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AFFINE CONNECTIONS FOR 2-OSCULATING VECTOR FIELDS AND GEODESICS *

L. DI TERLIZZI, F. VERROCA

Abstract. *We study the geodesics, with respect to the connections for 2-osculating vector fields, as projections of standard horizontal vector fields. Then, we consider the connections on the principal bundle of affine 2-osculating frames, and on its reductions. Finally, when it is possible we characterize the geodesics as the curves whose development is (an open interval of) a straight line.*

INTRODUCTION

It is well known that, given a linear connection on a manifold M, the geodesics are characterized as projections, on M, of integral curves of standard horizontal vector fields on the bundle of linear frames of M. Another characterization is obtained considering the affine fibre bundle of M: then the geodesics are the curves whose development is a straight line ([5], [3]). In [8] the geodesics with respect to a connection for 2-osculating vector fields which preserves the osculating order are studied; here it is proved that a theorem of existence and uniqueness for given initial conditions of 2-osculation does not exist. It follows that, characterizations as the classical ones, can not exist. The aim of this paper is to study the relationships between the geodesics with respect to the connections for 2-osculating vector fields and the curves which are projections of standard horizontal vector fields on the reduced bundles $\widetilde{P}$ and P' of the bundle P of 2-osculating frames of M (cfr. §2).

In §3 we consider the affine principal bundle $\mathscr{A}^2$, its reduced bundles $\widetilde{\mathscr{A}}^2$ and $\mathscr{A}'^2$ corresponding to $\widetilde{P}$ and P', and the affine vector bundle $A^2(M)$; we study the affine (generalized) connections and, in §4 we prove that, given an affine connection on the bundle $\mathscr{A}'^2$, the geodesics are the curves on M whose development in $A^2(M)$ is a straight line.

1. PRELIMINARIES

Let M be a C^∞-differentiable manifold, of dimension n. We denote with G_q the linear group $GL(q,R)$, with $L = (L(M), M, \pi_L, G_n)$ the principal bundle of linear frames of M and with $T_p(M)$ the tangent vector space at p to M. We use the same notation and terminology of [7]. The bundle $P = (P(M), M, \pi, G_t)$ of 2-osculating frames of M has been studied in [7]: here it is proved that the group G_t is reducible to its two subgroups:

$$\widetilde{G} = \left\{ \widetilde{a} \in G_t | \widetilde{a} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},\ a \in G_n,\ b \in M^n_m,\ c \in G_m \right\}$$

* Work partially supported by M.P.I.

$$G' = \left\{ a' \in G_t | a' = \begin{pmatrix} a & b \\ 0 & a^{(2)} \end{pmatrix}, \ a \in G_n, \ b \in M^n_m, \right\}$$

where $m = \dfrac{n(n+1)}{2}$ and $a^{(2)}$ has generic entry $\dfrac{2\, a^{(i}_{i'} a^{j)}_{j'}}{1+\delta^i_j}$, (cfr. [7]). The respective reduced boundles are denoted by $\widetilde{P} = (\widetilde{P}(M), M, \widetilde{\pi}, \widetilde{G})$ and $P' = (P'(M), M, \pi', G')$ and the vector bundle, associated with $P, \widetilde{P}$ and P' is denoted by $\mathscr{O}^2(M)$. We refer to [7] for the properties of these bundles and of the connections on them. We recall that, if (U, ϕ) is a coordinate neighborhood of M, and $p \in U$, the family $\underline{e}_p = \left((e_i)_p, \frac{1}{2}(e_{ij})_p\right), i \in I = \{1, 2, \ldots, n\}, (i,j) \in J = \{(i,j) \in I \times I | i \leqq j\}$, where $e_i = \dfrac{\partial}{\partial x^i}, e_{ij} = \dfrac{\partial^2}{\partial x^i \partial x^j}$ is a basis of $\mathscr{O}^2_p(M)$, which is called natural 2-osculating frame in p corresponding to (U, ϕ). Finally, the q-osculating vector field, $q \leqq 2$, to a C^∞-differentiable curve, $\tau : [a, b] \to M$, is defined putting, for each $s \in [a, b]$, $\tau^2(s) = \tau^2_s \left(e_0 + \frac{1}{2}\, e_{00}\right)$, where τ^2_s is the differential of order 2 of τ in s and $\left(e_0, \frac{1}{2}\, e_{00}\right)$ is the natural 2-osculating frame of $\mathbb{R}$. Moreover, we have:

$$(1.1) \qquad \tau^2(s) = \left(\frac{d\tau^i}{ds} + \frac{1}{2}\frac{d^2\tau^i}{ds^2}\right)(e_i)_{\tau(s)} + \frac{1}{2}\left(\frac{d\tau^i}{ds}\frac{d\tau^j}{ds}\right)(e_{ij})_{\tau(s)}$$

where $\tau^i = x^i \circ \tau$ and $i, j \in I$. τ is called a geodesic for a connection Γ on $\widetilde{P}$ (on P') if the field $\tau^2(s)$ is parallel along τ with respect to Γ.

2. GEODESICS AS PROJECTIONS OF INTEGRAL CURVES OF STANDARD HORIZONTAL VECTOR FIELDS

As in §8 of [7] we consider the morphism $\widetilde{f} : \widetilde{P} \to L$ defined as follows: for each $\widetilde{u} \in \widetilde{P}$ fixed a coordinate neighborhood (U, ϕ) such that $p = \widetilde{\pi}(\widetilde{u}) \in U$ and $\widetilde{a} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \widetilde{G}$ such that $\widetilde{u} = \underline{e}_p \widetilde{a}$, where $\underline{e}_p$ is the natural 2-osculating frame in p corresponding to (U, ϕ), we put $\widetilde{f}(\widetilde{u}) = \left\{\left(\sum_{i=1}^n (e_i)_p a^i_j\right)_{j=1,2,\ldots,n}\right\}$, and for each $\widetilde{a} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \widetilde{G}$, we put $\widetilde{f}(\widetilde{a}) = a$. If $j : P' \to \widetilde{P}$ denote the morphism of inclusion, we put $f' = \widetilde{f} \circ j$.

Definition 2.1. *We call canonical 1-form of P', the 1-form θ' on P' taking values in $\mathbf{R}^t$, defined by $\theta'_{u'}(X) = u'_{-1}(\pi'_{u'}(X))$, for each $X \in T_{u'}(P')$, where $u' \in P'$ is considered as isomorphism of $\mathbf{R}^t$ on $\mathscr{O}^2_{\pi'(u')}(M)$.*

Definition 2.2. *Let Γ' be a connection on P'. For each $\xi = (\xi,0) \in \mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^t$, we call standard horizontal vector field corresponding to ξ the horizontal vector field $B'(\xi)$ such that, for each $u' \in P'$, $\pi'_{u'}(B'(\xi)_{u'}) = u'(\xi)$.*

It is easy to verify that $\theta'(B'(\xi)) = \xi; \xi \neq 0 \Rightarrow u' \in P' : B'(\xi)_{u'} \neq 0; \forall a' \in G' : (R_{a'})_*(B'(\xi)) = B'(a'^{-1}\xi)$. We denote by $\widetilde{B}(\xi)$ the standard horizontal vector field corresponding to $\xi = (\xi,0) \in \mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^t$, with respect to a connection $\widetilde{\Gamma}$ on $\widetilde{P}$ (cfr. [7]).

Proposition 2.1. *Let $\widetilde{\Gamma}$ be a connection on $\widetilde{P}$ and Γ the connection induced on L by $\widetilde{f}$. Then, for each $\xi = (\xi,0) \in \mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^t$, we have:*

a) *$\widetilde{f}$ maps $\widetilde{B}(\xi)$ into the standard horizontal vector field $B(\xi)$ corresponding to ξ, with respect to Γ.*

b) *The projection on M of any integral curve of $\widetilde{B}(\xi)$ is a geodesic for Γ.*

a) We observe that for each $\widetilde{u} \in \widetilde{P}, \widetilde{f}(\widetilde{u})(\xi) = \widetilde{u}(\xi)$, so

$$(\pi_L)_{\widetilde{f}(\widetilde{u})}(\widetilde{f}_{\widetilde{u}}(\widetilde{B}(\xi)_{\widetilde{u}})) = (\pi_L \circ \widetilde{f})_{\widetilde{u}}(\widetilde{B}(\xi)_{\widetilde{u}}) = \widetilde{\pi}_{\widetilde{u}}(\widetilde{B}(\xi)_{\widetilde{u}}) = \widetilde{u}(\xi) = (\widetilde{f}(\widetilde{u}))(\xi).$$

Moreover $\widetilde{f}_*(\widetilde{B}(\xi))$ is horizontal with respect to Γ and then $\widetilde{f}_*(\widetilde{B}(\xi)) = B(\xi)$.

b) Let $\gamma : [a,b] \to \widetilde{P}$ be a integral curve of $\widetilde{B}(\xi)$. Then, for the curve $c = \widetilde{f} \circ \gamma : [a,b] \to L$ we have, for each $s \in [a,b]$:

$$\dot{c}(s) = \widetilde{f}_{\gamma(s)}(\dot{\gamma}(s)) = \widetilde{f}_{\gamma(s)}(\widetilde{B}(\xi)_{\gamma(s)}) = B(\xi)_{c(s)}.$$

Therefore c is an integral curve of $B(\xi)$. By prop. 6.3 of chap. III of [5], it follows that $\widetilde{\pi} \circ \gamma = \pi_L \circ c$ is a geodesic for Γ.

Remark 2.1. An analogous result to prop. 2.1 in the case of a connection Γ' on P' and for the connection induced on L by f' is true.

Proposition 2.2. *Let Γ' be a connection on P' and $\tau : [a,b] \to M$ be a differentiable curve of M. If τ is a geodesic for Γ', then τ is the projection on M of an integral curve of a standard horizontal vector field in P' with respect to Γ'.*

Suppose that $0 \in [a,b]$ and put $p = \tau(0)$. Let us denote with $(x_s), s \in [a,b]$, the curve τ. We consider $u_0 \in \pi'^{-1}(p)$ and we put $\tau' = (u_s), s \in [a,b]$, the horizontal lift of τ to P' with origin at u_0. There exists then, a coordinate neighborhood (U, ϕ) such that $p \in U$ and $u_0 = \underline{e}_p$, where $\underline{e}_p$ is the natural 2-osculating frame in p corresponding to (U, ϕ). We suppose that $\tau([a,b]) \subset U$ and, for each $s \in [a,b]$, let $\underline{e}_{x_s}$ be the 2-osculating natural frame in x_s corresponding to (U, ϕ). We put $\dot{x}_0 = \xi^i (e_i)_p$ and we denote, for each $s \in [a,b]$ by x_s^2 the q-osculating vector to τ in $x_s, q \leqq 2$. By (1.1) we have:

$$x_0^2 = (\xi^i + \eta^i)(e_i)_p + \frac{1}{2}\, \xi^i \xi^j (e_{ij})_p = (\xi^i + \eta^i)(e_i)_p + \frac{1}{2} \sum_{(i,j)\in J} \frac{2\xi^i\xi^j}{1+\delta^i_j}\,(e_{ij})_p$$

Moreover, for each $s \in [a,b]$, there exists $a'_s = \begin{pmatrix} a_s & b_s \\ 0 & a_s^{(2)} \end{pmatrix} \in G'$ with a'_0 identity matrix and $u_s = \underline{e}_{x_s} a'_s$. Denoted by $\tilde{\tau}$ the parallel displacement in $\mathscr{O}^2(M)$, determined by Γ' along τ, since τ is a geodesic, for each $s \in [a,b]$, we have $x_s^2 = \tilde{\tau}_s^0 (x_0^2) = u_s(u_0^{-1}(x_0^2))$. Then, $u_s^{-1}(x_s^2) = u_0^{-1}(x_0^2)$, that is, for each $s \in [a,b]$, with respect to the frame u_s, x_s^2 has the same components as x_0^2 with respect to the frame $u_0 = \underline{e}_p$. That is, if we put $u_s = ((u_i)_s, (u_{ij})_s), i \in I, (i,j) \in J$, we have:

$$x_s^2 = (\xi^i + \eta^i)(u_i)_s + \sum_{(i,j)\in J} \frac{2\xi^i\xi^j}{1+\delta^i_j}\,(u_{ij})_s$$

It follows that, with respect to the frame $\underline{e}_{x_s} = u_s(a'_s)^{-1}$, we have, for each $s \in [a,b]$:

$$x_s^2 = \left((a_s)^i_j(\xi^j + \eta^j) + 2 \sum_{(h,k)\in J} \frac{b^i_{hk}\xi^h\xi^k}{1+\delta^h_k} \right) (e_i)_{x_s} + \frac{1}{2} \sum_{\substack{(i,j)\\(h,k)}\in J} (a_s^{(2)})^{ij}_{hk}\, \frac{2\xi^h\xi^k}{1+\delta^h_k}\,(e_{ij})_{x_s}.$$

On the other hand, if we put $\dot{x}_s = \overline{\xi}^i_s (e_i)_{x_s}$, we have, with respect to the frame $\underline{e}_{x_s}$:

$$x_s^2 = (\overline{\xi}^i + \overline{\eta}^i)(e_i)_{x_s} + \frac{1}{2} \sum_{(i,j)\in J} \frac{2\overline{\xi}^i_s\overline{\xi}^j_s}{1+\delta^i_j}\,(e_{ij})_{x_s}.$$

By the last equalities, we obtain, with a direct computation, that for each $i \in I$, for each $s \in [a,b], (a_s)^i_h\xi^h = \overline{\xi}^i_s$ and then, for each $s \in [a,b]$, with respect to the frame u_s, we have $\dot{x}_s = \overline{\xi}^j_s(a_s^{-1})^i_j(u_i)_s = \xi^i(u_i)_s$, that is, with respect to the frame u_s, $\dot{x}_s$ has the same components as $\dot{x}_0$ with respect to the frame $u_0 = \underline{e}_p$. Then $\dot{x}_s = u_s(u_0^{-1}(x_0))$. If we put $\xi = (\xi^i) \in \mathbf{R}^n$; let $B'(\xi)$ be the standard horizontal vector field corresponding to $\xi = (\xi, 0) \in \mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^t$ with respect to Γ' on P'. We have, $u_s^{-1}(\pi'_{u_s}(\dot{u}_s)) = u_s^{-1}(\dot{x}_s) = \xi$. It follows that $\pi'_{u_s}(\dot{u}_s) = u_s(\xi)$ and, by the definition of $B'(\xi)$, $B'(\xi)_{u_s} = \dot{u}_s$, i.e. the curve τ' is an integral curve of a standard horizontal vector field. Since $\pi'(u_s) = x_s$ the result follows.

Corollary 2.1. *Let Γ' be a connection on P' and Γ the connection induced on L by f'. If τ is a geodesic for Γ', then τ is a geodesic for Γ.*

The proof follows by prop. 2.2 and by remark 2.1.

3. GENERALIZED AFFINE CONNECTIONS AND AFFINE CONNECTIONS

Let $\mathscr{A}^2_p(M)$ be the vector space $\mathscr{O}^2_p(M)$ considered as an affine real space and A^t the vector space $\mathbf{R}^t$ considered as an affine space. We consider the groups:

$$A(t,\mathbf{R}) = \left\{ \overline{a} \in G_{t+1} | \overline{a} = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix}, a \in G_t, \xi \in \mathbf{R}^t \right\}$$

$$\widetilde{A}(t,\mathbf{R}) = \left\{ \overline{a} \in A(t,\mathbf{R}) | \overline{a} = \begin{pmatrix} \widetilde{a} & \xi \\ 0 & 1 \end{pmatrix}, \widetilde{a} \in \widetilde{G}, \xi \in \mathbf{R}^n \right\}$$

$$A'(t,\mathbf{R}) = \left\{ \overline{a} \in A(t,\mathbf{R}) | \overline{a} = \begin{pmatrix} a' & \xi \\ 0 & 1 \end{pmatrix}, a' \in G', \xi \in \mathbf{R}^n \right\}, \textit{ where } \xi = \begin{pmatrix} \xi \\ 0 \end{pmatrix} \in \mathbf{R}^t.$$

It is well known that we can identify $A(t,\mathbf{R})$ with the group of the affine transformations of A^t and $\widetilde{A}(t,\mathbf{R})$ with the subgroup of the affine transformations of A^t that preserve $\mathbf{R}^n$. Moreover the sequence

$$(1) \qquad 0 \to \mathbf{R}^t \xrightarrow{\alpha} A(t,\mathbf{R}) \xrightarrow{\beta} G_t \to 1$$

with $\alpha(\xi) = \begin{pmatrix} U & \xi \\ 0 & 1 \end{pmatrix}$ *and* $\beta \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix} = a$ is a splitting exact sequence. The splitting morphism $\gamma : G_t \to A(t,\mathbf{R})$ is given by $\gamma(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. It follows that $A(t,\mathbf{R})$

is a semidirect product of $\mathbf{R}^t$ and G_t and the Lie algebra $a(t,\mathbf{R})$ is semidirect sum of $gl(t,\mathbf{R})$ and $\mathbf{R}^t$. Moreover we have $ad(G_t)(\mathbf{R}^t) = \mathbf{R}^t$. Analogous results hold for the sequences:

$$0 \to \mathbf{R}^n \xrightarrow{\widetilde{\alpha}} \widetilde{A}(t,\mathbf{R}) \xrightarrow{\widetilde{\beta}} \widetilde{G} \to 1; \quad 0 \to \mathbf{R}^n \xrightarrow{\alpha'} A'(t,\mathbf{R}) \xrightarrow{\beta'} G' \to 1$$

where $\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\alpha',\beta',\gamma'$ are defined as α,β,γ.

Definition 3.1. *We call 2-osculating affine frame at a point $p \in M$, any $(F_0;F_1,\ldots,F_t) \in (\mathcal{O}^2_p(M))^{t+1}$ such that $(F_1,\ldots,F_t)$ is a 2-osculating frame in p, that is a basis of $\mathcal{O}^2_p(M)$.*

Definition 3.2. *We call natural 2-osculating affine frame at a point $p \in M$, a 2-osculating affine frame in p $(F_0;F_1,\ldots,F_t)$, such that F_0 is a tangent vector at p and $(F_1,\ldots,F_t)$ is a 2-osculating natural frame at p.*

It is easy to verify that the set $\mathcal{A}^2_p$ of 2-osculating affine frames at p is bjective to the set of the affine transformations between A^t and $A^2_p(M)$. Moreover, if we put $\mathcal{A}^2(M) = \bigcup_{p\in M} \mathcal{A}^2_p$ and define $\overline{\pi} : \mathcal{A}^2(M) \to M$ so that for each $\overline{u} \in \mathcal{A}^2_p, \overline{\pi}(\overline{u}) = p$ (cfr. [5], [7]) it is easy to prove the following results:

Proposition 3.1. $\mathcal{A}^2 = (\mathcal{A}^2(M), M, \overline{\pi}, A(t,\mathbf{R}))$ *is a principal fibre boundle with base M, total space $\mathcal{A}^2(M)$ and structure group $A(t,\mathbf{R})$. This boundle is called the principal fibre bundle of 2-osculating affine frames of M.*

Proposition 3.2. *The structure group $A(t,\mathbf{R})$ of the bundle $\mathcal{A}^2$ is reducible to its subgroups $\widetilde{A}(t,\mathbf{R})$ and $A'(t,\mathbf{R})$.*

Definition 3.3. *The reduced bundle $\widetilde{\mathcal{A}}^2 = (\widetilde{\mathcal{A}}^2(M), M, \widetilde{\overline{\pi}}, \widetilde{A}(t,\mathbf{R}))$ is called principal reduced fibre bundle of the 2-osculating affine frames of M. The reduced bundle $\mathcal{A}'^2 = (\mathcal{A}'^2(M), M, \overline{\pi}', A'(t,\mathbf{R}))$ is called principal reduced fibre bundle of natural 2-osculating affine frames of M.*

It is easy to see that the map $\gamma : P(M) \to \mathcal{A}^2(M)$ such that $\gamma(u) = (0_p;u)$ together with the homomorphism γ of the sequence (1) gives rise to an injective bundle morphism and then P is a subbundle of $\mathcal{A}^2$. Analogously the map $\beta : \mathcal{A}^2(M) \to P(M)$ such that $\beta(F_0;F_1,\ldots,F_t) = (F_1,\ldots,F_t)$ together with the homomorphism β of the sequence (1), gives rise to a bundle morphism and, moreover $\beta \circ \gamma = id$. Analogously $\widetilde{P}$ is a subbundle of $\widetilde{\mathscr{A}}^2$ and P' is a subbundle of $\mathscr{A}'^2$. The relative morphisms $\widetilde{\gamma}, \widetilde{\beta}, \gamma', \beta'$ are defined in an analogous way and $\widetilde{\beta} \circ \widetilde{\gamma} = id, \beta' \circ \gamma' = id$. Since $\mathfrak{a}(t, \mathbf{R}) = \mathfrak{gl}(t, \mathbf{R}) \oplus \mathbf{R}^t$ and $ad(G_t)(\mathbf{R}^t) = \mathbf{R}^t$, using the theory of connections on a principal fibre bundle it is easy to see that, for any connection 1-form $\overline{\omega}$ on $\mathscr{A}^2$ we have $\gamma^* \overline{\omega} = \omega + \phi$, where ω is a connection 1-form on P and ϕ is a tensorial 1-form on P of type $(G_t, \mathbf{R}^t)$. Moreover the morphism $\beta : \mathscr{A}^2 \to P$ maps the connection $\overline{\Gamma}$ whose 1-form is $\overline{\omega}$ in the connection whose 1-form is ω. Analogousely, since $\widetilde{\mathfrak{a}}(t, \mathbf{R}) = \widetilde{\mathfrak{g}} \oplus \mathbf{R}^n$ and $ad(\widetilde{G})(\mathbf{R}^n) = \mathbf{R}^n$ ($\mathfrak{a}'(t, \mathbf{R}) = \mathfrak{g}' \oplus \mathbf{R}^n$ and $ad(G')(\mathbf{R}^n) = \mathbf{R}^n$) for any connection 1-form $\widetilde{\omega}$ on $\widetilde{\mathscr{A}}^2$ (on $\mathscr{A}'^2$) we have $\widetilde{\gamma}^* \widetilde{\omega} = \omega + \phi$ ($\gamma'^* \widetilde{\omega} = \omega + \phi$) where ω is a connection 1-form on $\widetilde{P}$ (on P') and ϕ is a tensorial 1-form on $\widetilde{P}$ (on P') of type $(\widetilde{G}, \mathbf{R}^t)$ (of type $(G', \mathbf{R}^t)$) taking values in $\mathbf{R}^n$. In each case the 1-form ϕ determines a tensor field of type (1, 1) on M, cfr. [5].

Definition 3.4. *A connection on the bundle $\widetilde{\mathscr{A}}^2$ or on the bundle $\mathscr{A}'^2$ is called generalized affine connection for 2-osculating vector fields. Any generalized affine connection which determines the Kronecker tensor field on M is called affine connection for 2-osculating vector fields.*

Using the morphism $\widetilde{\beta} : \widetilde{\mathscr{A}}^2 \to \widetilde{P}$, it is easy to verify that the set of the connections on $\widetilde{\mathscr{A}}^2$ is bijective to the set of the pairs whose first component is a connection on $\widetilde{P}$ and the second component is a tensorial field on M of type (1, 1). Moreover the set of the affine connections on $\widetilde{\mathscr{A}}^2$ is bijective to the set of the connections on $\widetilde{P}$. An analogous result holds for the connections on $\mathscr{A}'^2$ and P' using the morphism β'.

Proposition 3.3. *Let Γ be a connection on $\widetilde{P}$ (on P') and $\overline{\Gamma}$ be the connection on $\widetilde{\mathscr{A}}^2$ (on $\mathscr{A}'^2$) induced by Γ by means of $\widetilde{\gamma}$ (of γ') and $\overline{\Gamma}'$ be the affine connection on $\widetilde{\mathscr{A}}^2$ (on $\mathscr{A}'^2$) corresponding to Γ by means of $\widetilde{\beta}$, (of β'). We have $\overline{\Gamma} = \overline{\Gamma}'$.*

From the prop. 6.1. of chap. II of [5] we have to prove that $\widetilde{\beta}$ maps the horizontal space $\overline{Q}_{\widetilde{u}}$ of $\widetilde{\mathscr{A}}^2$ with respect to $\overline{\Gamma}$ into the horizontal space Q_u of $\widetilde{P}$ with respect to Γ.

If we take $\widetilde{u} \in \widetilde{\mathscr{A}}^2(M), u \in \widetilde{P}(M), \overline{a} \in A(t, \mathbf{R})$ such that $\widetilde{u} = \widetilde{\gamma}(u)\overline{a}$ we have: $\overline{Q}_{\widetilde{u}} = (R_{\overline{a}})_{\widetilde{\gamma}(u)}(\widetilde{\gamma}_u(Q_u))$ and then $\widetilde{\beta}_{\widetilde{u}}(\overline{Q}_{\widetilde{u}}) = Q_{u\widetilde{\beta}(a)} = Q_{\widetilde{\beta}(\widetilde{u})}$, since $\widetilde{\beta}(\widetilde{u}) = u\widetilde{\beta}(\overline{a})$. Let $\widetilde{E}$ be the vector bundle, associated with $\widetilde{\mathscr{A}}^2$, with standard fibre A^t and projection $\pi_{\widetilde{E}} : \widetilde{E} \to M$. As in prop. 7 of [7] the fibre of $\widetilde{E}$ over $p \in M$ is an affine space of dimension

t isomorphic to $A^2_p(M)$. So, the bundle $\widetilde{E}$ has $\bigcup_{p\in M} A^2_p(M)$ as total space and it will be denoted by $A^2(M)$.

Definition 3.5. *The fibre bundle $A^2(M)$, with base M, structure group $\widetilde{A}(t,\mathbf{R})$ and standard fibre A^t is called, affine 2-osculating vector bundle of M.*

Remark 3.1. Since the structure groups of the reduced bundles $\widetilde{\mathscr{A}}^2$ and $\mathscr{A}'^2$ are closed subgroups of $A(t,\mathbf{R})$, the vector bundles, associated with $\widetilde{\mathscr{A}}^2$ and $\mathscr{A}'^2$ are isomorphic to the vector bundle E, associated with $\widetilde{\mathscr{A}}^2$ (cfr. [4]). As in the case of the tangent bundle $T(M)$, we have that the bundle $A^2(M)$ is isomorphic to the bundle $\mathscr{O}^2(M)$ (cfr. [5]).

Definition 3.6. *Any (differentiable) section of $A^2(M)$ is called pointed (differentiable) q-osculating vector field, $q = 1, 2$.*

Let Γ be a connection on $\mathscr{A}^2$ (or on $\widetilde{\mathscr{A}}^2$, or on $\mathscr{A}'^2$). Given a curve τ on M, we consider the notions of parallel displacement of the fibres of $\mathscr{A}^2$ (or of $\widetilde{\mathscr{A}}^2$, or of $\mathscr{A}'^2$) along τ, and of parallel displacement along τ in $A^2(M)$ in the usual way (cfr. [5]). Since the parallel displacement gives rise to an affinity between the fibres of $A^2(M)$, it will be called *affine parallel displacement (a.p.d.)*, while the parallel displacement in $\mathscr{O}^2(M)$ will be called *linear parallel displacement (l.p.d.)*.

Proposition 3.4. *The a.p.d. determined by a connection $\widetilde{\Gamma}$ on $\widetilde{\mathscr{A}}^2$, preserves the osculating order of the pointed osculating vectors.*

The proof follows as in prop. 14 of [7].

Proposition 3.5. *Let $\overline{\Gamma}$ be a connection on $\mathscr{A}^2$ and Γ the connection on P induced by $\overline{\Gamma}$ by means of β. If the a.p.d. determined by $\overline{\Gamma}$ preserves the osculating order of the pointed osculating vectors, then the l.p.d. determined by Γ preserves the osculating order of the osculating vectors.*

Let $\tau = (x_s), s \in [0,1]$, be a curve of $M, \omega_0 \in \mathscr{O}^2_{x_0}(M), \omega_0 = \phi(u_0,\xi)$, with $u_0 \in P(M), \xi \in \mathbf{R}^t$, where $\phi : P(M) \times \mathbf{R}^t \to \mathscr{O}^2(M)$ is the canonical projection. Let $(\omega_s) = (\phi(u_s,\xi)), s \in [0,1]$, be the horizontal lift of τ to P with origin at u_0. Moreover put $\overline{\omega}_0 = \Psi(\omega_0)$, where Ψ is the isomorphism of $\mathscr{O}^2(M)$ on $A^2(M)$, and $(\overline{\omega}_s), s \in [0,1]$, the horizontal lift of τ to $A^2(M)$ with origin $\overline{\omega}_0$. If $\omega_0 \in T_{x_0}(M) \subset \mathscr{O}^2_{x_0}(M)$, then $\Psi(\omega_0) = \overline{\omega}_0 \in A_{x_0}(M) \subset A^2_{x_0}(M)$ and by the hypothesis, for each $s \in [0,1] \overline{\omega}_s \in$

$A_{x_s}(M)$, so that $\xi \in \mathbf{R}^n \subset \mathbf{R}^t$ and then, for each $s \in [0,1], \omega_s \in T_{x_s}(M)$. Finally, the l.p.d. $\tilde{\tau}$ determined by Γ, maps 2-osculating non tangent vectors in vectors of the same type, since the l.p.d. corresponding to the opposite curve of τ is the inverse isomorphism of $\tilde{\tau}$.

Proposition 3.6. *The l.p.d. determined by a connection Γ' on P' preserves the osculating order of the osculating vectors.*

The a.p.d. determined by a connection Γ' on $\mathscr{A}'^2$ preserves the osculating order of the pointed osculating vectors.

Since P' is a subbundle of $\tilde{P}, \Gamma'$ determines a unique connection $\tilde{\Gamma}$ on $\tilde{P}$ such that the reduction morphism maps horizontal subspaces of Γ' in horizontal subspaces of $\tilde{\Gamma}$. The l.p.d. determined by Γ' is the same as that determined by $\tilde{\Gamma}$, if the initial point belongs to P'. By the prop. 14 of [7], the first result follows. The last result can be proved in the same way.

Proposition 3.7. *Let Γ be a connection on P and consider the following properties:*

a) *Γ is reducible to a connection Γ' on P'.*

b) *The l.p.d. determined by Γ along a differentiable curve of M preserves the osculating order of the osculating vectors.*

c) *The parallel displacement determined in P by Γ along a differentiable curve of M maps natural 2-osculating frames in natural 2-osculating frames.*

One has: a) $\Leftrightarrow$ c) *and* a) $\Rightarrow$ b).

Since P' is a subbundle of P, if Γ is reducible to Γ' the horizontal subspaces of Γ' are mapped in the horizontal subspaces of Γ by the reduction morphism. Then, if $\tau = (x_s), s \in [0,1]$, is a curve of M, the horizontal lifts of τ in P' and in P are the same if the initial point is chosen in P'; then c) follows immediately. Moreover, if $\omega_0 \in \mathscr{O}^2(M), \omega_0 = \phi'(u_0, \xi)$ with $u_0 \in P'(M), \xi \in \mathbf{R}^t$, where $\phi' : P'(M) \times \mathbf{R}^t \to \mathscr{O}^2(M)$ is the canonical projection, and $\pi_{\mathscr{O}^2(M)}(\omega_0) = x_0 = \pi'(u_0)$; then the horizontal lift of τ to $\mathscr{O}^2(M)$, with initial point ω_0, is given by $(\omega_s), s \in [0,1], \omega_s = \phi'(u_s, \xi)$, where $(u_s), s \in [0,1]$, is the horizontal lift of τ to P' with initial point u_0. By the prop. 3.6, it follows that the l.p.d. determined by Γ preserves the osculating order of the osculating vectors, that is a) $\Rightarrow$ b). To prove that c) $\Rightarrow$ a), we verify that for each $u \in P'(M)$, the horizontal subspace $Q_u(P(M))$ of $T_u(P(M))$ is tangent to $P'(M)$. For it, fix $u_0 \in P'(M) \subset P(M), X \in Q_{u_0}(P(M)), x_0 = \pi'(u_0), \overline{X} = \pi'_{u_0}(X) \in T_{x_0}(M)$, and let $\tau = (x_s), s \in]-\varepsilon, \varepsilon[, \varepsilon \in \mathbf{R}^*_+$, be a curve of M through x_0 and such that $\dot{x}_0 = \overline{X}$. Let $(u_s), s \in]-\varepsilon, \varepsilon[$, be the horizontal lift of τ in P through u_0 such that $\dot{u}_0 = X$. By c), for each $s \in]-\varepsilon, \varepsilon[$, we have $u_s \in P'(M)$, so that the curve $(u_s), s \in]-\varepsilon, \varepsilon[$, lies in $P'(M)$ and so, the vector X is tangent to $P'(M)$ at u_0.

Analogously, we have:

Proposition 3.8. *Let Γ be a connection on $\mathscr{A}'^2$ and consider the properties:*

a) *Γ is reducible to a connection Γ' on $\mathscr{A}'^2$.*

b) *The a.p.d. determined by Γ along a differentiable curve of M preserves the osculating order of the pointed osculating vectors.*

c) *The parallel displacement in $\mathscr{A}^2$ determined by Γ along a differentiable curve of M, maps natural affine 2-osculating frames in natural affine 2-osculating frames.*

One has: a) $\Leftrightarrow$ c) *and* a)$\Rightarrow$ b).

4. DEVELOPMENTS

Let Γ be an affine connection on $\widetilde{\mathscr{A}}^2$, $\widetilde{\Gamma}$ the connection on $\widetilde{P}$ induced by Γ, and $\tau = (x_s), s \in [0,1]$, a differentiable curve of M. We denote with $\widetilde{\tau}$ the l.p.d. determined by $\widetilde{\Gamma}$ along τ and with $\widetilde{\tau}'$, the a.p.d. determined by Γ along τ. Finally identifying the points of τ with the zero pointed vector field along τ we obtain a curve $c_s = \widetilde{\tau}'^s_0(x_s), s \in [0,1]$, in $A^2_{x_0}(M)$.

Definition 4.1. *The curve $c_s = \widetilde{\tau}'_0(x_s), s \in [0,1]$, is called development of τ in $A^2_{x_0}(M)$.*

Proposition 4.1. *With the introduced notation, if we put $Y_s = \widetilde{\tau}^s_0(\dot{x}_s), s \in [0,1]$, we have $\dot{c}_s = Y_s$.*

The proof follows as in prop. 4.1., chap. III of [5].

In a similar way, given an affine connection Γ on $\mathscr{A}'^2$, we can consider the development in $A^2_{x_0}(M)$ of a curve $\tau = (x_s), s \in [0,1]$. Let Γ' be the connection on P', induced by Γ, $\widetilde{\tau}$ the l.p.d. determined by Γ' along τ and $\widetilde{\tau}'$ the a.p.d. determined by Γ along τ.

Definition 4.2. *The curve $c_s = \widetilde{\tau}'_0(x_s), s \in [0,1]$, is called development of τ in $A^2_{x_0}(M)$.*

Proposition 4.2. *With the introduced notation, if we put $Y_s = \widetilde{\tau}^s_0(\dot{x}_s), Z_s = \widetilde{\tau}^s_0(x^2_s), s \in [0,1]$, then $\dot{c}_s = Y_s$ and $c^2_s = Z_s$.*

We suppose that $x_s \in U$, for each $s \in [0,1]$, where (U, ϕ) is a neighborhood with coordinates y^i. Fixed $u_0 \in P'(M)$ such that $\pi'(u_0) = x_0$, let $(u_s), s \in [0,1]$, be the horizontal lift of τ to P', with origin at u_0. If we put $u'_0 = \gamma'(u_0) = (0, u_0) \in \mathscr{A}'^2(M)$, we have $\overline{\pi}'(u'_0) = x_0$. Let $(u'_s), s \in [0,1]$, be the horizontal lift of τ to $\mathscr{A}'^2$ with origin at u'_0. Then, there exists a curve of $\mathbb{R}^n, (\xi_s), s \in [0,1]$, such that $u'_s = u_s \overline{a}_s$, where for each $s \in [0,1], \alpha(\xi_s) = \overline{a}_s$, and u_s is identified with $\gamma'(u_s)$. Moreover, since $u'_0 = \gamma'(u_0) = u_0, \overline{a}_0$ is the identity matrix. As in prop. 4.1. of chap. III of [5], using the connection 1-forms ω' of Γ', and ω of Γ, we obtain $\dot{c}_s = Y_s$ for each $s \in [0,1]$, let $\underline{e}_{x_s}$ be the natural 2-osculating frame in x_s corresponding to (U, ϕ). Let us observe that, for each $s \in [0,1]$, $u_s \in P'(M)$ and so there exists a system (U', ϕ') of local coordinates $y^{i'}$ such that u_s is the natural 2-osculating frame corresponding to (U', ϕ'). It follows that $u_s = \underline{e}_{x_s} \sigma_s$ where

$$\sigma_s = \begin{pmatrix} \theta(s) & \overline{\theta}(s) \\ 0 & \theta^{(2)}(s) \end{pmatrix} \in G' \text{ and } \theta(s) = (\theta^i_{i'}(s)) = \left(\frac{\partial y^i}{\partial y^{i'}}\right) \in G_n,$$

$$\overline{\theta}(s) = \left(\frac{1}{2}\theta^i_{i'j'}(s)\right) = \left(\frac{1}{2}\,\frac{\partial^2 y^i}{\partial y^{i'}\partial y^{j'}}(s)\right) \in M^n_m.$$

Obviously σ_0 is the identity matrix. For each $s \in [0,1]$, the components of the curve c_s in the affine 2-osculating frame $\underline{v}_{x_0} = (0; \underline{e}_{x_0}) = \gamma'(\underline{e}_{x_0})$ are $c^i_s = -\xi^i_s, i \in I; c^{ij}_s = -\xi^{ij}_s, (i,j) \in J$. Moreover the components of c_s and Y_s in $\underline{v}_{x_0}$ are:

$$\frac{dc^i_s}{ds} = -\frac{d\xi^i_s}{ds}, i \in I; \quad \frac{dc^{ij}_s}{ds} = -\frac{d\xi^{ij}_s}{ds}, (i,j) \in J$$

and by (1.1), the components of c^2_s in $\underline{v}_{x_0}$ are:

$$(c^2_s)^i = -\frac{d\xi^i_s}{ds} - \frac{1}{2}\,\frac{d^2\xi^i_s}{ds^2}, i \in I; \quad (c^2_s)^{ij} = \frac{d\xi^i_s}{ds}\,\frac{d\xi^j_s}{ds}, i,j \in I.$$

Since σ_0 is the identity matrix, we have $c^2_s = u_0(-\xi^2_s)$. On the other hand, computing the components of $\dot{x}_s$ and x^2_s in the frame $\underline{e}_{x_s}$ we have $x^2_s = u_s(-\xi^2_s)$. By the last two equalities if follows that:

$$Z_s = \overline{\tau}^s_0(x^2_s) = u_0(u_s^{-1}(x^2_s)) = c^2_s.$$

Corollary 4.1. *With the notation in prop. 4.2, the curve τ is a geodesic with respect to Γ' if and only if its development in $A^2_{x_0}(M)$ is a straight line.*

We suppose that $\tau([0,1]) \subset U$, where (U, ϕ) is a coordinate neighborhood of M. If τ is a geodesic with respect to Γ', for each $s \in [0,1]$, we have $c^2_s = \overline{\tau}^s_0(x^2_s) = x^2_0$. By (1.1) it follows that:

$$\begin{cases} \dfrac{dc^i_s}{ds} + \dfrac{1}{2}\,\dfrac{d^2 c^i_s}{ds^2} = const., \ i \in I \\ \dfrac{dc^i_s}{ds}\,\dfrac{dc^j_s}{ds} = const., \ i,j \in I \end{cases}$$

and so $\dfrac{dc^i_s}{ds}$ is a constant, for each $i \in I$, and c is a straight line.

The viceversa is obvious.

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Luigia Di Terlizzi, Francesca Verroca
Dipartimento di Matematica
Campus Universitario
Via G. Fortunato
Bari, Italy