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SEMIGROUPS IN CARTAN DOMAINS OF TYPE FOUR

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1. J^*-ALGEBRAS AND BOUNDED SYMMETRIC DOMAINS

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, and let $\mathcal{L}(\mathcal{K},\mathcal{H})$ be the complex Banach space of all bounded linear operators $\mathcal{K} \to \mathcal{H}$. For $A \in \mathcal{L}(\mathcal{K},\mathcal{H}), A^* \in \mathcal{L}(\mathcal{H},\mathcal{K})$ will indicate the adjoint operator of A.

A J^*-algebra is a closed linear subspace $\mathfrak{F}$ of $\mathcal{L}(\mathcal{K},\mathcal{H})$ such that, if $A \in \mathfrak{F}$ then $AA^*A \in \mathfrak{F}$. The notion of J^*-algebra has been introduced by L.A. Harris in [5]. We refer to this paper for all basic facts on J^*- algebras, and to [6], [7], [8] for further developments.

For example, $\mathfrak{F} = \mathcal{L}(\mathcal{K},\mathcal{H})$ is a J^*-algebra. If $n = dim_{\mathbb{C}}\mathcal{H} < \infty$ and $m = dim_{\mathbb{C}}\mathcal{K} < \infty, \mathfrak{F} = \mathcal{L}(\mathcal{K},\mathcal{H})$ (can be identified with $\mathbb{C}^{nm}$ and) is called *Cartan factor of type one*. This terminology has been extended by L.A. Harris to the infinite dimensional case.

If $\mathcal{K} = \mathbb{C}, \mathcal{L}(\mathbb{C},\mathcal{H})$ can be canonically identified with the Hilbert space $\mathcal{H}$ which is then a J^*-algebra. If $\mathcal{H} = \mathcal{K}$, the Banach space $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H},\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is a J^*-algebra. By the Gelfand-Naimark theorem [23], every C^*-algebra is therefore a J^*-algebra.

A conjugation in the complex Hilbert space $\mathcal{H}$ is, by definition, a continuous anti-linear map $x \to \overline{x}$ of $\mathcal{H}$ into itself, which is involutory and has norm ≤ 1. It turns out that a conjugation is necessarily a surjective isometry of $\mathcal{H}$. Conjugations always exist in all complex Hilbert spaces and may be so chosen to coincide with their adjoints. Given a conjugation $x \to \overline{x}$ on $\mathcal{H}$, the *linear* operator ${}^tA$ defined for $A \in \mathcal{L}(\mathcal{H})$ by $\overline{{}^tAx} = A^*\overline{x}$ is continuous on $\mathcal{H}$ and is called the *transposed* operator of A. The space

$$\mathfrak{F} = \{A \in \mathcal{L}(\mathcal{H}) : {}^tA = A\}$$

is a J^*-algebra, which is called a *Cartan factor of type two* by an extension of the familiar terminology introduced when $dim_{\mathbb{C}}\mathcal{H} < \infty$.

Similarly, the space

$$\mathfrak{F} = \{A \in \mathcal{L}(\mathcal{H}) : {}^tA = -A\}$$

is a J^*-algebra, called a *Cartan factor of type three*.

A *Cartan factor of type four* is a self-adjoint (i.e. *-invariant) closed subspace $\mathfrak{F}$ of $\mathcal{L}(\mathcal{H})$ such that $A \in \mathfrak{F}$ implies that A^2 is a scalar multiple of the identity $I = I_{\mathcal{H}}$ on $\mathcal{H}$:

$$A^2 = aI$$

for some $a \in \mathbb{C}$. By definition, $A^* \in \mathfrak{F}$, and then

$$A^{*2} = \bar{a} I.$$

If $B \in \mathfrak{F}$, then $B^2 = bI$ and $(A+B)^2 = cI$ for some b and c in $\mathbb{C}$. Hence $AB + BA$ is a scalar multiple of the identity:

$$AB + BA = (A+B)^2 - A^2 - B^2 = (c - a - b)I. \tag{1.1}$$

For $B = A^*$, then

$$AA^*A = (AA^* + A^*A)A - A^*A^2 = (c - 2Re a)A - aA^* \in \mathfrak{F},$$

proving thereby that a Cartan factor of type four is a J^*-algebra [5].

The open unit ball D of a J^*-algebra $\mathfrak{F}$ is a bounded, homogeneous, symmetric domain [5], i.e. the group $AutD$ of all holomorphic automorphisms of D acts transitively on D, and every point of D is an isolated fixed point of an involutory element of $AutD$. In the case in which $\mathfrak{F}$ is a Cartan factor, D is called a *Cartan domain.* In the finite-dimensional case, the Cartan domains of the four types listed above exhaust all finite-dimensional bounded symmetric domains, except for two domains of complex dimensions 16 and 27 respectively. It was shown by O. Loos and K. McCrimmon [19] (cf. also [20]) that these domains are not the open unit balls of J^*-algebras.

Since D is a homogeneous ball, the Kobayashi differential metric κ of D coincides with the Carathéodory metric [(1)].

Let $IsoD$ be the semigroup of all holomorphic maps of D into D which are isometries for κ. The invariance property of κ implies that $AutD$ is a subgroup of $IsoD$. Contrary to what happens in the finite-dimensional case, if $dim_{\mathbb{C}} \mathscr{E} = \infty$, $AutD$ is properly contained in $IsoD$:

$$AutD \subsetneq IsoD. \tag{1.2}$$

In the finite-dimensional case, the Cartan domains were described by É. Cartan [1] in 1935 as quotient of connected simple Lie groups, and their homogeneity was a direct consequence of the construction. However (leaving obviously aside the cases of the unit disc of $\mathbb{C}$ and of the polydisc in $\mathbb{C}^n$), the determination of $AutD$ came only later. The first general result is due to C.L. Siegel [26] who in 1943 determined AutD for any Cartan domain D of type two. Siegel's ground-breaking paper inspired further research by H. Klingen [16], [17], [18] for

(1) For all properties of invariant metrics referred to in this paper, see e.g. [4].

domains of type one and three, and by U. Hirzebruch [13] for domains of type four. In the case of the open unit ball D of any J^*-algebra $\mathfrak{F}$, similar question arise for $AutD$, and - in the infinite dimensional case - generate, in view of (1.2), parallel and still largely unsolved problems for the semigroup $IsoD$.

The description of $AutD$ for an infinite dimensional Cartan domain D was first carried out, for the open unit ball of a complex Hilbert space, by A. Renaud [22] (who extended to any dimension results established by M. Hervé [10] in the finite-dimensional case) and by T.L. Hayden and J.T. Suffridge [9]. The group $AutD$ was investigated by T. Franzoni [3] for any Cartan domain of type one, by J. Hervés [11] for the types two and three, and by L.A. Harris [5] (cf. also [12]) for type four. All these papers follow the same pattern, which consists of two steps. The first one exhibits a group of holomorphic automorphism acting transitively on D. This group consists of "fractional" transformations which, in analogy to the classical terminology for the unit disc of $\mathbb{C}$, are often called "Moebius transformations". The second step yields an explicit construction of the isotropy group $(AutD)_0$ of 0 in $AutD$. An essential tool is here H. Cartan's linearity theorem (cf. e.g. [4]) whereby every element of $(AutD)_0$ is the restriction to D of a linear automorphism of the Banach space $\mathfrak{F}$.

The knowledge of the subgroup of $AutD$ consisting of the Moebius transformations reduces the construction of $IsoD$ to the determination of the isotropy semigroup $(IsoD)_0$ of 0 in $IsoD$. At this point, however, the similarity to the case $AutD$ ends, and the investigation is made much harder by the fact that H. Cartan's linearity theorem fails to hold for $(IsoD)_0$, as examples show [28]. So far, the only case which has been exhaustively dealt with is that of the unit ball of any complex Hilbert space [4, 28].

Let D be now a bounded domain in $\mathbb{C}^n$. According to a classical theorem of H. Cartan [2] (cf. e.g. [21]), the topology of uniform convergence on compact sets in D is the underlying topology of a real Lie group structure in $AutD$ for which the canonical map $AutD \times D \to D$ defined by the action of $AutD$ in D is continuous (actually real-analytic). Hence the investigation of the structure of the group $AutD$, and more specifically the description of the one-parameter subgroups of $AutD$, lies within the framework of Lie algebras.

In the infinite dimensional case different topologies may be considered in $IsoD$ and $AutD$, leading to different one parameter semigroups. These latter have been investigated in [27] when D is the open unit ball of any complex Hilbert space, and in [29] in the case of any Cartan factor $\mathscr{L}(\mathscr{K},\mathscr{H})$ when at least one of the two Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ has finite dimension.

The present paper will report on a similar investigation in the case of Cartan domains of type four.

2. CARTAN FACTORS OF TYPE FOUR AS HILBERT SPACES

The spectral representation of continuous linear operators whose squares are scalar multiples of the identity has been investigated in [30] in the general case in which the operators act on a complex Banach space $\mathscr{U}$.

Let $A \in \mathscr{L}(\mathscr{U})$. The following lemma has been established in [30].

Lemma 2.1. *If $A^2 = aI$ and $a \neq 0$, at least one of the two square-roots of a is contained in the spectrum $\sigma(A)$ of A. If $\sqrt{a} \in \sigma(A)$ and if Q is the spectral projector associated to $\sqrt{a}$, A is expressed by*

$$A = \sqrt{a}(2Q - I).$$

If $a = 0$, then

$$(\xi I - A)^{-1} = \frac{1}{\xi}\left(I + \frac{1}{\xi} A\right) \quad \text{for all } \xi \in \mathbb{C} \setminus \{0\}.$$

If $a \neq 0$, A is invertible in $\mathscr{L}(\mathscr{U})$ and

$$A^{-1} = \frac{1}{\sqrt{a}}(2Q - I). \tag{2.1}$$

Corollary 2.2. *If $a \neq 0$ and $\sigma(A) = \{\sqrt{a}\}$, then A is a scalar multiple of the identity: $A = \sqrt{a}I$.*

Now, let A and B be two non-vanishing elements of $\mathscr{L}(\mathscr{U})$ such that

$$A^2 = aI, \; B^2 = bI, (A + B)^2 = cI, \tag{2.2}$$

for some a, b, c in $\mathbb{C}$. Then (1.1) holds, and, for all $\xi \in \mathbb{C}$.

$$(\xi I - AB)(\xi I - BA) = (\xi - \xi_1)(\xi - \xi_2)I, \tag{2.3}$$

where ξ_1 and ξ_2 are the roots of the quadratic equation

$$\xi^2 + (a + b - c)\xi + ab = 0.$$

By (2.3), $\sigma(AB) \subset \{\xi_1, \xi_2\}$. Since $(\xi_1 I - AB)(\xi_1 I - BA) = 0$, if $\xi_1 \notin \sigma(AB)$, then $BA = \xi_1 I$, and (2.3) yields

$$(\xi - \xi_1)(\xi I - AB) = (\xi - \xi_1)(\xi - \xi_2)I$$

for all $\xi \in \mathbb{C}$, whence $AB = \xi_2 I$. Because $\xi_1 \notin \sigma(AB)$, then $\xi_1 \neq \xi_2$, contradicting the fact that

$$\xi_2 A = ABA = A(BA) = \xi_1 A,$$

proving thereby that

$$\sigma(AB) = \sigma(BA) = \{\xi_1, \xi_2\}.$$

A direct computation using (2.3) shows that, if $\xi_1 \neq \xi_2$,

$$(\xi I - AB)^{-1} = \frac{1}{(\xi - \xi_1)(\xi_1 - \xi_2)}(\xi_1 I - BA) - \frac{1}{(\xi - \xi_2)(\xi_1 - \xi_2)}(\xi_2 I - BA),$$

for all $\xi \neq \xi_1, \xi \neq \xi_2$, proving

Lemma 2.3. *If $\xi_1 \neq \xi_2, \xi_1$ is a pole or order one of $(\cdot I - AB)^{-1}$ and of $(\cdot I - BA)^{-1}$ with residues $\frac{1}{\xi_1 - \xi_2}(\xi_1 I - BA)$ and $\frac{1}{\xi_1 - \xi_2}(\xi_1 I - AB)$.*

Similarly, ξ_2 is a pole of order one of $(\cdot I - AB)^{-1}$ and of $(\cdot I - BA)^{-1}$ with residues $\frac{1}{\xi_2 - \xi_1}(\xi_2 I - BA)$ and $\frac{1}{\xi_2 - \xi_1}(\xi_2 I - AB)$.

Lemma 2.4. *If $\xi_1 = \xi_2, \xi_1$ is a pole of order two of $(\cdot I - AB)^{-1}$ and of $(\cdot I - BA)^{-1}$.*

Now, let $\mathfrak{E}$ be a Cartan factor of type four and let $A, B \in \mathfrak{E}$. By (1.1), $AB^* + B^*A$ is a scalar multiple of the identity. Setting

$$AB^* + B^*A = 2(A|B)I, \tag{2.4}$$

the function $A, B \to (A|B) \in \mathbb{C}$ is a positive-definite inner product on $\mathfrak{E}$, such that

$$\frac{1}{2}||A||^2 \leq (A|A) \leq u||A||^2 \quad for\ all\ A \in \mathfrak{E},$$

where $|| \ ||$ denotes the norm in the Banach space $\mathfrak{E}$ [5]. Thus the inner product expressed by (2.4) defines in $\mathscr{E}$ a complex Hilbert space norm which is equivalent to $|| \ ||$. Conversely, as was shown by L.A. Harris in [5], any complex Hilbert space endowed with a conjugation can be obtained as the Hilbert space associated to a Cartan factor of type four.

If A is orthogonal to I, then

$$0 = 2(A|I)I = AI + IA = 2A.$$

Corollary 2.2 yields

Lemma 2.5. *If $\mathfrak{T}$ contains an element whose spectrum consists of a single non-vanishing complex number, then $\mathfrak{T} \cong \mathbb{C}$.*

If $A \in \mathfrak{T}$ is a normal operator, (2.4) implies that A is a scalar multiple of a unitary operator. If A itself is unitary, by Lemma 2.1 A is expressed by

$$A = e^{i\theta}(2Q - I),$$

where $\theta \in \mathbb{R}$, $\sigma(A) = \{e^{i\theta}, e^{-i\theta}\}$ and, by (2.1), the fact that A is unitary is equivalent to Q being an orthogonal projector. Hence, every unitary operator in $\mathfrak{T}$ is a scalar multiple of a self-adjoint, unitary operator. In conclusion, the following lemma holds [5]:

Lemma 2.6. *Every normal operator in a Cartan factor of type four is a scalar multiple of a self-adjoint unitary operator.*

Since $A^2 = 0$ is equivalent to the condition

$$KerA \supset RanA,$$

every isometry contained in $\mathfrak{T}$ is a unitary operator.

According to a general result of Kadison-Harris [5] the complex extreme points of the closed unit ball of any J^*-algebra $\mathfrak{T}$ coincide with the real extreme points and are those operators $A \in \mathfrak{T}$ satisfying

$$(I - AA^*)Z(I - A^*A) = 0 \tag{2.6}$$

identically for all $Z \in \mathfrak{T}$. In particular, every extreme point is a partial isometry, and (2.6) implies that, if A is a linear isometry, then A is an extreme point of the closed unit ball.

Let now D be the open unit ball of the Cartan factor of type four $\mathfrak{T}$, and let A be an extreme point of $\overline{D}$. Then A is a partial isometry. If A is not an isometry, then $KerA \neq \{0\}$, and therefore $A^2 = 0$. Hence $A^{*2} = 0$, and choosing $Z = A^*$, (2.6) yields $A^* = 0$, i.e., $A = 0$. That proves

Proposition 2.7. *The extreme points of $\overline{D}$ are all the unitary operators in $\mathfrak{T}$.*

Since the Banch space $\mathfrak{T}$ is reflexive [25], the Kreĭn-Milman theorem, Proposition 2.7 and Lemma 2.6 yield

Proposition 2.8. *Any Cartan factor of type four is spanned by its self-adjoint unitary operators.*

The set of these operators is called a spin-system.

Some of the results established above will now be instrumental in expressing the norm $\|\ \|$ in terms of the inner product $(\,|\,)$. Choosing in (2.2) $B = A^*$, then $b = \overline{a}$,

$$c = 2(Re a + (A|A))$$

and $\sigma(A^*A)$ consists of the roots of the quadratic equation

$$\xi^2 - 2(A|A)\xi + |a|^2 = 0. \tag{2.7}$$

Since A^*A is hermitian, $\sigma(A^*A) \subset \mathbb{R}$, i.e.,

$$(A|A)^2 - |a|^2 \geq 0,$$

$$|a| \leq (A|A).$$

But $|a|$ is the spectral radius $\rho(A^2) = \rho(A)^2$ of A^2. Hence

$$\rho(A) = \sqrt{|a|} \leq (A|A)^{1/2} \quad for\ all \quad A \in \mathfrak{F}. \tag{2.8}$$

For $A \neq 0$, the equation (2.7) has one positive and one non-negative real root, and $\rho(A^*A)$ must be the largest of these roots, i.e.,

$$\rho(A^*A) = (A|A) + \sqrt{(A|A)^2 - |a|^2}. \tag{2.9}$$

Because $\rho(A^*A) = \|A^*A\| = \|A\|^2$, and

$$2(A|A^*)I = AA + AA = 2aI, \tag{2.10}$$

then

$$(A|A^*) = a, \tag{2.11}$$

and (2.9) becomes

$$\|A\|^2 = (A|A) + \sqrt{(A|A)^2 - |(A|A^*)|^2}.$$

The unit ball D in the Banach space $\mathfrak{F}$ is defined by the inequality

$$\sqrt{(A|A)^2 - |(A|A^*)|^2} < 1 - (A|A).$$

If $A \in D$, then by (2.5)

$$(A|A)^2 - |(A|A^*)|^2 < (1 - (A|A))^2.$$

Hence $A \in D$ if, and only if,

$$(A|A) < 1, 1 - 2(A|A) + |(A|A^*)|^2 > 0.$$

Since, by (2.11) and (2.8), $|(A|A^*)| \leq (A|A)$, then

$$D = \left\{ A \in \mathfrak{T} : (A|A) < \frac{1}{2}(1 + |(A|A^*)|^2) < 1 \right\}.$$

3. HOLOMORPHIC AUTOMORPHISMS OF CARTAN DOMAINS OF TYPE FOUR

Changing notations, let $\mathscr{A}$ be a real Hilbert space, and let $\mathscr{H} = \mathscr{A} + i\mathscr{A}$ be its complexification with the corresponding conjugation $\overline{\xi + i\eta} = \xi - i\eta$ ($\xi, \eta \in \mathscr{A}$) and the complex Hilbert space structure defined by

$$||\xi + i\eta||^2 = ||\xi||^2 + ||\eta||^2.$$

Let $(\,|\,)$ be the corresponding inner product in $\mathscr{H}$, and let D be the domain in $\mathscr{H}$ defined by

$$D = \left\{ z \in \mathscr{H} : ||z||^2 < \frac{1}{2}(1 + |(z|\overline{z})|^2) < 1 \right\}.$$

In view of the results of §2, D is a Cartan domain of type four. As such, D is a bounded, convex, homogeneous, symmetric domain.

Assume on $\mathbb{C}^2$ the canonical conjugation. The Hilbert space direct sum $\mathscr{H} \oplus \mathbb{C}^2$ is endowed with a conjugation leaving $\mathscr{H}$ and $\mathbb{C}^2$ invariant, and whose restrictions to $\mathscr{H}$ and to $\mathbb{C}^2$ coincide with the given ones.

Let $(\,|\,)$ denote also the inner product in $\mathscr{H} \oplus \mathbb{C}^2$, and let $J \in \mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$ be the real operator defined by the matrix

$$J = \begin{pmatrix} I & 0 \\ 0 & -I_2 \end{pmatrix},$$

where I denotes, as before, the identity operator on $\mathscr{H}$ and I_2 is the identity operator in $\mathbb{C}^2$. For $p = {}^t({}^t x, u_1, u_2) \in \mathscr{H} \oplus \mathbb{C}^2, (x \in \mathscr{H}, u_1, u_2 \in \mathbb{C})$, let D' be the set in $\mathscr{H} \oplus \mathbb{C}^2$ defined by

$$(Jp|\overline{p}) = 0, \quad (Jp|p) < 0,$$

i.e.,

$$(x|\overline{x}) = u_1^2 + u_2^2, \|x\|^2 = (x|x) < |u_1|^2 + |u_2|^2. \tag{3.1}$$

If $p \in D'$, then $u_1 u_2 \neq 0$. Furthermore $Im \dfrac{u_1}{u_2} \neq 0$, since otherwise

$$|u_1|^2 + |u_2|^2 = |u_1 + u_2|^2,$$

while, by (3.1) and the Schwarz inequality,

$$|u_1^2 + u_2^2| = |(x|\overline{x})| \leq \|x\|^2 < |u_1|^2 + |u_2|^2.$$

Let $\widetilde{D}$ be the set

$$\widetilde{D} = \left\{ p = {}^t({}^t x, u_1, u_2) \in D' : Im \frac{u_1}{u_2} > 0 \right\}.$$

For any point in this set, let $z = \dfrac{1}{u_1 + iu_2} x$. Then

$$(z|\overline{z}) = \frac{u_1 - iu_2}{u_1 + iu_2}, \tag{3.2}$$

$$\|z\|^2 = (z|z) < \frac{|u_1|^2 + |u_2|^2}{|u_1 + iu_2|^2} = \frac{|u_1 - iu_2|^2 + |u_1 + iu_2|^2}{2|u_1 + iu_2|^2} = \frac{1}{2}\left(1 + \left|\frac{u_1 - iu_2}{u_1 + iu_2}\right|^2\right),$$

$$Im \frac{u_1}{u_2} = \frac{1}{|u_2|^2}(Im u_1 Re u_2 - Re u_1 Im u_2).$$

Since, on the other hand,

$$1 - \left|\frac{u_1 - iu_2}{u_1 + iu_2}\right|^2 = \frac{4}{|u_1 + iu_2|^2}(Im u_1 Re u_2 - Re u_1 Im u_2),$$

the inequality $Im \dfrac{u_1}{u_2} > 0$ is equivalent to

$$\left|\frac{u_1 - iu_2}{u_1 + iu_2}\right| < 1.$$

In conclusion, the map $p \to z = \dfrac{1}{u_1 + iu_2} x$ transforms bijectively the set of all complex lines in $\widetilde{D}$ onto the domain D. Thus, letting $D_1 = \{p \in \widetilde{D} : u_1 + iu_2 = 1\}$, the map $p \to z$ is a bi-holomorphic map of D_1 onto D.

It is easily seen that the boundary ∂D of D consists of those points $z \in \mathscr{H}$ at which at least one of the two inequalities

$$\|z\| \leq 1, \quad 1 - 2\|z\|^2 + |(z|\overline{z})|^2 \geq 0$$

becomes an equality. Denoting by $\partial_0 D$ the closed subset of ∂D where *both* inequalities become equalities, the Schwarz inequality shows that

$$\partial_0 D = \{z \in \mathscr{H} : \|z\| = 1, \overline{z} = e^{i\theta} z \quad for\ some \quad \theta \in R\}.$$

Going back to the representation of $\mathscr{H}$ as a Cartan factor of type four, by Proposition 2.7 $\partial_0 D$ is the set of all complex extreme points of $\overline{D}$.

In the finite-dimensional case $\partial_0 D$ is the Schilov boundary of D.

If $\|z\| = 1$ and $1 - 2\|z\|^2 + |(z|\overline{z})|^2 \geq 0$, then

$$0 \leq 1 - 2\|z\|^2 + |(z|\overline{z})|^2 = -1 + |(z|\overline{z})|^2 \leq -1 + \|z\|^4 = 0,$$

whence $|(z|\overline{z})| = 1$ and $1 - 2\|z\|^2 + |(z|\overline{z})|^2 = 0$, showing that

$$\partial D \cap \{z \in \mathscr{H} : \|z\| = 1\} = \partial_0 D.$$

If $p \in \partial \widetilde{D}$, then

$$(x|\overline{x}) = u_1^2 + u_2^2, \|x\|^2 = (x|x) = |u_1|^2 + |u_2|^2,$$

and there is a sequence $\{p^{(v)} =^t ({}^t x^{(v)}, u_1^v, u_2^v)\}$ in $\widetilde{D}$ converging to p, i.e.,

$$x = \lim x^{(v)}, \quad u_1 = \lim u_1^{(v)}, \quad u_2 = \lim u_2^{(v)}.$$

If $u_2 \neq 0$ and $u_1 + iu_2 = 0$, then

$$-i = \frac{u_1}{u_2} = \lim \frac{u_1^{(v)}}{u_2^{(v)}}$$

contradicting the fact that, being $p^{(v)} \in \widetilde{D}$, then $Im\ \dfrac{u_1^{(v)}}{u_2^{(v)}} > 0$. If $u_2 = 0$, then $u_1 = 0$, and therefore $\|x\|^2 = |u_1|^2 + |u_2|^2 = 0$. That proves

Lemma 3.1. *If* $0 \neq p \in \partial\tilde{D}$, *then* $u_1 + iu_2 \neq 0$.

Let Λ be the semigroup consisting of all linear, real, continuous operators G on $\mathscr{H} \oplus \mathbb{C}^2$ such that

$$ {}^tGJG = J. \tag{3.3} $$

Let Γ be the maximum subgroup of Λ consisting of those $G \in \Lambda$ which are invertible in $\mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$. Any linear operator G on $\mathscr{H} \oplus \mathbb{C}^2$ is represented by a matrix

$$ G = \begin{pmatrix} A & B_1 & B_2 \\ (\cdot|C_1) & E_{11} & E_{12} \\ (\cdot|C_2) & E_{21} & E_{22} \end{pmatrix}, $$

where A is a linear, real operator on $\mathscr{H}$, B_1, B_2, C_1, C_2 are real vectors in $\mathscr{H}$, E_{11}, E_{12}, E_{21}, E_{22} are real scalars. The operator G is bounded on $\mathscr{H} \oplus \mathbb{C}^2$ if, and only if, $A \in \mathscr{L}(\mathscr{H})$.

Setting

$$ B = (B_1, B_2) \in \mathscr{L}(\mathbb{C}^2, \mathscr{H}), $$

$$ C = {}^t((\cdot|C_1), (\cdot|C_2)) \in \mathscr{L}(\mathscr{H}, \mathbb{C}^2), $$

$$ E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \in \mathscr{L}(\mathbb{C}^2), $$

then $G \in \Lambda$ if, and only if, $A \in \mathscr{L}(\mathscr{H})$ and furthermore

$$ {}^tAA - {}^tCC = I, \tag{3.4} $$

$$ {}^tAB = {}^tCE, \tag{3.5} $$

$$ {}^tEE - {}^tBB = I_2. \tag{3.6} $$

These three conditions can also be written as follows:

$$ {}^tAA - (\cdot|C_1)C_1 - (\cdot|C_2)C_2 = I, $$

$$^tAB_1 = E_{11}C_1 + E_{21}C_2,$$

$$^tAB_2 = E_{12}C_1 + E_{22}C_2;$$

$$E_{11}^2 + E_{21}^2 - ||B_1||^2 = 1, E_{11}E_{12} + E_{21}E_{22} = (B_2|B_1), E_{12}^2 + E_{22}^2 - ||B_2||^2 = 1.$$

Equation (3.6) implies that E is injective, i.e., det $E \neq 0$. Since E is a continuous function of G for the strong topology, det E has the same sign on each connected component of Λ for the latter topology.

Under which conditions on $G \in \Lambda$ is $G\widetilde{D} \subset \widetilde{D}$?

Since $\widetilde{D}$ is a connected component of D' and $GD' \subset D'$, then $G\widetilde{D} \subset \widetilde{D}$, if, and only if, $G\widetilde{D} \cap \widetilde{D} \neq \emptyset$. For $p = {}^t({}^tx, u_1, u_2) \in D'$, the point $p' = Gp$ is given by $p' = {}^t({}^tx', u_1', u_2')$, where

$$x' = Ax + u_1 B_1 + u_2 B_2,$$

$$u_1' = (x|C_1) + E_{11}u_1 + E_{12}u_2, \tag{3.8}$$

$$u_2' = (x|C_2) + E_{21}u_1 + E_{22}u_2.$$

Choosing $x = 0$, then

$$u_1' = E_{11}u_1 + E_{12}u_2, \quad u_2' = E_{21}u_1 + E_{22}u_2,$$

and $u_2' \neq 0$ because otherwise $\frac{u_1}{u_2} = -\frac{E_{22}}{E_{21}} \in \mathbb{R}$, contrary to the fact that $Im\ \frac{u_1}{u_2} \neq 0$. Since

$$Im\ \frac{u_1'}{u_2'} = \frac{1}{|u_2'|^2} Im\ u_1'\overline{u_2'} = \frac{|u_2|^2}{|u_2'|^2} det\ E\ Im\ \frac{u_1}{u_2},$$

then $G\widetilde{D} \cap \widetilde{D} \neq \emptyset$ if, and only if, det $E > 0$. This shows that the set Λ_0 defined by

$$\Lambda_0 = \{G \in \Lambda : detE > 0\}$$

is a subsemigroup of Λ. Hence the set $\Gamma_0 = \Lambda_0 \cap \Gamma$ is a subgroup of Γ. Lemma 3.1 implies that, if det $E > 0$, $u_1' + iu_2' \neq 0$ for all $p \in \overline{\widetilde{D}} \setminus \{0\}$, where $\overline{\widetilde{D}}$ is the closure of $\widetilde{D}$.

Setting $z = \frac{1}{u_1 + iu_2}\ x$, (3.2) yields

$$\frac{u_1}{u_1 + iu_2} = \frac{1 + (z|\overline{z})}{2}, \frac{u_2}{u_1 + iu_2} = \frac{1 - (z|\overline{z})}{2i},$$

and, by (3.7) and (3.8),

$$u_1' + iu_2' = \frac{u_1 + iu_2}{2}\,\delta(G,z)$$

where $\delta(G,z)$ is defined for $z \in \mathscr{H}$ by

$$\delta(G,z) = 2(z|C_1 - iC_2) + (E_{11} - E_{22} + i(E_{12} + E_{21})(z|\overline{z}) + E_{11} + E_{22} + i(E_{21} - E_{12}))\,.$$

Hence $\delta(G,z) \neq 0$ for all z contained in a neighborhood of $\overline{D}$. The function

$$\widehat{G} : z \to \frac{1}{\delta(G,z)}(2Az + (1 + (z|\overline{z}))B_1 - i(1 - (z|\overline{z}))B_2) \tag{3.9}$$

is holomorphic on a neighborhood of $\overline{D}$, and furthermore $\widehat{G}(D) \subset D$. A direct computation shows that $G \to \widehat{G}$ is a homomorphism of Λ_0 into the semigroup $Hol(D,D)$ of all holomorphic mappings of D into itself. Hence the image $\widehat{\Gamma}_0$ of Γ_0 is a subgroup of $AutD$. It will be shown now that $\widehat{\Gamma}_0$ is the entire group $AutD$, i.e. that every holomorphic automorphism of D is given by (3.9) for some $G \in \Gamma_0$.

The following lemma is due to L.A. Harris [5] (and to U. Hirzebruch [13] in the finite dimensional case; cf. also [12])

Lemma 3.2. *If $g \in (AutD)_0$, there exist $\theta \in \mathbb{R}$ and a real unitary operator U on $\mathscr{H}$ such that $g(z) = e^{i\theta}Uz$ for all $z \in D$.*

On the other hand, if $G \in \Lambda_0$ is such that $\widehat{G}(0) = 0$, then $B_1 = B_2 = C_1 = C_2 = 0$ and $E_{11} - E_{22} + i(E_{12} + E_{21}) = 0$ $E_{11} + E_{22} + i(E_{21} - E_{12}) = 2e^{-i\theta}$ for some $\theta \in \mathbb{R}$, so that, by (3.9),

$$\widehat{G}(z) = e^{i\theta}Az.$$

Hence, if furthermore $G \in \Gamma_0$, then $\widehat{G} \in (AutD)_0$, i.e.,

$$(AutD)_0 \subset \widehat{\Gamma}_0.$$

It will be shown now - following essentially L.K. Hua [14, pp. 86-87] - that

Lemma 3.3. *The group $\widehat{\Gamma}_0$ acts transitively on D.*

Proof. a) Let $z_0 \in D$. Since, as was just noticed, the map $z \to e^{i\theta}z$ is contained in $\widehat{G}$ for any $\theta \in \mathbb{R}$, replacing z_0 by $e^{i\theta}z_0$ and choosing a suitable $\theta \in \mathbb{R}$, it can be assumed that $(Rez_0|Imz_0) = 0$, so that $w_0 = (z_0|\overline{z}_0) \in \mathbb{R}$. Consider the non-singular 2×2 matrix

$$M_0 = \begin{pmatrix} w_0 + 1 & w_0 + 1 \\ i(w_0 - 1) & -i(w_0 - 1) \end{pmatrix},$$

and let $L_0 \in \mathscr{L}(\mathbb{C}^2, \mathscr{H})$ be the real (continuous) operator defined by

$$L_0 = 2(z_0, \bar{z}_0) M_0^{-1} = \frac{1}{1 - w_0^2}((1 - w_0)(z_0 + \bar{z}_0), i(1 + w_0)(z_0 - \bar{z}_0)).$$

Since

$$I_2 -{}^t L_0 L_0 = I_2 - L_0^* L_0 = {}^t\overline{M_0}^{-1} \left({}^t\overline{M}_0 M_0 - 4 \begin{pmatrix} \|z_0\|^2 & w_0 \\ w_0 & \|z_0\|^2 \end{pmatrix} \right) M_0^{-1} =$$

$$\frac{1}{2(1 - 2\|z_0\|^2 + w_0^2)} (M_0 \, {}^t\overline{M}_0)^{-1} =$$

$$= \frac{1}{1 - 2\|z_0\|^2 + w_0^2} \begin{pmatrix} \dfrac{1}{(1 + w_0)^2} & 0 \\ 0 & \dfrac{1}{(1 - w_0)^2} \end{pmatrix} > 0,$$

then, denoting by E the positive square root of the positive hermitian operator $(I_2 - {}^tL_0 L_0)^{-1}$:

$$E = (1 - 2\|z_0\|^2 + w_0^2)^{1/2} \begin{pmatrix} 1 + w_0 & 0 \\ 0 & 1 - w_0 \end{pmatrix},$$

one has

$$E(I_2 -{}^t L_0 L_0) E = I_2 .$$

Because $I_2 -{}^t L_0 L_0 > 0$, then $\|L_0\| < 1$. Hence $\|{}^tL_0\| = \|L_0^*\| = \|L_0\| < 1$, and therefore $I - L_0 \, {}^tL_0 > 0$. If A is the positive-definite real hermitian operator which is the positive square root of $(I - L_0 \, {}^tL_0)^{-1}$, then

$$A(I - L_0 \, {}^tL_0) A = I.$$

Letting

$$C = {}^t L_0 A, \quad B = L_0 E,$$

(3.4), (3.5), (3.6) are fulfilled, i.e., $G := \begin{pmatrix} A & B \\ C & E \end{pmatrix} \in \Lambda_0$ (because $\det E > 0$).

b) It will be shown now that $G \in \Gamma_0$, proving more specifically that

$$(3.10) \qquad G^{-1} = \begin{pmatrix} A & -{}^tC \\ -{}^tB & E \end{pmatrix}$$

First of all

$$\begin{pmatrix} A & -{}^tC \\ -{}^tB & E \end{pmatrix}\begin{pmatrix} A & B \\ C & E \end{pmatrix} = \begin{pmatrix} A^2 - {}^tCC & AB - {}^tCE \\ -{}^tBA + {}^tEC & -{}^tBB + E^2 \end{pmatrix} = \begin{pmatrix} I & \\ & -I_2 \end{pmatrix}$$

by (3.4), (3.5), (3.6). Furthermore,

$$A{}^tA - B{}^tB = A^2 - B\,{}^tB = (I - L_0\,{}^tL_0)^{-1} - L_0E^2\,{}^tL_0 =$$

$$= (I - L_0\,{}^tL_0)^{-1} - L_0(I_2 - {}^tL_0L_0)^{-1}\,{}^tL_0 = (I - L_0{}^tL_0)^{-1} - (I - L_0{}^tL_0)^{-1}L_0{}^tL_0 =$$

$$= (I - L_0{}^tL_0)^{-1}(I - L_0{}^tL_0) = I,$$

$$D\,{}^tD - C{}^tC = D^2 - C{}^tC = (I_2 - {}^tL_0L_0)^{-1} - {}^tL_0A^2L_0 =$$

$$= (I_2 - {}^tL_0L_0)^{-1} - {}^tL_0(I - L_0{}^tL_0)^{-1}L_0 =$$

$$= (I_2 - {}^tL_0L_0)^{-1} - (I_2 - {}^tL_0L_0)^{-1}\,{}^tL_0L_0 = (I_2 - {}^tL_0L_0)^{-1}(I_2 - {}^tL_0L_0) = I_2,$$

$$A{}^tC - BE = A^2L_0 - L_0E^2 = (I - L_0{}^tL_0)^{-1}L_0 - L_0(I_2 - {}^tL_0L_0)^{-1} =$$

$$= (I - L_0{}^tL_0)^{-1}L_0 - (I - L_0{}^tL_0)^{-1}L_0 = 0,$$

proving that G is invertible in $\mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$ and that G^{-1} is given by (3.10).

c) Since

$$2Az_0 - {}^tC\begin{pmatrix} 1 + w_0 \\ -i(1 - w_0) \end{pmatrix} =$$

$$= A\left(2z_0 - \frac{1}{w_0^2 - 1}((w_0^2 - 1)(z_0 + \overline{z}_0) + (w_0^2 - 1)(z_0 - \overline{z}_0))\right)$$

$$= A(2z_0 - 2z_0) = 0,$$

then, by (3.10) and (3.9), $\widehat{G}^{-1}(z_0) = 0$. That proves that $\widehat{\Gamma}_0$ acts transitively on D.

QED

In conclusion, the following theorem holds.

Theorem 3.4. *The map $G \to \widehat{G}$ is a surjective homomorphism of Γ_0 onto $AutD$.*

The kernel consists of $\pm$ the identity operator on $\mathscr{H} \oplus \mathbb{C}^2$.

Let $\kappa(z; \cdot)$ be the Kobayashi differential metric of D at the point $z \in D$. For every $v \in \mathscr{H}$

$$\kappa(0; v) = \|v\|^2 + \sqrt{\|v\|^4 - |(v|\overline{v})|^2}.$$

This shows that, if $G \in \Lambda_0$ is such that $\widehat{G}(0) = 0$, then

$$\kappa(0; d\widehat{G}(0)v) = \kappa(0; v)$$

for all $v \in \mathscr{H}$. This fact, together with lemma 3.3, yields

$$\widehat{\Lambda}_0 \subset IsoD.$$

4. ONE-PARAMETER SEMIGROUPS OF HOLOMORPHIC ISOMETRIES

Let $T : \mathbb{R}_+ \to \mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$ be a strongly continuous semigroup such that $T(t) \in \Lambda$ for all $t \geq 0$. Since $T(0) = I_{\mathscr{H} \oplus \mathbb{C}^2} \subset \Lambda_0$, then $T(t) \in \Lambda_0$ for all $t \geq 0$. Since $T(t)$ is a real operator for any $t \geq 0$, the infinitesimal generator X of T is real and in particular its domain $\mathscr{D}(X)$ is conjugation-invariant. Viceversa, if the infinitesimal generator X of a strongly continuous semigroup T is real (hence $\mathscr{D}(X)$ is a conjugation-invariant), then $T(t)$ is a real operator for all $t \geq 0$. In fact for all p and all $t \geq 0$ the exponential formula yields

$$\overline{T(t)p} = \lim_{n \to +\infty} \overline{(I - \frac{t}{n} X)^{-n} p} = \lim_{n \to +\infty} (I - \frac{t}{n} X)^{-n} \overline{p} = T(t)\overline{p}.$$

Since $T(t)$ is real, then ${}^tT(t) = T(t)^*$, and thus (3.3) yields

$$T(t)^* J T(t) = J \quad for\ all \quad t \geq 0. \tag{4.1}$$

By theorem III of [29], if $T : \mathbb{R}_+ \to \mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$ is any strongly continuous semigroup satisfying (4.1), there is a dense linear subvariety $\mathscr{D}$ of $\mathscr{H}$ such that $\mathscr{D}(X) = \mathscr{D} \oplus \mathbb{C}^2$ and X is represented by the matrix

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ (\cdot|X_{12}) & X_{22} & X_{23} \\ (\cdot|X_{13}) & X_{32} & X_{33} \end{pmatrix}, \tag{4.2}$$

where X_{12}, X_{13} are vectors in $\mathscr{H}$; X_{11} is a closed operator on $\mathscr{H}$ with domain $\mathscr{D}(X_{11}) = \mathscr{D}$ such that iX_{11} is symmetric and $\sigma X_{11} \subset \{\xi \in C : Re\xi \leq 0\}$; $\begin{pmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{pmatrix}$ is a 2×2 complex matrix such that $i \begin{pmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{pmatrix}$ is hermitian. Moreover, according to theorem III of [29], if $X_{11}, \ldots, X_{33}$ satisfy the above conditions, and $\mathscr{D}(X_{11})$ is dense in $\mathscr{H}$, the operator X defined by (4.2) on the domain $\mathscr{D}(X) = \mathscr{D}(X_{11}) \oplus \mathbb{C}^2$ is the infinitesimal generator of a strongly continuous semigroup T satisfying (4.1). Moreover, T is the restriction to $\mathbf{R}_+$ of a strongly continuous group $\mathbf{R} \to \mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$ if, and only if, iX_{11} is self-adjoint.

The operator X given by (4.2) is real if, and only if, X_{11} is a real operator (and therefore $\mathscr{D}(X_{11})$ is conjugation invariant), the vectors X_{12} and X_{13} are real and $X_{22}, X_{23}, X_{32}, X_{33}$ are real numbers. The fact that $i \begin{pmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{pmatrix}$ is hermitian implies then that $X_{22} = X_{33} = 0, X_{32} = -X_{23}$. In conclusion, the following theorem holds:

Teorem 4.1. *Let X be a linear operator on $\mathscr{H} \oplus \mathbb{C}^2$. Then X is the infinitesimal generator of a strongly continuous linear semigroup $T : \mathbf{R}_+ \to \mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$ such that $T(t) \in \Lambda$ (hence $T(t) \in \Lambda_0$) for all $t \geq 0$ if, and only if, there is a dense linear manifold $\mathscr{D} \subset \mathscr{H}$ such that $\mathscr{D}(X) = \mathscr{D} \oplus \mathbb{C}^2$, and X is represented by the matrix*

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ (\cdot|X_{12}) & 0 & X_{23} \\ (\cdot|X_{13}) & -X_{23} & 0 \end{pmatrix}, \tag{4.3}$$

where: $X_{23} \in \mathbf{R}, X_{12}$ and X_{13} are real vectors in $\mathscr{H}$; X_{11} is a real, closed operator with domain $\mathscr{D}(X_{11}) = \mathscr{D}$, such that iX_{11} is symmetric and $\sigma(X_{11}) \subset \{\xi \in C : Re\xi \leq 0\}$.

Furthermore, the semigroup T is the restriction to $\mathbf{R}_+$ of a strongly continuous group if, and only if, iX_{11} is self-adjoint.

The fact that for $G \in \Lambda$ (3.3) is equivalent to

$$G^* J G = J,$$

and Proposition 4.2 of [29] imply

Proposition 4.2. *Let M be a domain in $\mathbb{C}$. There are no non-constant holomorphic functions $F : M \to \mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$ such that $F(\xi) \in \Lambda$ for all $\xi \in M$.*

In particular, there are no non-trivial holomorphic semigroups with values in Λ.

The general results established in [29] for Cartan factors of type one provide some information on the spectral structure of X in terms of the spectrum of the operator X_{11} appearing in (4.3).

There exists in the open right half-plane $\{\xi \in \mathbb{C} : Re\xi > 0\}$ a set C consisting of two points at most and possibly empty, such that

$$\sigma(X) = C \cup \{\xi \in C : Re\xi \leq 0\},$$

if iX_{11} is symmetric but not self-adjoint. If iX_{11} is self-adjoint, denoting by C' the image of C by the reflection $\xi \to -\overline{\xi}$ around the imaginary axis, then

$$\sigma(X) \setminus i\mathbb{R} = C \cup C'.$$

The set C and, if iX_{11} is self-adjoint, the set C' consist of polar singularities of the resolvent function $(\cdot I - X)^{-1}$.

If iX_{11} is symmetric but not self-adjoint, then the set $\{\xi \in \mathbb{C} : Re\xi < 0\} \setminus C'$ is contained in the residual spectrum of X.

Let $\Phi \in Hol(\mathbb{C} \setminus \sigma(X_{11}), \mathscr{L}(\mathbb{C}^2))$ be defined by

$$\Phi(\xi) = \begin{pmatrix} \xi - (((\xi I - X_{11})^{-1} X_{12} | X_{12}) & -X_{23} - (((\xi I - X_{11})^{-1} X_{13} | X_{12}) \\ X_{23} - (((\xi I - X_{11})^{-1} X_{12} | X_{13}) & \xi - (((\xi I - X_{11})^{-1} X_{13} | X_{13}) \end{pmatrix}.$$

Then, by n. 8 of [29], the set C is the zero-set of the restriction of the holomorphic function $\xi \to det\Phi(\xi)$ to the open right half-plane.

By Theorem 4.1, $\Phi(\overline{\xi}) = \overline{\Phi(\xi)}$. Thas shows that the set C (and thus also the set C') is invariant by conjugation.

5. A RICCATI EQUATION

With the same notations as before, let $T : \mathbb{R}_+ \to \mathscr{L}(\mathscr{H} \oplus \mathbb{C}^2)$ be a strongly continuous semigroup such that $T(t) \in \Lambda$ for all $t \geq 0$. The infinitesimal generator X of T is given by the matrix (4.3). Let $p^0 =^t ({}^t x^0, u_1^0, u_2^0) \in \mathscr{D}(X)$, and consider the Cauchy problem

$$(5.1) \qquad \dot{p}(t) = Xp(t) \qquad (t > 0),$$

with the initial condition

$$p(0) = p^0. \tag{5.2}$$

Setting $p(t) = {}^t({}^t x(t), u_1(t), u_2(t))$, (5.1) is equivalent to

$$\dot{x}(t) = X_{11}x(t) + u_1(t)X_{12} + u_2(t)X_{13},$$

$$\dot{u}_1(t) = (x(t)|X_{12}) + X_{23}u_2(t),$$

$$\dot{u}_2(t) = (x(t)|X_{13}) - X_{23}u_1(t),$$

and (5.2) is equivalent to

$$x(0) = x^0, u_1(0) = u_1^0, u_2(0) = u_2^0.$$

Let $Im\frac{u_1^0}{u_2^0} > 0$ and let $z^0 = \frac{1}{u_1^0 + iu_2^0} x^0 \in D \cap \mathscr{D}(X_{11})$. Thus $Im\frac{u_1(t)}{u_2(t)} > 0$ for all $t \geq 0$, and, setting

$$z(t) = \frac{1}{u_1(t) + iu_2(t)} x(t), \tag{5.3}$$

then

$$z(t) \in D \cap \mathscr{D}(X_{11}) \quad \textit{for all} \quad t \geq 0.$$

Furthermore

$$\frac{u_1(t) - iu_2(t)}{u_1(t) + iu_2(t)} = (z(t)|\overline{z(t)}),$$

and $z(t)$ satisfies the Riccati equation

$$\dot{z}(t) = (X_{11} + iX_{23}I)z(t) + \frac{1}{2}(X_{12} + iX_{13})(z(t)|\overline{z(t)}) \tag{5.4}$$
$$-(z(t)|X_{12} - iX_{13})z(t) + \frac{1}{2}(X_{12} - iX_{13})$$

with the initial condition

$$z(0) = z^0. \tag{5.5}$$

The function $t \to z(t)$ is continuous for the graph-norm

$$z \to ||z|| + ||X_{11}z|| \tag{5.6}$$

on $\mathscr{D}(X_{11})$. A similar argument to the proof of Theorem VII of [29] (cf. also [27]) yields:

Theorem 5.1. *For any* $\gamma > 0$ *and any choice of* $z^0 = \dfrac{1}{u_1^0 + iu_2^0}\, x^0 \in D \cap \mathscr{D}(X_{11})$, *the function* $t \to z(t)$ *defined by (5.3) for* $0 \leq t \leq \gamma$ *is the unique solution of the Riccati equation (5.4) with the initial condition (5.5) and with* $z([0,\gamma]) \subset D \cap \mathscr{D}(X_{11})$, *which is contained in* $C^1([0,\gamma], \mathscr{L}(\mathbb{C}^2, \mathscr{H}))$ *and is continuous for the graph-norm (5.6).*

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