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THE WEAK TOPOLOGY IN CERTAIN RIGHT-INJECTIVE TENSOR PRODUCTS OF BANACH SPACES

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Abstract. *We present in this paper some results on the weak topology of some right injective tensor products of Banach spaces. We study weak compactness, weak sequential completeness and related questions.*

The reflexivity and weak sequential completeness in components of various Banach operator ideals and tensor products of Banach spaces have been treated by several authors. Saphar [8] deals with the reflexivity in spaces of p-integral and absolutely p-summing operators. Lewis [5] and Heinrich [3] obtain conditions of weak sequential completeness in different topological tensor products of Banach spaces. This paper is concerned with the study of properties of certain α-tensor products of Banach spaces. Similar questions have been treated by Lewis [4] in injective tensor products of Banach spaces.

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NOTATIONS, DEFINITIONS AND PREVIOUS NOTES

The terminology concerning Banach spaces, operators and tensor norms is standard. We denote by (B) and (RNP) the classes of all Banach spaces and Banach spaces with the Radon-Nikodym property respectively. If $E \in (B)$, then E' represent its topological dual and B_E its closed unit ball. If $M \subset E$, we say that M is conditionally compact if every sequence in M has a weakly Cauchy subsequence.

For $E, F \in (B)$ over the same scalar field, we consider the algebraic tensor product $E \otimes F$ as canonically embedded in $L(E', F)$. The notations and basic properties of tensor products used hereafter are primarily that (2). For α a tensor norm, $\alpha\backslash$, α^t and α' denote respectively the right injective hull, trasposed and dual tensor norms of α. We denote by $E \underset{\alpha}{\otimes} F$ the normed α-tensor product and by $E \underset{\alpha}{\hat{\otimes}} F$ the còmpletion space of $E \underset{\alpha}{\otimes} F$. Following the terminology of (1), if $E \in (B), x_1, x_2, \ldots, x_n \in E$, and $1 \leq p \leq \infty$

$$l_p(x_i) = \|(\|x_i\|)_{i=1,\ldots,n}\|_{l_p^n} \qquad \text{(strong } l_p\text{-norm)}$$

$$w_p(x_i) = \sup_{x' \in B_{E'}} l_p(\langle x_i, x' \rangle) \qquad \text{(weak } l_p\text{-norm)}$$

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We represent by p' the conjugate number of p which is determined by

$$\frac{1}{p}+\frac{1}{p'}=1$$

For $1\leq p,q\leq\infty$ with $\frac{1}{p}+\frac{1}{q}\geq 1$ define $1\leq r\leq\infty$ by

$$\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$$

The Lapreste's α_{pq}-tensor norms are defined in the following way: if $E,F\in(B)$ and $z\in E\otimes F$,

$$\alpha_{pq}(z)=\inf\left\{l_r(\lambda_i)\cdot w_{q'}(x_i)\cdot w_{q'}(y_i)/z=\sum_{i=1}^{n}\lambda_i x_i\otimes y_i\right\}$$

In particular $\alpha_{p1}=g_p,\alpha_{1p}=d_p$ (the Saphar's tensor norms (7)) and $\varepsilon_p=(d_{p'})'=g_p\backslash$, see [7].

Lewis [5] says that a tensor norm α has the Radon-Nikodym property if $E'\hat{\otimes}_{\alpha}l_1=(E\otimes_{\alpha'}c_0)'$, for every $E\in(B)$. Exemples of tensor norms with the Radon-Nikodym property [(5), theorem 3]: α_{pq} with $1\leq q<\infty$ α'_{pq} with $1<p\leq\infty, d_p$ with $1\leq p<\infty$ and g'_p with $1<p\leq\infty$.

Proposition 1. *Let α be a right-injective tensor norm and $E,F\in(B)$ such that every bounded $\sigma(E\hat{\otimes}_{\alpha}F,E'\otimes F')$-convergent sequence in $E\hat{\otimes}_{\alpha}F$ is weakly convergent. Then a bounded subset $M\subset E\hat{\otimes}_{\alpha}F$ is weakly conditionally compact if*

1) $\{u(x'),u\in M\}\subset F$ *is weakly relatively compact for each* $x'\in E'$
2) $\{u^t(y'),u\in M\}\subset E$ *is weakly conditionally compact for each* $y'\in F'$.

Proof. If F is separable, let $D=\{y'_1,y'_2,\ldots\}$ be a countable set of F' such that $\langle D\rangle$ is $\sigma(F',F)$-dense in F'. Then $\langle D\rangle$ is also dense in F' with the Mackey topology $\mu(F',F)$. If $\{u_n\}$ is a sequence of M, by 2) the sequence $\{u^t_n(y'_1)\}$ has a weakly Cauchy subsequence $\{u^t_{n1}(y'_1)\}$. Also by 2) the sequence $\{u^t_{n1}(y'_2)\}$ has a weakly Cauchy subsequence $\{u^t_{n2}(y'_2)\}$. Proceeding in this way, for every $y'\in D,\{u^t_{nn}(y')\}$ is a weakly Cauchy sequence. Given $x'\in E'$ and $z'\in F'$, the condition 1) shows that given $\varepsilon>0$ t here exist an $y'\in\langle D\rangle$ such that

$$\sup_n|\langle u_{nn}(x'),z'-y'\rangle|\leq\varepsilon/3.$$

Moreover there is an $n_0 \in \mathbf{N}$ such that

$$\forall i,\ j \geq n_0, |\langle (u^t_{ii} - u^t_{jj})(y'), x'\rangle| \leq \varepsilon/3.$$

In consequence $|\langle u_{ii} - u_{jj}, x' \otimes z'\rangle| \leq \varepsilon$, and from the property of $\alpha, \{u_{nn}\}$ is a weakly Cauchy subsequence of $\{u_n\}$ (note that the assumption holds for Cauchy sequences if it holds for convergent sequences). If F is not separable since α is a right-injective tensor norm, we can consider $\{u_n\}$ as a sequence in $E \overset{\wedge}{\underset{\alpha}{\otimes}} G$, where G is a closed and separable subspace of F which contains the rang of every $\{u_n, n \in \mathbf{N}\}$. Then we use the former arguments to prove that $\{u_n\}$ has a weakly Cauchy subsequence. ∎

The assumption over α is satisfied:

a) If $\alpha = \varepsilon$ (the injective tensor norm) and $E, F \in (B)$ arbitrary (this was done in (4) with the representation of $(E \underset{\varepsilon}{\otimes} F)'$ by integrals and Lebesgue's dominated convergence theorem), or

b) If $E' \otimes F' \subset (E \underset{\alpha}{\otimes} F)'$ is norm-dense. Lewis [(5), theorem 3] showed that the natural map $E' \otimes F' \to (E \underset{\beta'\backslash}{\otimes} F)'$ has norm-dense range if $F' \in (RNP)$ and β is a tensor norm with the Radon-Nikodym property (the proof shows that the approximation property is not necessary for this statement). Then the assumption holds for $\alpha = \beta'\backslash$. Exemples for this are $\alpha = \varepsilon_p$ with $1 < p \leq \infty$, because in this case $\beta = g'_p$ has the Radon-Nikodym property and $\alpha = \beta'\backslash = (g'_p)'\backslash = g_p\backslash = \varepsilon_p$.

From now on, $\alpha = \beta'\backslash$, with β having the Radon-Nikodym property. We obtain the following characterization of he weakly conditionally compact subsets of several α-tensor products of Banach spaces.

Proposition 2. *If $E, F \in (B)$ and F is reflexive, then a bounded set M of $E \overset{\wedge}{\underset{\alpha}{\otimes}} F$ is weakly conditionally compact if and only if on verifies the condition 2) of Proposition 1.*

Propositions 3 and 4 are immediate consequences of Proposition 2.

Proposition 3. *With the hypothesis of Proposition 2, $l_1 \not\subset E \overset{\wedge}{\underset{\alpha}{\otimes}} F$ iff $l_1 \not\subset E$.*

Proof. It is evident from Proposition 2 and Rosenthal's [6]: $l_1 \not\subset E$ if and only if every bounded set of E is weakly conditionally compact.

Proposition 4. *If E and F are reflexive Banach spaces, then $E \overset{\wedge}{\underset{\alpha}{\otimes}} F$ is reflexive if and only if it is weakly sequentially complete.*

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