b := ab - ba \quad \text{for all } a,b\in A.$$

Let $L \subseteq A$ be the Lie algebra generated by $x_1, \ldots, x_n$. By Witt's theorem, $L$ is free and freely generated by $x_1, \ldots, x_n$. For all $m, k_1, \ldots, k_n \in \mathbb{N}_0$, we set

$$L_m := L \cap A_m$$
$$L(k_1, \ldots, k_n) := L \cap A(k_1, \ldots, k_n).$$

Clearly $L_0 = \{0\}$ and $L_1 = A_1$. For $m \geq 2$, things become more difficult. The $R$-rank of $L_m$ is given by Witt's formula ([1,3.1.3]). For example, the $R$-rank of $L_2$ is $\binom{n}{2}$, whereas the $R$-rank of $A_2$ is $n^2$. The Specht-Wever theorem ([1,2.5.5]) is a criterion for an element of $A_m$ to belong to $L_m$ in the case that $R$ is a field of characteristic $0$. The crucial mapping of $A_m$ onto $L_m$ in its proof will also be important for our investigations: We set $\omega_0 := 0$ and for $m \geq 1$, as in [4,(4)],

$$\omega_m := \prod_{j=m}^{2} (id - (j, \ldots, 1)) =$$
$$=(id - (m, \ldots, 1)) \cdot \ldots \cdot (id - (2,1)) \in RS_m.$$

Then

(11) $$x_{i_1} \circ \ldots \circ x_{i_m} = \omega_m(x_{i_1} \ldots x_{i_m}) \quad \text{for all } m \in \mathbb{N}$$

where, by inductive definition, the «left normed Lie monomial» $x_{i_1} \circ \ldots \circ x_{i_m}$ is defined to be $(x_{i_1} \circ \ldots \circ x_{i_{m-1}}) \circ x_{i_m}$ for $m > 2$. We prove (11) by induction on $m$. As $\omega_1 = id$, (11) holds for $m = 1$. For $m \geq 2$, we have (using the induction hypothesis in the third step),

$$\begin{aligned}\omega_m(x_{i_1} \ldots x_{i_m}) &=(id - (m, \ldots, 1))(\omega_{m-1}(x_{i_1} \ldots x_{i_m})) \\ &=(id - (m, \ldots, 1))((\omega_{m-1}(x_{i_1} \ldots x_{i_{m-1}}))x_{i_m}) \\ &=(id - (m, \ldots, 1))((x_{i_1} \ldots x_{i_{m-1}})x_{i_m}) \\ &=(x_{i_1} \circ \ldots \circ x_{i_{m-1}})x_{i_m} - (m, \ldots, 1)(x_{i_1} \circ \ldots \circ x_{i_{m-1}})x_{i_m}.\end{aligned}$$

But $(m, \ldots, 1)(x_{i_1} \ldots x_{i_m}) = x_{i_m} x_{i_1} \ldots x_{i_{m-1}}$. Hence

$$(m, \ldots, 1)(w x_{i_m}) = x_{i_m} w \quad \text{for all } w \in A_{m-1}.$$

In particular, $(m, \ldots, 1)((x_{i_1} \circ \ldots \circ x_{i_m})x_{i_m}) = x_{i_m}(x_{i_1} \circ \ldots \circ x_{i_{m-1}})$, which completes the inductive step, yielding (11).

As a consequence, we note

$$(12) \qquad (x_{i_1} \circ \ldots \circ x_{i_m})\gamma = (x_{i_1}\gamma) \circ \ldots \circ (x_{i_m}\gamma) \quad \text{for all } \gamma \in G.$$

This follows from (4) and (11), as $(x_{i_1} \circ \ldots \circ x_{i_m})\gamma = \omega_m((x_{i_1} \ldots x_{i_m})\gamma) = \omega_m((w_{i_1}\gamma) \ldots (x_{i_m}\gamma)) = (x_{i_1}\gamma) \circ \ldots \circ (x_{i_m}\gamma)$. In particular, $L_m$ is a $G$-submodule of $A_m$. Finally, applying (12) in the special case of $m = n$, we conclude

$$(13) \qquad (x_{1\tau} \circ \ldots \circ x_{n\tau})\sigma = x_{1\tau\sigma} \circ \ldots \circ x_{n\tau\sigma} \quad \text{for all } \sigma, \tau \in S_n.$$

In particular, $L(1, \ldots, 1)$ is a submodule of $A(1, \ldots, 1)$, considered as an $RS_n$-right module (cf. (6)). (It should be noted that this is not true with respect to the left action (2).)

It is well known (see, e.g., [4, §1]) that $L(1, \ldots, 1)$ is the $R$-span of the left normed Lie monomials $x_{1\sigma} \circ \ldots \circ x_{n\sigma} (\sigma \in S_n)$. Therefore, as an $RS_n$-right module, $L(1, \ldots, 1)$ is generated by $x_1 \circ \ldots \circ x_n$ and hence cyclic.

## 3. THE RIGHT IDEAL OF GENERALIZED JACOBI IDENTITIES

We are ready for the key definition of this paper:

**Definition.** *A generalized Jacobi identity of degree $n$ over $R$ is an element $\iota \in RS_n$ such that $(x_1 \circ \ldots \circ x_n)\iota = 0$.*

If $\iota = \sum_{\sigma \in S_n} r_\sigma \sigma \; (r_\sigma \in R)$, this means that $\sum_{\sigma \in S_n} r_\sigma x_{1\sigma} \circ \ldots \circ x_{n\sigma} = 0$.

The mapping $RS_n \to L(1, \ldots, 1), \alpha \to (x_1 \circ \ldots \circ x_n)\alpha$ is an $RS_n$-right module epimorphism. Hence its kernel $\mathcal{J}$ is a right ideal of $RS_n$ and obviously consists of all generalized Jacobi identities of degree $n$ over $R$. Replacing $x_1 \circ \ldots \circ x_n$ by another left normed generator $x_{1\tau} \circ \ldots \circ x_{n\tau} (\tau \in S_n)$ leads to $\tau^{-1}\mathcal{J}$ as the kernel of the corresponding $RS_n$-right module epimorphism. This simple observation may justify the recurrence on the special Lie monomial $x_1 \circ \ldots \circ x_n$ in our definition of a generalized Jacobi identity. Our notion does not essentially depend on this particular monomial.

Our aim is to show that $\mathcal{J}$ is a free $R$-module of rank $(n-1)!(n-1)$ and to determine an $R$-basis of $\mathcal{J}$. For all $j \in \{1, \ldots, n\}$ we set

$$\eta_j := \omega_j - \omega_{j-1}.$$

For example, $\eta_1 = id$, $\eta_2 = -(2,1)$, $\eta_3 = -(3,2,1) + (3,1)$, etc. In general, one readily verifies

$$(14) \qquad \eta_j = -(j,\ldots,1) \cdot \prod_{k=j-1}^{2} (id - (k,\ldots,1)) \quad \text{for } j \geq 2.$$

The description of $\mathcal{J}$ rests on the following

**Lemma.** *$id - \eta_j \in \mathcal{J}$ for all $j \in \{1, \ldots, n\}$.*

*Proof* . We show first

$$(15) \qquad (x_1 \circ \ldots \circ x_n)\omega_j = j x_1 \circ \ldots \circ x_n \quad \text{for all } j \in \{1,\ldots,n\}.$$

Indeed one has

$$\begin{aligned}
(x_1 \circ \ldots \circ x_n)\omega_j &= ((x_1 \circ \ldots \circ x_j)\omega_j) \circ x_{j+1} \circ \ldots \circ x_n \\
&= ((\omega_j(x_1 \ldots x_j))\omega_j) \circ x_{j+1} \circ \ldots \circ x_n && \text{by (11)} \\
&= \omega_j((x_1 \ldots x_j)\omega_j) \circ x_{j+1} \circ \ldots \circ x_n && \text{by (7)} \\
&= (\omega_j^2(x_1 \ldots x_j)) \circ x_{j+1} \circ \ldots \circ x_n && \text{by (10)} \\
&= j(\omega_j(x_1 \ldots x_j)) \circ x_{j+1} \circ \ldots \circ x_n && \text{by [4, §3]} \\
&= j(x_1 \circ \ldots \circ x_j) \circ x_{j+1} \circ \ldots \circ x_n && \text{by (11).}
\end{aligned}$$

Using (15) and the fact that $\omega_0 = 0$, we conclude that

$$(x_1 \circ \ldots \circ x_n)\eta_j = (x_1 \circ \ldots \circ x_n)\omega_j - (x_1 \circ \ldots \circ x_n)\omega_{j-1} = x_1 \circ \ldots \circ x_n.$$

This proves our claim.

**Theorem.** *$\mathcal{J}$ is a free R-module, freely generated by the elements $\sigma - \eta_{1\sigma^{-1}}\sigma$ where $\sigma \in S_n$ such that $1\sigma \neq 1$.*

*Proof* . As $\mathcal{J}$ is a right ideal of $RS_n$, our Lemma implies that $\sigma - \eta_j\sigma = (id - \eta_j)\sigma \in \mathcal{J}$ for all $\sigma \in S_n$, $j \in \{1, \ldots, n\}$. Now let $T := \{\sigma | \sigma \in S_n, 1\sigma = 1\}$ and $B := \{\beta |$ there exists $\sigma \in S_n \backslash T$ such that $\beta = \sigma - \eta_{1\sigma^{-1}}\sigma\}$. Then (14) shows that

$$(16) \qquad \eta_{1\sigma^{-1}}\sigma \in RT \quad \text{for all } \sigma \in S_n.$$

Let $\langle B\rangle$ be the $R$-linear span of $B$. By (16), $\langle B\rangle + RT = RS_n$. If $c_\sigma \in R$ such that $\sum_{\sigma\in S_n\setminus T} c_\sigma(\sigma - \eta_{1\sigma^{-1}}\sigma) \in RT$, then $\sum_{\sigma\in S_n\setminus T} c_\sigma\sigma \in RT$, by (16). But this implies that $c_\sigma = 0$ for all $\sigma \in S_n\setminus T$. We conclude that

$$RS_n = \langle B\rangle \oplus RT \tag{17}$$

and, moreover, that $B$ is an $R$-free set of $(n-1)!(n-1)$ elements. As $B \subseteq \mathcal{J}$, all that remains to do is to prove that $\mathcal{J} \cap RT = 0$. It is well known that the elements $x_{1\sigma} \circ \ldots \circ x_{n\sigma}$ where $\sigma \in T$ form an $R$-basis of $L(1, \ldots, 1)$ (cf., e.g., [1,4.8.1. Lemma 5]). Therefore, the mapping $RT \to L(1, \ldots, 1), \alpha \to (x_1 \circ \ldots \circ x_n)\alpha$, is bijective. Hence $\mathcal{J} \cap RT = 0$.

Finally we want to express our basis elements of $\mathcal{J}$ as $R$-linear combinations of elements of $S_n$. We need the following

**Proposition.** *Let $n \geq j_1 > \ldots > j_k > 1$. Then, in $S_n$, we have*

$$(j_1, \ldots, 1)(j_2, \ldots, 1)\ldots(j_k, \ldots, 1) =$$

$$= \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & k+2 & \ldots & n \\ j_1 & j_2 & \ldots & j_k & 1 & i_{k+2} & \ldots & i_n \end{pmatrix}$$

*where $1 < i_{k+2} < \ldots < i_n \leq n$.*

*Proof by induction on $k$.* The case $k = 1$ is obvious. As for the step from $k$ to $k+1$, we calculate, using the induction hypothesis,

$$(j_1, \ldots, 1)\ldots(j_{k+1}, \ldots, 1) =$$

$$= \begin{pmatrix} 1 & \ldots & k & k+1 & k+2 & \ldots & n \\ j_1 & \ldots & j_k & 1 & i_{k+2} & \ldots & i_n \end{pmatrix}(j_{k+1}, \ldots, 1)$$

$$= \begin{pmatrix} 1 & \ldots & k & k+1 & k+2 & \ldots & k+j_k-1 & k+j_k & \ldots & n \\ j_1 & \ldots & j_k & 1 & 2 & \ldots & j_k-1 & i_{k+j_k} & \ldots & i_n \end{pmatrix}(j_{k+1}, \ldots, 1)$$

$$= \begin{pmatrix} 1 & \ldots & k & k+1 & k+2 & \ldots & k+j_{k+1} & k+j_{k+1}+1 & \ldots & n \\ j_1 & \ldots & j_k & j_{k+1} & 1 & \ldots & j_{k+1}-1 & i_{k+j_{k+1}+1} & \ldots & i_n \end{pmatrix},$$

and indeed $1 < \ldots < j_{k+1} - 1 < i_{k+j_{k+1}+1} < \ldots < i_n \leq n$.

**Corollary.** *Let* $X := \{\pi | \pi \in S_n, 1\pi > 2\pi > \ldots > (1\pi^{-1})\pi < (1\pi^{-1}+1)\pi < \ldots < n\pi\}$.

(a) $\eta_j = \sum_{\substack{\pi \in X \\ 1\pi = j}} (-1)^{1\pi^{-1}} \pi$ *for* $n \geq j \geq 1$.

(b) *If* $\sigma \in S_n$ *such that* $1\sigma \neq 1$, *then*

$$\sigma - \eta_{1\sigma^{-1}}\sigma = \sigma + \sum_{\substack{\pi \in X \\ 1\pi\sigma = 1}} (-1)^{1\pi^{-1}} \pi\sigma.$$

*Proof* . By (14) and our Proposition, we have for $2 \leq j \leq n$

$$\eta_j = -\sum_{\substack{\pi \in X \\ 1\pi = j}} (-1)^{1\pi^{-1}-2} \pi = -\sum_{\substack{\pi \in X \\ 1\pi = j}} (-1)^{1\pi^{-1}} \pi,$$

proving (a). Therefore, if $\sigma \in S_n$ such that $1\sigma \neq 1$, then $\sigma - \eta_{1\sigma^{-1}}\sigma = \sigma + \sum_{\substack{\pi \in X \\ 1\pi = 1\sigma^{-1}}} (-1)^{1\pi^{-1}} \pi\sigma$, as claimed in (b).

In particular, in the representation of a basis element $\beta$ $(= \sigma - \eta_{1\sigma^{-1}}\sigma)$ as an $R$-linear combination of the elements of $S_n$ only 1 and $-1$ occur as coefficients. Thus, in a certain sense, one has always the same basis for the right ideal of generalized Jacobi identities, independent of the structure of $R$.

For an arbitrary element of $RS_n$, given as an $R$-linear combination of elements of $S_n$, it is easy to decide if it belongs to $\mathcal{J}$ and then to write it as an $R$-linear combination of our basis elements of $\mathcal{J}$:

**Remark.** Let $c_\sigma \in R$ for every $\sigma \in S_n$. The following assertions are equivalent:

$(i)$
$$\sum_{\sigma \in S_n} c_\sigma \sigma \in \mathcal{J}$$

$(ii)$
$$\sum_{\sigma \in S_n} c_\sigma \sigma = \sum_{\substack{\sigma \in S_n \\ 1\sigma \neq 1}} c_\sigma (\sigma - \eta_{1\sigma^{-1}}\sigma)$$

$(iii)$
$$\sum_{\substack{\sigma \in S_n \\ 1\sigma = 1}} c_\sigma \sigma = -\sum_{\substack{\sigma \in S_n \\ 1\sigma \neq 1}} c_\sigma \eta_{1\sigma^{-1}}\sigma$$

(We note as a consequence: The mapping $\sum_{\sigma\in S_n} c_\sigma\sigma \to \sum_{\sigma\in S_n} c_\sigma \eta_{1\sigma^{-1}}\sigma$ is the projection of $RS_n$ onto $RT$ with respect to the direct decomposition (17)).

*Proof*. Let $T := \{\sigma | \sigma \in S_n, 1\sigma = 1\}$. By (16), $\sum_{\sigma\in S_n} c_\sigma\sigma - \sum_{\substack{\sigma\in S_n \\ 1\sigma\neq 1}} c_\sigma\,(\sigma - \eta_{1\sigma^{-1}}\sigma) \in RT$.

This implies the equivalence of (i) and (ii) as, by (17), $\mathcal{J} \cap RT = 0$. The equivalence of (ii) and (iii) is obvious.

For example, the three identities corresponding to the equations at the very beginning of this paper are represented in the following way:

$$id + (1,2) = (1,2) - \eta_2(1,2),$$

$$\begin{aligned} id &+ (1,3,2) + (1,2,3) = \\ &= [(1,3,2) - \eta_2(1,3,2)] + \\ &+ [(1,2,3) - \eta_3(1,2,3)] \qquad \text{(Jacobi identity)}, \end{aligned}$$

$$\begin{aligned} id &+ (1,2)(3,4) + (1,3)(2,4) + (1,4)(2,3) = \\ &= [(1,2)(3,4) - \eta_2(1,2)(3,4)] + \\ &+ [(1,3)(2,4) - \eta_3(1,3)(2,4)] + \\ &+ [(1,4)(2,3) - \eta_4(1,4)(2,3)]. \end{aligned}$$

REFERENCES

[1] Y.A. BAHTURIN, *Identical Relations in Lie Algebras*, Utrecht 1987.

[2] N. JACOBSON, *Lie Algebras*, New York 1962.

[3] R. KEOWN, *An Introduction to Group Representation Theory*, New York - London 1975.

[4] F. WEVER, *Über Invarianten in Lie'schen Ringen*, Math. Ann. **120** (1949) pp. 563-580.

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